Combinatorial Applications of Ordinal Sum Decompositions

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Abstract. For any double sequence $(q_{k,n})$ with $q_{k,0} = 0$, the "summatorial sequence" $(p_{k,n}) = \sum(q_{k,n})$ is defined by $p_{0,0} = 1$ and $p_{k,n} = \sum_{j=0}^{k} \sum_{m=1}^{n} q_{j,m} p_{k-j,n-m}$. If $q_{k,n} = 0$ for $k < n-1$ then there exists a unique sequence (c_j) satisfying the recurrence $p_{k,n} = \sum_{j=0}^{k} c_j p_{k-j,n-j-1}$ for $k < n$. We apply this combinatorial recursion to certain counting functions on finite posets. For example, given a set A of positive integers, let $p_{k,n}$ denote the number of unlabeled posets with n points and exactly k antichains whose cardinality belongs to A, and let $q_{k,n}$ denote the corresponding number of ordinally indecomposable posets. Then $(p_{k,n})$ is the summatorial sequence of $(q_{k,n})$. If $2 \in A$ then $(p_{k,n})$ enjoys the above recurrence for $k < n$. In particular, for fixed k, there is a polynomial p_k of degree k such that $p_{k,n} = p_k(n)$ for all $n \geq k$, and $p_{k,n}$ is asymptotically equal to $\binom{n-1}{k}$. For some special classes A and small k, we determine the numbers c_k and the polynomials p_k explicitly. Moreover, we show that, at least for small k, the remainder sequences $p_{k,n} - p_k(n)$ satisfy certain Fibonacci recursions, proving a conjecture of Culberson and Rawlins. Similar results are obtained for labeled posets and for naturally ordered sets.

Introduction

The enumeration of all posets with a given finite number of points by means of a reasonable explicit or recursive formula is still an open problem (for a survey and recent numerical results, see [8]). However, the enumeration of finite posets with a relatively small number of "doubletons", i. e. two-element antichains, has made some progress in the last years. In [2] Culberson and Rawlins had discovered that the numbers $p_{k,n}$ of unlabeled posets with n points and k doubletons satisfy a recursion of the form

$$p_{k,n} = \sum_{j=0}^{k} c_j p_{k-j,n-j-1}$$

as long as $k < n$, while this recurrence formula fails for $k \geq n$. The formula was proved in [6] and [7], and it was shown that this phenomenon mainly relies on the fact that the corresponding numbers $q_{k,n}$ of *ordinally indecomposable* posets (see Section 2) are zero for $k < n-1$, and that the computation of the numbers $p_{k,n}$ may be reduced to that of the numbers $q_{k,n}$ (cf. Stanley [11]).

This observation suggests to consider such triangular double sequences in general and to derive analogous recurrences for other types of counting functions on finite posets. As the above formula shows, it suffices to know the diagonal sequences $p_{k,k}$ and $p_{k,k+1}$ in order to compute the numbers c_k and the values $p_{k,n}$ for all $n \geq k$.

Many counting problems concerning antichains in finite posets may be attacked by this method. For example, generalizing the situation of two-element antichains, consider any set A of positive integers containing the number 2, and denote by $p_{k,n}$ the number of all unlabeled posets having exactly k antichains whose size belongs to A. Then we are able to show that the numbers $p_{k,n}$ satisfy a recursion of the above type, for $k < n$. After having developed the general machinery for the aforementioned recurrences, it is not hard to find explicit expressions for $p_{k,n}$ when k is small. It turns out that for fixed k and variable $n \geq k$,

$$p_{k,n} = p_k(n)$$

where p_k is a polynomial of degree k with leading coefficient $1/k!$. Similarly, we shall see that for $n \geq k$, the corresponding numbers $P_{k,n}$ of *labeled posets* may be expressed as $n!$ times a polynomial of degree k in the variable n, with leading coefficient $1/(k!2^k)$.

Although the polynomial values $p_k(n)$ differ from the exact values $p_{k,n}$ for $k > n$, it is possible to express the differences

$$d_{j,k} = p_{k,k-j-1} - p_k(k-j-1)$$

in terms of the numbers $s_{j,m} = q_{j+m,m+1}$ and we have explicit formulae for these numbers if j is small. This enables us to show that at least for $j = 0$ and $j = 1$ (and probably also for greater values of j), the differences $d_{j,k}$ satisfy certain generalized Fibonacci recursions, confirming a conjecture of Culberson and Rawlins [2] based on their numerical computations for small k.

At the end of the paper, we use recent numerical tables from [12] to determine the polynomials p_k explicitly for $k \leq 11$ and some important choices of A.

1. Identities for triangular double sequences

We denote by ω the set of all natural numbers including 0, and by $\leq$ the usual order on ω. For $n \in \omega$, it will be convenient to denote by $\underline{n}$ the set of natural numbers $1, \ldots, n$ (in particular $\underline{0} = \emptyset$).

Let $(q_{k,n}) = (q_{k,n} : k, n \in \omega)$ be any double sequence (of real numbers), and let

$$q(x,y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q_{k,n} x^k y^n$$

denote its generating function (considered as a formal power series without any convergence restrictions). If $q_{k,0} = 0$ for all k, define the *summatorial (double) sequence* $(p_{k,n}) = \sum(q_{k,n})$ by the recursion

$$\begin{aligned} p_{0,0} &= 1 \\ p_{k,n} &= \sum_{j=0}^{k} \sum_{m=1}^{n} q_{j,m} p_{k-j,n-m} \qquad \text{for } (k,n) \neq (0,0) \end{aligned} \tag{1.1}$$

A straightforward comparison of coefficients shows that the generating function

$$p(x,y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p_{k,n} x^k y^n$$

is given by

$$p(x,y) = (1 - q(x,y))^{-1} = 1 + q(x,y) + q(x,y)^2 + \dots \tag{1.2}$$

The combinatorial significance of this construction is evident: suppose $\mathcal{J}$ is a class of unlabeled (i.e. isomorphism classes of) finite structures (e.g. topological spaces, ordered sets, graphs, etc.) If f and g are functions from $\mathcal{J}$ to ω and $q_{k,n}$ is the number of structures $S \in \mathcal{J}$ with $f(S) = k$ and $g(S) = n$, then the summatorial sequence $(p_{k,n}) = \sum(q_{k,n})$ counts the number of all finite sequences $(S_1, \dots, S_r)$ in $\mathcal{J}$ such that $f(S_1) + \dots + f(S_r) = k$ and $g(S_1) + \dots + g(S_r) = n$.

By a *1-triangular sequence* we mean a double sequence $(q_{k,n})$ such that

$$q_{0,1} = 1, \quad q_{k,0} = 0 \text{ for all } k, \quad \text{and } q_{k,n} = 0 \text{ for } k < n-1.$$

Let $(q_{k,n})$ be any such 1-triangular sequence. Then the generating function

$$s(x,y) = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} s_{k,n} x^k y^n = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} q_{n+k,n+1}\, x^k y^n \tag{1.3}$$

satisfies the functional equation

$$q(x,y) = y \cdot s(x, xy) \tag{1.4}$$

and conversely, any such equation forces $(q_{k,n})$ to be 1-triangular if only $s_{0,0} = 0$. Using the coefficients $s_{j,m}^{(i)}$ of the i-th power

$$s(x,y)^i = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} s_{j,m}^{(i)} x^j y^m,$$

we obtain for the generating function of the summatorial sequence $(p_{k,n}) = \sum(q_{k,n})$:

$$p(x,y) = (1 - q(x,y))^{-1} = (1 - y \cdot s(x,xy))^{-1}$$
$$= \sum_{i=0}^{\infty} y^i s(x,xy)^i = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} s_{j,m}^{(i)} x^{j+m} y^{i+m},$$

whence

$$p_{k,n} = \sum_{i=\max(n-k,0)}^{n} s_{k-n+i,n-i}^{(i)} = \sum_{i=\max(0,k-n)}^{k} s_{i,k-i}^{(i+n-k)} \tag{1.5}$$

As $q_{0,1} = 1$, we may write the generating function $s(x,y)$ in the form

$$s(x,y) = 1 + \overset{\circ}{s}(x,y),$$

where the constant term of $\overset{\circ}{s}(x,y)$ is zero. As before, we use the powers

$$\overset{\circ}{s}(x,y)^u = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \overset{\circ}{s}{}_{j,m}^{(u)} x^j y^m$$

to obtain

$$s(x,y)^i = \sum_{u=0}^{i} \binom{i}{u} \overset{\circ}{s}(x,y)^u = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{u=0}^{j+m} \binom{i}{u} \overset{\circ}{s}{}_{j,m}^{(u)} x^j y^m$$

becauses $\overset{\circ}{s}{}_{j,m}^{(u)}$ vanishes for $u > j + m$, and $\binom{i}{u} = 0$ for $u > i \geq 0$. Taking

$$s_{j,m}^{(i)} = \sum_{u=0}^{j+m} \binom{i}{u} \overset{\circ}{s}{}_{j,m}^{(u)}$$

as the defining equation of the coefficients $s_{j,m}^{(i)}$ also if i is a negative integer, we see that they are polynomials in i of degree at most $j + m$. Now, we may rewrite (1.5) in the following form:

$$p_{k,n} = \sum_{i=0}^{k} s_{i,k-i}^{(i+n-k)} - \sum_{i=0}^{k-n-1} s_{i,k-i}^{(i+n-k)}$$

where the first sum is a polynomial $p_k(n)$ of degree at most k. Thus, setting $p_{k,n} = 0$ for negative n, and

$$d_{j,k} = -\sum_{i=0}^{j} s_{i,k-i}^{(i-j-1)} = p_{k,k-j-1} - p_k(k-j-1) \tag{1.6}$$

we arrive at

Proposition 1.1. *Let $(q_{k,n})$ be any 1-triangular sequence. Then the corresponding summatorial sequence $(p_{k,n})$ satisfies an equation*

$$p_{k,n} = p_k(n) + d_{k-n-1,k},$$

where p_k is a polynomial of degree $\leq k$, and the differences $d_{j,k}$ vanish for $j < 0$. The generating function $d(x,y) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} d_{j,k} x^j y^k$ and the generating function $s(x,y)$ with $q(x,y) = y \cdot s(x,xy)$ are related by the following functional equation:

$$d(x,y) \cdot (x - s(xy,y)) = 1.$$

Proof. The last assertion can be deduced from (1.6) by a somewhat tedious comparison of coefficients. An alternative and perhaps more instructive way is the following. Observe that

$$p_{k,n} = \sum_{j=0}^{k} \sum_{m=1}^{j+1} q_{j,m} p_{k-j,n-m} \qquad \text{for } (k,n) \neq (0,0)$$

and

$$p_k(n) = \sum_{j=0}^{k} \sum_{m=1}^{j+1} q_{j,m} p_{k-j}(n-m).$$

Indeed, the first equation is true by (1.1) (recall that $q_{j,m} = 0$ for $j < m-1$); consequently, the second equation holds for fixed k and all $n > k$, since then $p_k(n) = p_{k,n}$ and $p_{k-j}(n-m) = p_{k-j,n-m}$ for $m \leq j+1$. But two polynomials having the same values for infinitely many entries must be equal. Thus

$$d_{k-n-1,k} = p_{k,n} - p_k(n) = \sum_{j=0}^{k} \sum_{m=1}^{j+1} q_{j,m} d_{k-j-n+m-1,k-j}$$

$$= \sum_{j=0}^{k} \sum_{m=0}^{j} q_{j,m+1} d_{k-j-n+m,k-j}$$

unless $(k,n) = (0,0)$, where $d_{-1,0} = 0 \neq -1 = q_{0,1} d_{0,0}$. Replacing j with $i+m$ and observing that $d_{k-i-n,k-i-m} = 0$ for $i > k-n$, we obtain

$$d_{k-n-1,k} = \sum_{i=0}^{k-n} \sum_{m=0}^{k-i} q_{i+m,m+1} d_{k-i-n,k-i-m} = \sum_{i=0}^{k-n} \sum_{m=0}^{k-i} s_{i,m} d_{k-i-n,k-i-m}$$

Then, replacing $k-n$ with j, we arrive at the equation

$$d_{j-1,k} = \sum_{i=0}^{j} \sum_{m=0}^{k-i} s_{i,m} d_{j-i,k-i-m} \quad \text{for } (j,k) \neq (0,0).$$

Finally, after insertion of suitable powers of x and y, a summation over all terms gives the desired functional equation

$$-1 + x \cdot d(x,y) = s(xy,y) \cdot d(x,y). \qquad \blacksquare$$

Our next (and crucial) observation is that for any 1-triangular sequence $(q_{k,n})$ the lower triangular part of the summatorial sequence $(p_{k,n}) = \sum(q_{k,n})$ can be computed if only the diagonal elements $p_{k,k}$ and $p_{k,k+1}$ are known (cf. [2] and [7]).

Proposition 1.2. *Let $(q_{k,n})$ be any 1-triangular sequence. Then there exists a unique formal power series*

$$c(x) = \sum_{j=0}^{\infty} c_j x^j$$

such that the summatorial sequence $(p_{k,n}) = \sum(q_{k,n})$ satisfies the following recursion:

$$p_{k,n} = \sum_{j=0}^{k} c_j p_{k-j,n-j-1} \qquad (k < n) \tag{1.7}$$

In fact, the generating function $c(x)$ is the unique solution f of the fixpoint equation

$$s(xf, f) = f,$$

where $s(x,y)$ is the formal power series determined by (1.3).

This was shown in [7] for a slightly more special situation, but the arguments carry over verbatim to the present general setting. The above fixpoint equation plays a role in the theory of Banach spaces (see [3], Chapter 10). The recursion below provides a method to compute the solutions of such fixpoint equations.

On account of the equations $q(x,y) = y \cdot s(x,xy)$ and $s(xy,y) = y$ for $y = c(x)$, the sequence (c_k) is determined by the formula (cf. [7])

$$c_k = \sum_{i=0}^{k} \sum_{j=0}^{k-i} c_j^{(i)} q_{k-j,k-j-i+1} \tag{1.8}$$

where the numbers $c_j^{(i)}$ are the coefficients of the i-th power

$$c(x)^i = \sum_{j=0}^{\infty} c_j^{(i)} x^j.$$

Hence these coefficients satisfy the following recursion:

$$c_0^{(0)} = 1, c_j^{(0)} = 0 \text{ for } j > 0, \text{ and } c_j^{(i+1)} = \sum_{k=0}^{j} c_k c_{j-k}^{(i)} \tag{1.9}$$

In particular, we obtain the following explicit expressions for the coefficients c_k:

$$c_0 = q_{0,1},$$
$$c_1 = q_{1,2} + q_{0,1}q_{1,1}$$
$$c_2 = q_{2,3} + q_{0,1}q_{2,2} + q_{1,2}q_{1,1} + q_{0,1}q_{1,1}^2 + q_{0,1}^2 q_{2,1},$$
etc.

Thus we have constructed the coefficients c_k from the numbers $q_{k,n}$, but it is easier to express them in terms of the diagonal sequences $(p_{k,k})$ and $(p_{k,k+1})$ by setting $n = k+1$ in (1.7):

$$c_k = p_{k,k+1} - \sum_{j=0}^{k-1} c_j p_{k-j,k-j} \tag{1.10}$$

Employing the generating functions

$$c(x) = \sum_{k=0}^{\infty} c_k x^k,$$
$$p^*(x) = \sum_{k=0}^{\infty} p_{k,k} x^k,$$
$$p_{\leq}(x,y) = \sum_{k\leq n} p_{k,n} x^k y^n,$$

the recurrence in Proposition 1.2 may be written as a functional equation:

$$p_{\leq}(x,y) = p^*(xy) + y \cdot c(xy) \cdot p_{\leq}(x,y),$$

which provides a representation of the two-parametric generating function $p_{\leq}(x,y)$ in terms of the one-parametric power series $c(x)$ and $p^*(x)$:

Corollary 1.3. *With the above notations and the same hypotheses as before,*

$$p_{\leq}(x,y) = p^*(xy) \cdot (1 - y \cdot c(xy))^{-1}.$$

Furthermore, one can derive the following binomial representation for $p_k(n)$ from 1.2, by a double induction argument (see [6] or [9]):

Corollary 1.4. *The polynomials $p_k(n)$ in Proposition 1.1 satisfy equations*

$$p_k(n) = \sum_{j=0}^{k} a_{k,j} \binom{n-j-1}{k-j}$$

where the coefficients $a_{k,j}$ can be determined recursively from the diagonal sequences $p_{k,k}$ and $p_{k,k+1}$, using formula (1.10):

$$a_{k,j} = \sum_{i=1}^{j+1} c_i(a_{k-i,\, j-i+1} - a_{k-i,\, j-i}) \text{ for } j < k \text{ (with } a_{k-i,-1} = 0),$$

$$a_{k,k} = p_{k,k}.$$

In particular, $a_{k,0} = c_1{}^k$.

Corollary 1.5. *If $c_1 \neq 0$ then the leading coefficient of the polynomial p_k is $c_1{}^k/k!$, and consequently, $p_{k,n}$ is asymptotically equal to $\binom{n-1}{k} c_1{}^k$.*

Of course, for fixed k, the binomial coefficient $\binom{n-1}{k}$ has the same asymptotical behavior as $n^k/k!$, but in concrete computations like those in Section 2, one observes that $\binom{n-1}{k}$ gives better approximations for $p_{k,n}$ than $n^k/k!$.

For explicit computations, the following reduction is often helpful:

Corollary 1.6. *Under the same hypotheses as before and the additional assumption $c_1 = 1$, the coefficients $a_{k,j}$ in Corollary 1.4 may be represented in the form*

$$a_{k,\, j} = \sum_{m=0}^{j} b_{m,\, j} \binom{k-j}{j-m},$$

where the numbers $b_{m,j}$ can be determined recursively from the diagonal sequences $p_{k,k}$ and $p_{k,k+1}$, using (1.10):

$$b_{m,\, j} = \sum_{i=1}^{m+1} (c_{i+1} - c_i)\, b_{m-i+1,\, j-i} - \sum_{i=1}^{m} c_i b_{m-i,\, j-i} \text{ for } m < j,$$

$$b_{j,\, j} = a_{j,\, j} = p_{j,\, j}.$$

Hence, for fixed j, the coefficients $a_{k,\, j}$ are polynomials in k. It may happen very well that the exact degree of these polynomials is strictly smaller than j. For example, if $c_0 = c_1 = c_2 = 1$ then the degree of a_j is not greater than $j/2$ (for details, see [6]).

In most of the situations we shall encounter, the sequences $(q_{k,n})$ and (c_k) enjoy certain monotonicity properties:

Lemma 1.7. *Let $(q_{k,n})$ be any 1-triangular sequence satisfying*

$$q_{k,n} \geq q_{k-1,n-1} \qquad (k, n > 0)$$

and let $(p_{k,n})$ *be its summatorial sequence. Then the sequence* (c_k) *in Proposition 1.2 is monotone increasing, and*

$$p_{k,n} \geq p_{k,n-1} + p_{k-1,n-1} \geq \binom{n-1}{k} \quad \textit{for all } k, n > 0.$$

Proof. Observe that the hypothesis ensures nonnegativity of the numbers $q_{k,n}$ and, consequently, of the numbers $p_{k,n}$. To prove the monotonicity assertion for the sequence (c_k), we use the recursions (1.8) and (1.9). First, by (1.9) and induction on $j + i$, we obtain $c_j^{(i)} \geq 0$ for all j and i. Then, by (1.8), we get

$$c_k = \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} c_j^{(i)} q_{k-j,k-j-i+1} + \sum_{i=0}^{k} c_{k-i}^{(i)} q_{i,1} \geq \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} c_j^{(i)} q_{k-j-1,k-j-i} = c_{k-1}.$$

The inequality $p_{k,n} \geq p_{k,n-1} + p_{k-1,n-1}$ follows by induction from the equation (1.1).

Finally, starting with the initial values $p_{0,n} = 1 = \binom{n-1}{0}$ and $p_{k,1} \geq p_{k,0} + p_{k-1,0} \geq 0 = \binom{0}{k}$ for $k > 0$, one obtains inductively $p_{k,n} \geq \binom{n-1}{k}$ for all $k \geq 0$ and $n > 0$, and consequently $p_{k,n-1} + p_{k-1,n-1} \geq \binom{n-2}{k} + \binom{n-2}{k-1} = \binom{n-1}{k}$ for $k \geq 1, n \geq 2$. ∎

2. Additive functions on the class of finite posets

The *ordinal sum* $Q \oplus R$ of two disjoint posets Q and R is their union, endowed with the partial order

$$x \leq_{Q \oplus R} y \Leftrightarrow x \leq_Q y \text{ or } x \leq_R y \text{ or } (x, y) \in Q \times R.$$

Of course, one may also form ordinal sums of non-disjoint posets, by replacing them in the usual way with disjoint isomorphic copies. A poset is said to be (ordinally) decomposable if it is empty or of the form $Q \oplus R$ for some nonempty posets Q and R; otherwise it is (ordinally) indecomposable. Each finite poset P has a unique decomposition

$$P = Q_1 \oplus \cdots \oplus Q_r$$

into indecomposable posets Q_i (with $r = 0$ for empty P). Hence finite posets are in one-to-one correspondence with finite sequences of indecomposable finite posets, and consequently, our considerations from Section 1 will apply to the present situation.

A function f assigning to each finite poset P a natural number is called *isomorphism-invariant* if it is constant on isomorphism classes. If, moreover, $f(\emptyset) = 0$

and $f(Q \oplus R) = f(Q) + f(R)$ for all pairs of disjoint finite posets Q, R then f is said to be *additive*. A few typical additive functions are:

- the *cardinality function*, assigning to each poset P the number $|P|$ of its points,
- the *height function*, assigning to each poset the maximal size of its chains,
- the *antichain function*, assigning to each poset the number of its nonempty antichains,
- the *incomparability function*, assigning to each poset the number of its two-element antichains ("incomparable pairs" or "doubletons"; cf. [7]).

Let f be any isomorphism-invariant function on the class of all finite posets. By $P^f_{k,n}$ we denote the number of posets $P = (\underline{n}, \leq_P)$ with $f(P) = k$, and by $p^f_{k,n}$ the corresponding number of unlabeled (i.e. isomorphism classes of) posets. Similarly, $Q^f_{k,n}$ and $q^f_{k,n}$ will denote the numbers of ordinally indecomposable posets with these properties. Notice that $Q^f_{0,0} = q^f_{0,0} = 0$, while $P^f_{0,0} = p^f_{0,0} = 1$ if f is additive. It will be convenient to use the symbols $\overline{p}^f_{k,n}$ and $\overline{q}^f_{k,n}$ for the quotients $P^f_{k,n}/n!$ and $Q^f_{k,n}/n!$, respectively. Further, we denote by $\tilde{p}^f_{k,n}$ the number of all naturally ordered sets $P = (\underline{n}, \leq_P)$ with $f(P) = k$ (where P is said to be *naturally ordered* if $x \leq_P y$ implies $x \leq y$ in the natural order on $\underline{n}$; cf. [1]), and by $\tilde{q}^f_{k,n}$ the corresponding number of ordinally indecomposable naturally ordered sets. Thus we have

$$\overline{q}^f_{k,n} \leq q^f_{k,n} \leq \tilde{q}^f_{k,n} \leq Q^f_{k,n} \quad \text{and} \quad \overline{p}^f_{k,n} \leq p^f_{k,n} \leq \tilde{p}^f_{k,n} \leq P^f_{k,n}.$$

Using the fact that every nonempty finite poset P is uniquely representable as a sum $Q \oplus R$ with an indecomposable poset Q, one arrives at the basic result that the counting functions $p^f_{k,n}$ etc. are the summatorial sequences of the counting functions $q^f_{k,n}$ for the corresponding indecomposable posets (see Section 1):

Proposition 2.1. *For any additive function f on finite posets,*

$$(p^f_{k,n}) = \sum(q^f_{k,n}),\ (\overline{p}^f_{k,n}) = \sum(\overline{q}^f_{k,n}),\ (\tilde{p}^f_{k,n}) = \sum(\tilde{q}^f_{k,n}).$$

Hence the generating functions

$$p^f(x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} p^f_{k,n} x^k y^n \quad \textit{and} \quad q^f(x,y) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} q^f_{k,n} x^k y^n$$

are related by the identity

$$p^f(x,y) = (1 - q^f(x,y))^{-1},$$

and similar equations hold for $\overline{p}^f$ and $\overline{q}^f$, as well as for $\tilde{p}^f$ and $\tilde{q}^f$.

Proof. It will suffice to consider the labeled case; the other cases are similar. Choose a set M of m distinct numbers between 1 and n. Then combine each indecomposable poset on M with each poset on $\underline{n} \setminus M$ and form the ordinal sum. In this way, each poset on $\underline{n}$ is obtained exactly once, and we get

$$P^f_{k,n} = \sum_{m=1}^{n} \binom{n}{m} \sum_{j=0}^{k} Q^f_{j,m} P^f_{k-j,n-m}.$$

Division by $n!$ yields the claimed equation (1.1) for $\overline{p}^f_{k,n}$ and $\overline{q}^f_{k,n}$ (recall that $\overline{p}^f_{0,0} = 1$). ∎

Henceforth $\mathcal{K}$ denotes any class of ordinally indecomposable posets which is closed under isomorphisms. Examples are (for a fixed natural number m):

$\mathcal{A}_m$, the class of all antichains with m elements,
$\mathcal{A}_{\le m}$, the class of all nonempty antichains with at most m elements,
$\mathcal{A}_{>m}$, the class of all antichains with more than m elements.

For any finite poset $P = (S, \le_P)$, we denote by $f^{\mathcal{K}}(P)$ the number of all subsets $R \subseteq S$ such that the induced poset $(R, \le_{P|R})$ belongs to $\mathcal{K}$. Furthermore, it will be convenient to write $P^{\mathcal{K}}_{k,n}$ for $P^{f^{\mathcal{K}}}_{k,n}$ etc. Hence, for example, $p^{\mathcal{K}}_{k,n}$ is the number of all unlabeled posets with n points and k subposets belonging to the class $\mathcal{K}$.

Proposition 2.2. *For any class $\mathcal{K}$ of indecomposable posets, the counting function $f^{\mathcal{K}}$ is additive. If $\mathcal{K}$ contains all two-element antichains then the double sequences $(q^{\mathcal{K}}_{k,n})$, $(\overline{q}^{\mathcal{K}}_{k,n})$ and $(\tilde{q}^{\mathcal{K}}_{k,n})$ are 1-triangular.*

Conversely, if one of these double sequences is 1-triangular and $\mathcal{K}$ contains no singleton then it must contain all two-element antichains.

Proof. The first statement is clear since any indecomposable subposet of an ordinal sum $Q \oplus R$ must be contained completely in Q or in R. If $\mathcal{A}_2 \subseteq \mathcal{K}$ and Q is any indecomposable poset then by Lemma 1 of [7],

$$f^{\mathcal{K}}(Q) \ge f^{\mathcal{A}_2}(Q) \ge |Q| - 1,$$

whence $q^{\mathcal{K}}_{k,n} = \overline{q}^{\mathcal{K}}_{k,n} = \tilde{q}^{\mathcal{K}}_{k,n} = 0$ for $k < n-1$. Furthermore, we have $q^{\mathcal{K}}_{k,0} = \overline{q}^{\mathcal{K}}_{k,0} = \tilde{q}^{\mathcal{K}}_{k,0} = 0$ because the empty set is ordinally decomposable.

Conversely, if one of these three double sequences is assumed to be 1-triangular then we must have $f^{\mathcal{K}}(Q) \ge |Q| - 1$ for each indecomposable poset Q, since

$$\overline{q}^{\mathcal{K}}_{k,n} \le q^{\mathcal{K}}_{k,n} \le \tilde{q}^{\mathcal{K}}_{k,n} \le Q^{\mathcal{K}}_{k,n} = n!\,\overline{q}^{\mathcal{K}}_{k,n},$$

and one of these numbers is zero whenever $k < n-1$. In particular,

$$f^{\mathcal{K}}(Q) \geq 1 \text{ for all } Q \in \mathcal{A}_2 .$$

If the singletons are not members of $\mathcal{K}$, then the only subset of $Q \in \mathcal{A}_2$ counted by $f^{\mathcal{K}}$ is Q itself, whence $Q \in \mathcal{K}$. ∎

Combining the previous result with Proposition 1. 2, we can now state a rather general recursion formula for posets (the case $\mathcal{K} = \mathcal{A}_2$ has been handled in [7]):

Proposition 2.3. *Let $\mathcal{K}$ be any isomorphism-closed class of ordinally indecomposable posets with $\mathcal{A}_2 \subseteq \mathcal{K}$. Then, for $n > k$, the numbers $p^{\mathcal{K}}_{k,n}$ of all unlabeled posets with n points and exactly k subposets in $\mathcal{K}$ satisfy a recursion of the form*

$$p^{\mathcal{K}}_{k,n} = \sum_{j=0}^{k} c^{\mathcal{K}}_j p^{\mathcal{K}}_{k-j,n-j-1} .$$

where the coefficients $c^{\mathcal{K}}_j$ can be determined recursively from the numbers $p^{\mathcal{K}}_{k,k}$ and $p^{\mathcal{K}}_{k,k+1}$. Analogous recurrences hold for the numbers $\tilde{p}^{\mathcal{K}}_{k,n}$ and $\bar{p}^{\mathcal{K}}_{k,n}$. Hence

$$P^{\mathcal{K}}_{k,n} = n \sum_{j=0}^{k} C^{\mathcal{K}}_j \binom{n-1}{j} P^{\mathcal{K}}_{k-j,n-j-1}$$

for suitable coefficients $C^{\mathcal{K}}_j = j!\bar{c}^{\mathcal{K}}_j$ and all $n > k$.

Next, we apply 1.1 and 1.5 to the present situation. Observing that

$$c^{\mathcal{K}}_0 = q^{\mathcal{K}}_{0,1} = \tilde{c}^{\mathcal{K}}_{0,1} = \bar{c}^{\mathcal{K}}_0 = \bar{q}^{\mathcal{K}}_{0,1} = 1,$$
$$c^{\mathcal{K}}_1 = q^{\mathcal{K}}_{1,2} = \tilde{c}^{\mathcal{K}}_1 = \tilde{q}^{\mathcal{K}}_{1,2} = 1,$$
$$\bar{c}^{\mathcal{K}}_1 = \bar{q}^{\mathcal{K}}_{1,2} = \frac{1}{2},$$

we arrive at

Corollary 2.4. *Under the same hypotheses as in Proposition 2.3, the numbers $p^{\mathcal{K}}_{k,n}$, $\tilde{p}^{\mathcal{K}}_{k,n}$ and $\bar{p}^{\mathcal{K}}_{k,n}$ are, for fixed k and all $n \geq k$, the values of certain polynomials of degree k in the variable n. The leading coefficient is $1/k!$ in the first two cases and $1/(k!2^k)$ in the last case. Thus we have the asymptotical equalities*

$$p^{\mathcal{K}}_{k,n} \sim \tilde{p}^{\mathcal{K}}_{k,n} \sim \bar{p}^{\mathcal{K}}_{k,n} \cdot 2^k \sim \binom{n-1}{k}.$$

The last statement is quite surprising with regard to the fact that the claimed asymptotical equality is independent of the chosen class $\mathcal{K}$ with $\mathcal{A}_2 \subseteq \mathcal{K}$.

For an application of Lemma 1.7, we need

Lemma 2.5. *If $\mathcal{A}_2 \subseteq \mathcal{K} \subseteq \mathcal{A}_{>1}$ then $q^{\mathcal{K}}_{k,n} \geq q^{\mathcal{K}}_{k-1,n-1}$, and similar inequalities hold for $\tilde{q}^{\mathcal{K}}_{k,n}$ and $\overline{q}^{\mathcal{K}}_{k,n}$.*

Proof. Let $Q = (\underline{n-1}, \leq_Q)$ be any indecomposable poset with $f^{\mathcal{K}}(Q) = k-1$, and choose a maximal element m of Q. We construct a poset $Q' = (\underline{n}, \leq_{Q'})$, by putting the point n above all elements except m. The resulting poset is indecomposable because any ordinal decomposition of Q' would induce an ordinal decomposition of Q. Furthermore, the only set $R \subseteq \underline{n}$ with $n \in R$ and $(R, \leq_{Q'|R}) \in \mathcal{K}$ is the doubleton $\{n, m\}$, as $\mathcal{K}$ consists of antichains with more than one element. Thus we have $f^{\mathcal{K}}(Q') = f^{\mathcal{K}}(Q) + 1 = k$. Since two posets Q_1 and Q_2 must be isomorphic if so are the posets Q_1' and Q_2' (the point n being fixed under isomorphisms), the asserted inequality for $q^{\mathcal{K}}_{k,n}$ follows. The other cases are treated similarly. ∎

Now we apply Lemma 1.7 and obtain

Corollary 2.6. *Under the same hypotheses as in Proposition 2.3 and the additional assumption $\mathcal{A}_2 \subseteq \mathcal{K} \subseteq \mathcal{A}_{>1}$, the sequences $(c^{\mathcal{K}}_k)$, $(\tilde{c}^{\mathcal{K}}_k)$ and $(\overline{c}^{\mathcal{K}}_k)$ are monotone increasing. Furthermore, one has the inequalities*

$$\tilde{p}^{\mathcal{K}}_{k,n} \geq p^{\mathcal{K}}_{k,n} \geq \binom{n-1}{k}$$

and

$$P^{\mathcal{K}}_{k,n} \geq n! \binom{n-1}{k} 2^{-k}$$

with asymptotical equality in all three cases.

Hence, for large n, the number of posets having n points and $n+k+1$ antichains with at most two elements is about the same as the number of posets with $n+k+1$ antichains (of unrestricted size).

Here are some of the polynomials we have computed explicitly (for more numerical material, see the end of this note):

k	0	1	2
$p_k^{\mathcal{A}_{>1}}(n)$	1	$n-1$	$\frac{1}{2}(n^2-3n+2)$
$p_k^{\mathcal{A}_2}(n)$	1	$n-1$	$\frac{1}{2}(n^2-3n+2)$
$\tilde{p}_k^{\mathcal{A}_{>1}}(n)$	1	$n-1$	$\frac{1}{2}(n^2+n-6)$
$\tilde{p}_k^{\mathcal{A}_2}(n)$	1	$n-1$	$\frac{1}{2}(n^2+n-6)$
$\overline{p}_k^{\mathcal{A}_{>1}}(n)$	1	$\frac{1}{2}(n-1)$	$\frac{1}{8}(n^2+3n-10)$
$\overline{p}_k^{\mathcal{A}_2}(n)$	1	$\frac{1}{2}(n-1)$	$\frac{1}{8}(n^2+3n-10)$

k	3	4
$p_k^{A>1}(n)$	$\frac{1}{6}(n^3-6n^2+17n-24)$	$\frac{1}{24}(n^4-10n^3+59n^2-146n+96)$
$p_k^{A_2}(n)$	$\frac{1}{6}(n^3-6n^2+23n-36)$	$\frac{1}{24}(n^4-10n^3+83n^2-290n+288)$
$\tilde{p}_k^{A>1}(n)$	$\frac{1}{6}(n^3+6n^2-25n-6)$	$\frac{1}{24}(n^4+14n^3-37n^2-146n+144)$
$\tilde{p}_k^{A_2}(n)$	$\frac{1}{6}(n^3+6n^2-19n-18)$	$\frac{1}{24}(n^4+14n^3-13n^2-146n-96)$
$\bar{p}_k^{A>1}(n)$	$\frac{1}{48}(n^3+12n^2-25n-60)$	$\frac{1}{384}(n^4+26n^3+35n^2-478n-248)$
$\bar{p}_k^{A_2}(n)$	$\frac{1}{48}(n^3+12n^2-17n-76)$	$\frac{1}{384}(n^4+26n^3+67n^2-382n-888)$

Let us shortly digress and translate the previous results into the language of finite topological spaces. Recall that a topological space satisfies the T_0-axiom iff different points have distinct closures, and that the finite T_0-spaces are in one-to-one correspondence with finite posets (while arbitrary finite topological spaces correspond to finite quasiordered sets). Denote by $T^0_{k,n}$ (resp. $t^0_{k,n}$) the number of all (homeomorphism classes of) T_0-topologies on n points with k open sets; it is well known that this is also the number of all (isomorphism classes of) posets with n points and k antichains (see, e.g., [5] or [[8]). Thus

$$T^0_{k,n+1+k} = P^{A>1}_{k,n} \qquad \text{and} \qquad t^0_{k,n+1+k} = p^{A>1}_{k,n}$$

Corollary 2.7. *There are polynomials R_k and r_k of degree k with leading coefficient* 1 *such that for all $n \geq k$,*

$$T^0_{k,n+1+k} = \frac{n!}{k!\,2^k}R_k(n) \qquad \textit{and} \qquad t^0_{k,n+1+k} = \frac{1}{k!}r_k(n).$$

It is well known that the enumeration of arbitrary topologies can be reduced to that of T_0-topologies (see, for example, [4] and [10]).

3. Generalized FIbonacci recursions

Recall from 1.1 that for any 1-triangular sequence $(q_{k,n})$ and its summatorial sequence $(p_{k,n})$, there is a representation

$$p_{k,n} = p_k(n) + d_{k-n-1,k}$$

where p_k is a polynomial of degree $\leq k$ and the remainder $d_{k-n-1,k}$ vanishes for $k \leq n$. On the base of numerical tables, Culberson and Rawlins [2] have observed that for the special case where $p_{k,n}$ denotes the number of unlabeled posets with n points and k doubletons (i.e. two-element antichains), the sequences $(d_{k,n})$ seem to satisfy certain generalized Fibonacci recursions, at least if k is small.

The key for a proof of this phenomenon lies in the observation that similar recursions hold for the sequences $(s_{k,n}) = (q_{n+k,n+1})$.

Proposition 3.1. *For $k = 0$ and $k = 1$, respectively, the numbers $s_{k,n} = q^{A_2}_{n+k,n+1}$ of all unlabeled ordinally indecomposable posets with $n+1$ points and $n+k$ doubletons obey the following recursions and explicit identities:*

$$\begin{aligned}
&s_{0,0} = s_{0,1} = s_{0,2} = 1, \\
&s_{0,n} = 2s_{0,n-1} = 2^{n-2}\,(n \geq 3), \\
&s_{1,0} = s_{1,1} = 0, s_{1,2} = 1, s_{1,3} = 3, s_{1,4} = 8, \\
&s_{1,n} = 2s_{1,n-1} + 3s_{0,n-2} = 2^{n-4}(3n-4)(n \geq 5).
\end{aligned}$$

While the equations for $s_{0,n}$ are easily obtained, the proof of those for $s_{1,n}$ is more involved (for details, see [6] or [12]). From the table at the end of this note, we read off the following values:

$$s_{2,0} = s_{2,1} = s_{2,2} = 0, s_{2,3} = 1, s_{2,4} = 8, s_{2,5} = 32, s_{2,6} = 105 \tag{3.1}$$

and it appears very likely that

$$s_{2,n} = 2s_{2,n-1} + 3s_{1,n-2} + 9s_{0,n-3} = 2^{n-7}(9n^2 - 15n - 24)(n \geq 7) \tag{3.2}$$

Probably, similar expressions can be found for all sequences $(s_{k,n})$ if n is sufficiently large (perhaps for all $n \geq 2k+3$). However, it seems hard to solve the general case explicitly.

A straightforward comparison of coefficients in Proposition 3.1 gives

Corollary 3.2. *The generating functions $s_k(y) = \sum_{n=0}^{\infty} s_{k,n} y^n$ satisfy the following identities for $k = 0$ and $k = 1$:*

$$s_0(y) = \frac{1-y-y^2}{1-2y}, \quad s_1(y) = \frac{y^2(1-y+2y^3)}{(1-2y)^2}.$$

Similarly, one finds that (3.1) together with (3.2) is equivalent to

$$s_2(y) = \frac{y^3(1+y^3)(1+2y-4y^2)}{(1-2y)^3} \tag{3.3}$$

and we conjecture that $s_k(y)$ is always a rational function, i.e., a quotient of two polynomials. If one can prove this conjecture then the generating functions $d_k(y)$ will be rational again, on account of the functional equation at the end of Proposition 1.1 and the following straightforward

Lemma 3.3. *Suppose $s(x,y) = \sum_{k=0}^{\infty} s_k(y)x^k$ and $d(x,y) = \sum_{j=0}^{\infty} d_j(y)x^j$ are formal power series satisfying the identity*

$$d(x,y)(x - s(xy,y)) = 1.$$

Then the functions $d_j(y)$ can be computed from the functions $s_k(y)$ by recursion:

$$d_0(y) = -s_0(y)^{-1}$$

$$d_j(y) = s_0(y)^{-1}\left(d_{j-1}(y) - \sum_{k=0}^{j-1} s_k(y)\cdot y^k \cdot d_{j-k}(y)\right) \quad (j>0).$$

In particular,

$$d_1(y) = s_0(y)^{-2}(s_1(y)\cdot y - 1).$$

Now it is easy to confirm two formulae conjectured in [2]:

Proposition 3.4. *Let $p_{k,n} = p_{k,n}^{A_2}$ denote the number of all unlabeled posets with n singletons and k two-element antichains; furthermore, let p_k denote the polynomial of degree k with $p_{k,n} = p_k(n)$ for $n \geq k$, and $(d_{j,k} = p_{k,k-j-1} - p_k(k-j-1)$. Then the sequences $(d_{0,k})$ and $(d_{1,k})$ obey the following Fibonacci recursions:*

$$d_{0,0} = -1,\quad d_{0,1} = 1,\qquad d_{0,k} = d_{0,k-1} + d_{0,k-2} \qquad (k \geq 2),$$
$$d_{1,0} = -1,\quad d_{1,1} = 2,\ d_{1,2} = -1,\quad d_{1,k} = d_{1,k-1} + d_{1,k-2} + d_{0,k-1} \quad (k \geq 5),$$

and their generating functions are given by

$$d_0(y) = \frac{-1+2y}{1-y-y^2},\quad d_1(y) = \frac{-1+4y-4y^2+y^3-y^4+2y^6}{(1-y-y^2)^2}.$$

Proof. The explicit representation of the generating functions follows at once from 3.2 and 3.3, and then the recursion formulae are easily obtained by comparing coefficients. ∎

If one can prove similar recursions as in 3.1 for other sequences $(s_{k,n})$ then the general proof scheme will follow the same pattern as before. In particular, under the hypothesis (3.2), one obtains:

$$d_{2,0} = -1, d_{2,1} = 3, d_{2,2} = -3, d_{2,3} = 6, d_{2,4} = -3, d_{2,5} = 8, d_{2,6} = 8 \tag{3.4}$$

$$d_{2,k} = d_{2,k-1} + d_{2,k-2} + d_{1,k-1} - d_{0,k-5} \quad (k \geq 7) \tag{3.5}$$

$$d_2(y) = \frac{-1+6y-12y^2+10y^3-6y^4+5y^5+6y^6-9y^7+y^9-y^{10}}{(1-y-y^2)^3} \tag{3.6}$$

For $k \leq 14$, the validity of the recursion for $d_{2,k}$ has been verified in [2].

Without proof, we note analogous results for the numbers $\overline{p}_{k,n}$ and $\tilde{p}_{k,n}$. Recall the following notational conventions:

$\tilde{p}_{k,n}^{A_2}$: number of naturally ordered sets with n points and k doubletons
$\overline{p}_{k,n}^{A_2}$: number of labeled posets with n points and k doubletons, divided by $n!$

The same symbols, with q instead of p, denote the corresponding numbers of indecomposable posets. Then one can prove (cf. [7] and the tables at the end):

Proposition 3.5. *The numbers of unlabeled ordinally indecomposable posets, resp. of naturall ordered sets, with n or $n+1$ doubletons satisfy the following identities:*

$$\overline{q}^{A_2}_{n,n+1} = 2^{n-2} \qquad \tilde{q}^{A_2}_{n,n+1} = 3^{n-1} \qquad (n \geq 1)$$
$$\overline{q}^{A_2}_{n+1,n+1} = 2^{n-5}(5n-9) \qquad \tilde{q}^{A_2}_{n+1,n+1} = 3^{n-4}(25n-42) \qquad (n \geq 3)$$

Using these formulae and similar arguments as in the proof of 3. 4, one obtains:

Proposition 3.6. *Let $\overline{p}_k$ denote the polynomial of degree k with $\overline{p}^{A_2}_{k,n} = \overline{p}_k(n)$ for $n \geq k$, and put $\overline{d}_{j,k} = \overline{p}^{A_2}_{k,k-j-1} - \overline{p}_k(k-j-1)$. Then*

$$\overline{d}_{0,n} = \overline{d}_{0,n-1} + \overline{d}_{0,n-2} \qquad (n \geq 2),$$
$$\overline{d}_{1,n} = \overline{d}_{1,n-1} + \overline{d}_{1,n-2} + \overline{d}_{0,n-1} \qquad (n \geq 3).$$

The corresponding equations for naturally ordered sets are

$$\tilde{d}_{0,n} = 2\,\tilde{d}_{0,n-1} = 2^n \qquad (n \geq 1),$$
$$\tilde{d}_{1,n} = 2\,\tilde{d}_{1,n-1} + \tilde{d}_{0,n} - \tilde{d}_{0,n-5} = 2^{n-3}(7n+19) \qquad (n \geq 3).$$

In [2], the validity of the last two formulae has been confirmed for $n \leq 14$.

Analogous results are obtained for $\mathcal{A}_{>2}$, the class of all antichains with at least two elements, instead of $\mathcal{A}_2$. For example, one can prove the following formulae:

$$q^{\mathcal{A}_{>1}}_{n,n+1} = 2^{n-2} \qquad (n \geq 2),$$
$$q^{\mathcal{A}_{>1}}_{n+1,n+1} = 2^{n-4}(n-2) \qquad (n \geq 4),$$
$$\overline{q}^{\mathcal{A}_{>1}}_{n,n+1} = 2^{n-2} \qquad (n \geq 1),$$
$$\overline{q}^{\mathcal{A}_{>1}}_{n+1,n+1} = 2^{n-4}(n-2) \qquad (n \geq 2).$$

and probably also

$$q^{\mathcal{A}_{>1}}_{n+2,n+1} = 2^{n-7}(n^2+17n-28) \qquad (n \geq 6),$$
$$\overline{q}^{\mathcal{A}_{>1}}_{n+2,n+1} = 2^{n-7}(n^2+13n-32) \qquad (n \geq 3).$$

Then one may proceed as in the case of $\mathcal{A}_2$ in order to derive analogous Fibonacci recursions.

On the next two pages, we list the following tables for $k \leq 11$ and $n \leq 12$:

$q^{\mathcal{K}}_{k,n}$, the number of unlabeled indecomposable posets with n points and k doubletons

$p^{\mathcal{K}}_{k,n}$, the number of unlabeled posets with n points and k doubletons

$c^{\mathcal{K}}_n$, the coefficients of the recursion $p^{\mathcal{K}}_{k,n} = \sum_{j=0}^{k} c^{\mathcal{K}}_j p^{\mathcal{K}}_{k-j,n-j-1}$ for $k < n$ (see 2.3), and the polynomial representation of $p^{\mathcal{K}}_{k,n}$ for $k \leq n$ (see 1.4):

$$p^{\mathcal{K}}_k(n) = \sum_{j=0}^{k} a^{\mathcal{K}}_{k,j} \binom{n-j-1}{k-j}.$$

On the pages after, the same numbers are listed for unlabeled (indecomposable) posets with $n+k+1$ antichains. Finally, we present the corresponding coefficients $\overline{c}_k$ and polynomials $\overline{p}_k(n)$ for the labeled case. The tables of explicit values for $\overline{q}_{k,n}$, $\overline{p}_{k,n}$ and $\overline{a}_{k,n}$ (contained in the preprint version [9]) have been omitted because of the somewhat confusing shape of the involved fractions (notice that the coefficients $\overline{c}_k$ are not always integers!) For the case of naturally ordered sets, the corresponding tables can be found in [2] and [7].

$\mathcal{K} = \mathcal{A}_2$ Unlabeled posets, antichains with 2 elements

n:	0	1	2	3	4	5	6	7	8	9	10	11	12
$q^{\mathcal{K}}_{0,n}$:	0	1	0	0	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{1,n}$:	0	0	1	0	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{2,n}$:	0	0	0	1	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{3,n}$:	0	0	0	1	2	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{4,n}$:	0	0	0	0	3	4	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{5,n}$:	0	0	0	0	1	8	8	0	0	0	0	0	0
$q^{\mathcal{K}}_{6,n}$:	0	0	0	0	1	8	22	16	0	0	0	0	0
$q^{\mathcal{K}}_{7,n}$:	0	0	0	0	0	6	32	56	32	0	0	0	0
$q^{\mathcal{K}}_{8,n}$:	0	0	0	0	0	3	37	105	136	64	0	0	0
$q^{\mathcal{K}}_{9,n}$:	0	0	0	0	0	1	34	160	312	320	128	0	0
$q^{\mathcal{K}}_{10,n}$:	0	0	0	0	0	1	23	198	568	864	736	256	0
$q^{\mathcal{K}}_{11,n}$:	0	0	0	0	0	0	16	209	874	1814	2280	1664	512

$p^{\mathcal{K}}_{0,n}$:	1	1	1	1	1	1	1	1	1	1	1	1	1
$p^{\mathcal{K}}_{1,n}$:	0	0	1	2	3	4	5	6	7	8	9	10	11
$p^{\mathcal{K}}_{2,n}$:	0	0	0	1	3	6	10	15	21	28	36	45	55
$p^{\mathcal{K}}_{3,n}$:	0	0	0	1	4	9	17	29	46	69	99	137	184
$p^{\mathcal{K}}_{4,n}$:	0	0	0	0	3	12	28	54	94	153	237	353	509
$p^{\mathcal{K}}_{5,n}$:	0	0	0	0	1	10	35	83	166	300	506	811	1249
$p^{\mathcal{K}}_{6,n}$:	0	0	0	0	1	10	44	123	274	541	986	1694	2779
$p^{\mathcal{K}}_{7,n}$:	0	0	0	0	0	6	46	168	434	939	1836	3351	5808
$p^{\mathcal{K}}_{8,n}$:	0	0	0	0	0	3	43	204	629	1528	3240	6306	11545
$p^{\mathcal{K}}_{9,n}$:	0	0	0	0	0	1	36	239	874	2386	5492	11385	21985
$p^{\mathcal{K}}_{10,n}$:	0	0	0	0	0	1	25	249	1136	3551	8974	19897	40515
$p^{\mathcal{K}}_{11,n}$:	0	0	0	0	0	0	16	243	1402	5054	14096	33614	72382

$\mathcal{K} = \mathcal{A}_2$ Unlabeled posets, antichains with 2 elements

n:	0	1	2	3	4	5	6	7	8	9	10	11
$c_n^{\mathcal{K}}$:	1	1	1	3	8	21	63	195	612	1971	6458	21426
$a_{0,n}^{\mathcal{K}}$:	1	1	1	1	1	1	1	1	1	1	1	1
$a_{1,n}^{\mathcal{K}}$:		0	0	0	0	0	0	0	0	0	0	0
$a_{2,n}^{\mathcal{K}}$:			0	2	4	6	8	10	12	14	16	18
$a_{3,n}^{\mathcal{K}}$:				1	4	7	10	13	16	19	22	25
$a_{4,n}^{\mathcal{K}}$:					3	11	23	39	59	83	111	143
$a_{5,n}^{\mathcal{K}}$:						10	37	76	127	190	265	352
$a_{6,n}^{\mathcal{K}}$:							44	127	251	424	654	949
$a_{7,n}^{\mathcal{K}}$:								168	433	850	1455	2284
$a_{8,n}^{\mathcal{K}}$:									629	1525	2955	5069
$a_{9,n}^{\mathcal{K}}$:										2386	5444	10375
$a_{10,n}^{\mathcal{K}}$:											8974	19552
$a_{11,n}^{\mathcal{K}}$:												33614

$$p_0^{\mathcal{K}}(n) = 1/0!$$
$$p_1^{\mathcal{K}}(n) = (n-1)/1!$$
$$p_2^{\mathcal{K}}(n) = (n^2 - 3n + 2)/2!$$
$$p_3^{\mathcal{K}}(n) = (n^3 - 6n^2 + 23n - 36)/3!$$
$$p_4^{\mathcal{K}}(n) = (n^4 - 10n^3 + 83n^2 - 290n + 288)/4!$$
$$p_5^{\mathcal{K}}(n) = (n^5 - 15n^4 + 205n^3 - 1245n^2 + 3454n - 4320)/5!$$
$$p_6^{\mathcal{K}}(n) = (n^6 - 21n^5 + 415n^4 - 3855n^3 + 20464n^2 - 59484n + 63360)/6!$$
$$p_7^{\mathcal{K}}(n) = (n^7 - 28n^6 + 742n^5 - 9730n^4 + 82369n^3 - 416122n^2 + 1083648n - 1239840)/7!$$
$$p_8^{\mathcal{K}}(n) = (n^8 - 36n^7 + 1218n^6 - 21336n^5 + 259329n^4 - 1989204n^3 + 9174332n^2 - 24157104n + 26288640)/8!$$
$$p_9^{\mathcal{K}}(n) = (n^9 - 45n^8 + 1878n^7 - 42210n^6 + 688737n^5 - 7393365n^4 + 52188632n^3 - 234242460n^2 + 586515312n - 646652160)/9!$$
$$p_{10}^{\mathcal{K}}(n) = (n^{10} - 55n^9 + 2760n^8 - 77190n^7 + 1614333n^6 - 22952055n^5 + 227439890n^4 - 1526250860n^3 + 6504181416n^2 - 16186479840n + 17635968000)/10!$$
$$p_{11}^{\mathcal{K}}(n) = (n^{11} - 66n^{10} + 3905n^9 - 132660n^8 + 3438303n^7 - 62221698n^6 + 817150235n^5 - 7587963240n^4 + 48222532996n^3 - 200934920736n^2 + 489015380160n - 533607782400)/11!$$

$\mathcal{K} = \mathcal{A}_{>1}$ Unlabeled posets,antichains with 2 or more elements

n:	0	1	2	3	4	5	6	7	8	9	10	11	12
$q^{\mathcal{K}}_{0,n}$:	0	1	0	0	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{1,n}$:	0	0	1	0	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{2,n}$:	0	0	0	1	0	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{3,n}$:	0	0	0	0	2	0	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{4,n}$:	0	0	0	1	1	4	0	0	0	0	0	0	0
$q^{\mathcal{K}}_{5,n}$:	0	0	0	0	2	2	8	0	0	0	0	0	0
$q^{\mathcal{K}}_{6,n}$:	0	0	0	0	0	7	6	16	0	0	0	0	0
$q^{\mathcal{K}}_{7,n}$:	0	0	0	0	1	2	21	16	32	0	0	0	0
$q^{\mathcal{K}}_{8,n}$:	0	0	0	0	0	5	10	55	40	64	0	0	0
$q^{\mathcal{K}}_{9,n}$:	0	0	0	0	0	2	22	40	140	96	128	0	0
$q^{\mathcal{K}}_{10,n}$:	0	0	0	0	0	2	13	84	133	344	224	256	0
$q^{\mathcal{K}}_{11,n}$:	0	0	0	0	1	0	20	60	276	396	824	512	512
$p^{\mathcal{K}}_{0,n}$:	1	1	1	1	1	1	1	1	1	1	1	1	1
$p^{\mathcal{K}}_{1,n}$:	0	0	1	2	3	4	5	6	7	8	9	10	11
$p^{\mathcal{K}}_{2,n}$:	0	0	0	1	3	6	10	15	21	28	36	45	55
$p^{\mathcal{K}}_{3,n}$:	0	0	0	0	2	6	13	24	40	62	91	128	174
$p^{\mathcal{K}}_{4,n}$:	0	0	0	1	3	9	20	39	70	118	189	290	429
$p^{\mathcal{K}}_{5,n}$:	0	0	0	0	2	8	26	60	119	216	369	602	946
$p^{\mathcal{K}}_{6,n}$:	0	0	0	0	0	7	26	76	175	354	658	1151	1922
$p^{\mathcal{K}}_{7,n}$:	0	0	0	0	1	4	28	90	238	532	1074	2018	3590
$p^{\mathcal{K}}_{8,n}$:	0	0	0	0	0	5	23	105	309	767	1672	3350	6311
$p^{\mathcal{K}}_{9,n}$:	0	0	0	0	0	2	26	106	380	1038	2472	5304	10582
$p^{\mathcal{K}}_{10,n}$:	0	0	0	0	0	2	17	120	436	1354	3482	8015	16954
$p^{\mathcal{K}}_{11,n}$:	0	0	0	0	1	2	23	110	515	1678	4744	11658	26139

$\mathcal{K} = \mathcal{A}_{>1}$ Unlabeled posets,antichains with 2 or more elements

n:	0	1	2	3	4	5	6	7	8	9	10	11
$c_n^{\mathcal{K}}$:	1	1	1	2	6	15	39	108	308	890	2613	7777
$a_{0,n}^{\mathcal{K}}$:	1	1	1	1	1	1	1	1	1	1	1	1
$a_{1,n}^{\mathcal{K}}$:		0	0	0	0	0	0	0	0	0	0	0
$a_{2,n}^{\mathcal{K}}$:			0	1	2	3	4	5	6	7	8	9
$a_{3,n}^{\mathcal{K}}$:				0	3	6	9	12	15	18	21	24
$a_{4,n}^{\mathcal{K}}$:					3	8	14	21	29	38	48	59
$a_{5,n}^{\mathcal{K}}$:						8	22	42	68	100	138	182
$a_{6,n}^{\mathcal{K}}$:							26	67	127	207	308	431
$a_{7,n}^{\mathcal{K}}$:								90	212	392	639	962
$a_{8,n}^{\mathcal{K}}$:									309	671	1221	2001
$a_{9,n}^{\mathcal{K}}$:										1038	2149	3864
$a_{10,n}^{\mathcal{K}}$:											3482	6948
$a_{11,n}^{\mathcal{K}}$:												11658

$p_0^{\mathcal{K}}(n) = 1/0!$

$p_1^{\mathcal{K}}(n) = (n-1)/1!$

$p_2^{\mathcal{K}}(n) = (n^2 - 3n + 2)/2!$

$p_3^{\mathcal{K}}(n) = (n^3 - 6n^2 + 17n - 24)/3!$

$p_4^{\mathcal{K}}(n) = (n^4 - 10n^3 + 59n^2 - 146n + 96)/4!$

$p_5^{\mathcal{K}}(n) = (n^5 - 15n^4 + 145n^3 - 585n^2 + 814n - 360)/5!$

$p_6^{\mathcal{K}}(n) = (n^6 - 21n^5 + 295n^4 - 1815n^3 + 4744n^2 - 2484n - 10800)/6!$

$p_7^{\mathcal{K}}(n) = (n^7 - 28n^6 + 532n^5 - 4690n^4 + 20419n^3 - 20482n^2 - 167112n + 413280)/7!$

$p_8^{\mathcal{K}}(n) = (n^8 - 36n^7 + 882n^6 - 10584n^5 + 69489n^4 - 141204n^3 - 1038052n^2 + 6522384n - 11289600)/8!$

$p_9^{\mathcal{K}}(n) = (n^9 - 45n^8 + 1374n^7 - 21546n^6 + 198345n^5 - 707805n^4 - 3802744n^3 + 49355316n^2 - 197432496n + 288126720)/9!$

$p_{10}^{\mathcal{K}}(n) = (n^1 0 - 55n^9 + 2040n^8 - 40470n^7 + 495453n^6 - 2751735n^5 - 8568190n^4 + 247461220n^3 - 1717423704n^2 + 5497907040n - 6960038400)/10!$

$p_{11}^{\mathcal{K}}(n) = (n^1 1 - 66n^1 0 + 2915n^9 - 71280n^8 + 1115763n^7 - 8860698n^6 - 6071395n^5 + 932291580n^4 - 10108904564n^3 + 53483238864n^2 - 145082661120n + 159707116800)/11!$

$\mathcal{K} = \mathcal{A}_2$ **Labeled posets, antichains with 2 elements**

n:	0	1	2	3	4	5	6
$\tilde{c}_n^{\mathcal{K}}$:	1	$^1/_2$	1	$2^1/_6$	$5^7/_{12}$	$14^{11}/_{12}$	$42^{29}/_{72}$

n:	7	8	9	10	11
$\tilde{c}_n^{\mathcal{K}}$:	$123^{137}/_{144}$	$372^{71}/_{288}$	$1139^{1529}/_{1728}$	$3547^{1621}/_{4320}$	$11184^{1069}/_{2160}$

$\tilde{p}_0^{\mathcal{K}}(n) = 1/2^0 0!$

$\tilde{p}_1^{\mathcal{K}}(n) = (n-1)/2^1 1!$

$\tilde{p}_2^{\mathcal{K}}(n) = (n^2 + 3 - 10)/2^2 2!$

$\tilde{p}_3^{\mathcal{K}}(n) = (n^3 + 12n^2 - 17n - 76)/2^3 3!$

$\tilde{p}_4^{\mathcal{K}}(n) = (n^4 + 26n^3 + 67n^2 - 382n - 888)/2^4 4!$

$\tilde{p}_5^{\mathcal{K}}(n) = (n^5 + 45n^4 + 405n^3 - 125n^2 - 9526n - 6160)/2^5 5!$

$\tilde{p}_6^{\mathcal{K}}(n) = (n^6 + 69n^5 + 1235n^4 + 5335n^3 - 30596n^2 - 149164n - 88480)/2^6 6!$

$\tilde{p}_7^{\mathcal{K}}(n) = (n^7 + 98n^6 + 2870n^5 + 29260n^4 + 15169n^3 - 976878n^2 - 3136120n + 463680)/2^7 7!$

$\tilde{p}_8^{\mathcal{K}}(n) = (n^8 + 132n^7 + 5698n^6 + 100184n^5 + 572369n^4 - 2245292n^3 - 29574868n^2 - 55519344n + 39365760)/2^8 8!$

$\tilde{p}_9^{\mathcal{K}}(n) = (n^9 + 171n^8 + 10182n^7 + 270438n^6 + 3123561n^5 + 7025739n^4 - 129642752n^3 - 869526108n^2 - 767035952n + 1565679360)/2^9 9!$

$\tilde{p}_{10}^{\mathcal{K}}(n) = (n^10 + 215n^9 + 16860n^8 + 626550n^7 + 11510373n^6 + 86795415n^5 - 141950810n^4 - 5802007300n^3 - 23374926824n^2 - 6700023520n + 67453263360)/2^{10} 10!$

$\tilde{p}_{11}^{\mathcal{K}}(n) = (n^{11} + 264n^{10} + 26345n^9 + 1301520n^8 + 34225983n^7 + 454162632n^6 + 1953784195n^5 - 18815591520n^4 - 228465202284n^3 - 651706340736n^2 + 483094289600n + 2501470540800)/2^1 11!$

$\mathcal{K} = \mathcal{A}_{>1}$ Labeled posets, antichains with 2 or more elements

n:	0	1	2	3	4	5	6	7	8	9	10	11
$\bar{c}_n^{\mathcal{K}}$:	1	$^1/_2$	1	2	$4^2/_3$	$11^5/_{12}$	$29^3/_8$	$77^7/_{12}$	$210^{55}/_{72}$	$582^3/_4$	$1637^{11}/_{16}$	$4661^{47}/_{288}$

$\bar{p}_0^{\mathcal{K}}(n) = 1/2^0 0!$

$\bar{p}_1^{\mathcal{K}}(n) = (n-1)/2^1 1!$

$\bar{p}_2^{\mathcal{K}}(n) = (n^2 + 3n - 10)/2^2 2!$

$\bar{p}_3^{\mathcal{K}}(n) = (n^3 + 12n^2 - 25n - 60)/2^3 3!$

$\bar{p}_4^{\mathcal{K}}(n) = (n^4 + 26n^3 + 35n^2 - 478n - 248)/2^4 4!$

$\bar{p}_5^{\mathcal{K}}(n) = (n^5 + 45n^4 + 325n^3 - 1165n^2 - 6166n + 4080)/2^5 5!$

$\bar{p}_6^{\mathcal{K}}(n) = (n^6 + 69n^5 + 1075n^4 + 855n^3 - 35876n^2 - 36204n + 100320)/2^6 6!$

$\bar{p}_7^{\mathcal{K}}(n) = (n^7 + 98n^6 + 2590n^5 + 15820n^4 - 90951n^3 - 627438n^2 + 276520n + 3017280)/2^7 7!$

$\bar{p}_8^{\mathcal{K}}(n) = (n^8 + 132n^7 + 5250n^6 + 67480n^5 + 14609n^4 - 3416812n^3 - 8121940n^2 + 40732560n + 12664960)/2^8 8!$

$\bar{p}_9^{\mathcal{K}}(n) = (n^9 + 171n^8 + 9510n^7 + 201222n^6 + 1137801n^5 - 9159381n^4 - 87184000n^3 + 99919908n^2 + 817844688n + 88784640)/2^9 9!$

$\bar{p}_{10}^{\mathcal{K}}(n) = (n^{10} + 215n^9 + 15900n^8 + 494070n^7 + 5838693n^6 - 1545705n^5 - 436356250n^4 - 1209846020n^3 + 7776402456n^2 + 15652311840n - 26158809600)/2^{10} 10!$

$\bar{p}_{11}^{\mathcal{K}}(n) = (n^{11} + 264n^{10} + 25025n^9 + 1066560n^8 + 20263023n^7 + 111432552n^6 - 1205744485n^5 - 13013666160n^4 + 14489212276n^3 + 264164117184n^2 - 11068415040n - 761346432000)/2^{11} 11!$

Added in proof. In May1993,we have computed the numbers for $n = 12$, too.

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