A Zero-Sum Conjecture for Trees

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Abstract. We propose the following conjecture: *Let $m \geq k \geq 2$ be integers such that $k \mid m$, and let T_m be a tree on m edges. Let G be a graph with $\delta(G) \geq m + k - 1$. Then for every Z_k-colouring of the edges of G there is a zero-sum* $\pmod{k}$ *copy of T_m in G.* We prove the conjecture for $m \geq k = 2$, and explore several relations to the zero-sum Turan numbers.

1. Introduction and Observations

In 1961, Erdös, Ginzburg and Ziv [16], proved the following theorem:

Theorem A. *Let $\{a_1, a_2, \ldots, a_{m+k-1}\}$ be a collection of integers and suppose $k \mid m$. Then there exits a subset $I \subset \{1, 2, \ldots, m + k - 1\}$, $|I| = m$, such that $\sum_{i \in I} a_i \equiv 0 \pmod{k}$.*

This theorem was the starting point of the seminal paper of Bialostocki and Dierker [3], in which they introduced the concept of zero-sum colouring.

Graphs in this paper are finite and have no multiple edges nor loops. By $R(G; k)$ we denote the least positive integer r such that in any k colouring of the edges of the complete graph K_r there is a monochromatic copy of G. In the sequel we assume that G is a graph with m edges. By *Zero-Sum Ramsey number* denoted $R(G; Z_k)$, $k \mid m$, we mean the least positive integer r such that in any colouring of the edges of the complete graph K_r, by Z_k, the additive group of integers modulo k, there is a copy of G such that the sum of the values on its edges is 0 $\pmod{k}$. Such a copy of G is called a *zero-sum copy* $\pmod{k}$ *of* G. The existence of $R(G; Z_k)$, follows from the existence of the classical Ramsey number $R(G; k)$.

By $T(n, G)$ we denote the classical *Turan number*, namely, the maximum possible number of edges in a graph H on n vertices without a copy of G. When using k colours the Turan number is denoted $T(n, G, k)$. By *Zero-Sum Turan*

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number, denoted $T(n, G, Z_k)$, $k \mid m$, we mean the maximum number of edges in a Z_k-colouring of a graph on n vertices that contains no zero-sum (mod k) copy of G. Following [1] and [10] we have,

(1) $T(n, G, Z_k) \leq T(n, G, k) \leq kT(n, G)$ if $k \mid m$.

(2) $T(n, G, 2) \leq T(n, G, Z_k)$ if $k = m$.

There is a rapidly growing literature on zero-sum problems as can be indicated from the list of references (which is by no means complete) [1]–[5], [8]–[14], [16]–[17] and [20]–[21].

It is natural to start a paper related to Turan numbers of trees with the celebrated, and yet unsolved, conjecture of Erdos and Sos [15].

Conjecture 1.1. *Let T_m be a tree on m edges. Then* $T(n, T_m) \leq \frac{(m-1)n}{2}$.

A solution of that conjecture implies solutions and improvements in many other related problems, including better upper bounds for the Ramsey numbers $R(T_m, T_n)$. The conjecture is known to be true for stars, paths and some family of trees with large maximal degree (see [22]). The best known general upper bound is based upon two simple observations (see [18]):

Observation 1.2. *If G is a graph with minimal degree $\delta(G) \geq m$, then G contains a copy of every tree T_m.*

Observation 1.3. *If the average degree of a graph G is d then G contains an induced subgraph H, such that $\delta(H) \geq \lceil \frac{d}{2} \rceil$.*

Using these observations and a result of Bollobas [7], it is easy to obtain the bounds:

(3) $T(n, T_m) \leq (m-1)n - \binom{m}{2} + 1$ and

(4) $T(n, T_m, Z_k) \leq k(m-1)n$.

Hence, in order to tackle with $T(n, T_m, Z_k)$, it is convenient to determine a parallel observation. One possibility is to modify a conjecture of Seymour and Schrijver [21]:

Conjecture 1.4. *If G is a graph such that every $\{0,1\}$-coloring of its edges implies a monochromatic spanning tree of G, then every Z_m-coloring of its edges implies a zero-sum spanning tree of G.*

This conjecture was proved for m a prime number ([21]). It is tempting to strengthen Conjecture 1.4:

Problem 1.5. *Prove that if G is a graph such that every $\{0,1\}$-coloring of its edges implies a monochromatic copy of a given tree T_m then every Z_m-coloring of its edges implies a zero-sum copy of T_m.*

If true, Problem 1.5 would imply that $T(n, T_m, Z_m) \leq T(n, T_m, 2) \leq 2T(n, T_m)$. Surprisingly the answer to Problem 1.5 is negative since we have,

Theorem 1.6. *Let $m \equiv 3 \pmod 4$ and P_m a path on m edges. Then,*

(i) $T(n, P_m, Z_m) \geq m(n-m)$ *for* $n > m$.

(ii) *For* $n > m^2, T(n, P_m, Z_m) > 2T(n, P_m)$.

Proof: Observe first that (ii) follows from (i) since $m(n-m) > (m-1)n$ for $n > m^2$, and $(m-1)n \geq 2T(n, P_m)$.

Now to prove (i), consider the bipartite graph $K_{n-4k+1,4k-1}$ where, $m = 4k-1$. Split the $4k-1$ vertices into three sets A, B, C such that $|A| = |B| = 2k-1$, $|C| = 1$ and let D be the rest of the vertices. Color all the edges between A and D with 0. Color all the edges between B and D with 1. Color all the edges between C and D with $2k$.

Clearly there is no monochromatic copy of P_m and as $|C| = 1$ a zero-sum mod m copy of P_m must have an edge sum equal to either m or to $2m$ depending upon the number of edges of color $2k$ taken in the path. One can see that this is impossible, completing the proof. ∎

The next conjecture is along the same line. The rest of the paper is devoted to support that conjecture.

Conjecture 1.7. *Let $m \geq k \geq 2$ be integers such that $k \mid m$ and T_m a tree on m edges. If G is a graph such that $\delta(G) \geq m + k - 1$, then every Z_k-coloring of its edges implies a zero-sum $\pmod k$ copy of T_m.*

To indicate the strength of Conjecture 1.7, observe that:

Fact 1. *Conjecture 1.7 implies $R(T_m; Z_k) \leq m+k$ and in particular $R(T_m; 2) \leq R(T_m; Z_m) \leq 2m$.*

It is easy to see that the Erdős–Sos Conjecture [15] implies $R(T_m, 2) < 2m$ but there is no known proof of this Ramsey bound avoiding that conjecture.

Fact 2. *Conjecture 1.7 implies $T(n, T_m, Z_k) \leq (m+k-2)n$.*

Proof: If G is a graph on n vertices and $n(m+k-2)+1$ edges then $d = \frac{2n(m+k-2)+2}{n}$, where $d = d(G)$ is the average degree of the graph G. Hence, by Observation 1.3, G contains an induced subgraph H such that $\delta(H) \geq \lceil \frac{d(G)}{2} \rceil = m+k-1$, which implies, by Conjecture 1.7, a zero-sum $\pmod k$ copy of T_m. ∎

One can see that the bound mentioned in Fact 2 is much better than the trivial bound in statement (4).

2. Results and Proofs

The zero-sum Turan numbers for stars were determined in [1] and [11].

Theorem 2.1. *Let $m \geq k \geq 2$ be integers, $k \mid m$. Suppose $n > 2(m-1)(k-1)$. Then,*

$$T(n, K_{1,m}, Z_k) = \begin{cases} \frac{(m+k-2)n}{2} - 1, & n-1 \equiv m \equiv k \equiv 0 \pmod 2 \\ \left\lfloor \frac{(m+k-2)n}{2} \right\rfloor, & \textit{otherwise.} \end{cases}$$

It is also clear that because of Theorem A, Conjecture 1.7 holds for stars. In order to extend the list of trees for which Conjecture 1.7 holds we need the following lemma.

Lemma 2.2. *Let G be a graph such that $\delta(G) \geq k$ and let T be a tree with k edges. Then for every vertex $u \in V(T)$ and every vertex $v \in V(G)$ there is an embedding of T in G such that u is mapped onto v.*

Proof: We use induction on k. For $k = 1, 2$ it is trivial. Let T be a tree with k edges and G a graph with $\delta(G) \geq k$, and let $u \in V(T)$, $v \in V(G)$.

Case 1: u is not an end-vertex of T.

In this case define $T' = T \backslash w$, where w is some end-vertex of T. Then T' has $k-1$ edges and by the induction hypothesis we can embed T' in G such that u is mapped onto v. As $\delta(G) \geq k$, it follows that every vertex in G has a neighbor in $V(G \backslash T')$ and we can add the missing end-vertex to obtain the desired embedding.

Case 2: u is an end-vertex of T.

Let w be the vertex adjacent to u in T. Define $T' = T \backslash u$. Embed T' in $G \backslash v$ such that w is mapped onto a vertex z adjacent to v in G. This is possible since $\delta(G \backslash v) \geq k - 1$ and using the induction hypothesis. Now add the edge (z, v) to T' to obtain an embedding of T in G in which u is mapped onto v. ∎

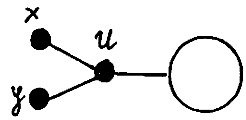
Theorem 2.3. *Let $T = T_m$ be a tree and $2 \mid m$. Let G be a graph such that $\delta(G) \geq m + 1$. Then in every Z_2-coloring of the edges of G, there exists a zero-sum $\pmod 2$ copy of T_m.*

Proof: As $\delta(G) \geq m+1$, there must exist a vertex v in G, in which there are edges from the two colors. Otherwise every component of G would be a monochromatic subgraph of minimal degree at least $m + 1$ and by Observation 1 we are done.

Hence, we may assume that v is incident with edges colored 0 and 1. Define $G_1 = G \backslash \{a, b\}$, where a and b are adjacent vertices of v such that $c(v, a) = 0$ and $c(v, b) = 1$, where c is the colouring function. Clearly, $\delta(G_1) \geq m - 1$ and by observation G_1 contains every tree T' on $m - 1$ edges.

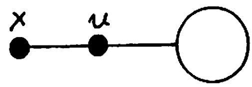
Consider two cases:

Case 1: In T there are two end-vertices of the form



In this case we choose T' to be: $T' = T\backslash\{y\}$. Embed T' in G_1 such that u is mapped onto v. If $\sum_{e\in E(T')} c(e) \equiv 1 \pmod 2$, then add to T' the edge (v, b). Otherwise add the edge (v, a) to obtain a zero-sum $\pmod 2$ copy of T.

Case 2: All the end-vertices of T are of the form

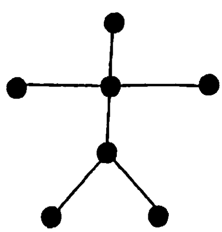


In this case we choose T' to be: $T'=T\backslash\{x\}$. Again embed T' in G_1 such that u is mapped onto v. If $\sum_{e\in E(T')} c(e) \equiv 1 \pmod 2$, then add to T' the edge (v, b). Otherwise add the edge (v, a) to obtain a zero-sum $\pmod 2$ copy of T. ∎

We already know Conjecture 1.7 is true for stars. We prove it also for a starlike trees.

Definition. *A tree T is called k-simple if there exists a sequence of subtrees $T = T_0, T_1, \ldots, T_n = \emptyset$, such that T_i is obtained from T_{i-1} by deleting k-endvertices having a common neighbor.*

Observe that a k-simple tree admits a $K_{1,k}$-decomposition.



Example: a 2-simple tree

Denote by $H(m, k)$ the family of all k-simple trees having m edges.

Theorem 2.4. *Let $m \geq k \geq 2$ be integers, $k \mid m$. Let G be a graph such that $\delta(G) \geq m + k - 1$. Then in every Z_k-coloring of G there is a zero-sum $\pmod k$ copy of every member $H \in H(m, k)$.*

Proof: Let $c : E(G) \to Z_k$, $k \mid m$. The proof is by induction on m.

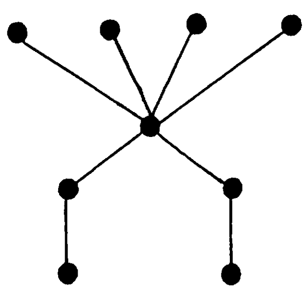
For $m = k$, $H(m,k) = \{K_{1,k}\}$ and $\delta(G) \geq 2k-1$. Hence, by Theorem A there is a zero-sum (mod k) copy of $K_{1,k}$ in G. Suppose we proved the theorem for $m = (t-1)k$ and let now $m = tk$. Let $H \in H(m,k)$. Delete from H, k end-vertices having common neighbor w to obtain a tree $H' \in H(m-k,k)$. As $\delta(G) \geq m+k-1$ and by the induction hypothesis there is a zero-sum (mod k) copy of H' in G. The vertex w in H' is adjacent to at least $2k-1$ vertices not belonging to H', since $|H'| = m-k+1$ implies $\deg'_H w \leq m-k$. But, $\deg_G w \geq m+k-1$. Hence, applying Theorem A to the vertices of $G \backslash H'$ which are adjacent to w we can find a zero- sum (mod k) copy of $K_{1,k}$ among those edges with a center at w. Adding these edges we have a zero-sum (mod k) copy of H in G. ∎

Another family of trees for which we can prove Conjecture 1.7 is the family of k-symmetric trees.

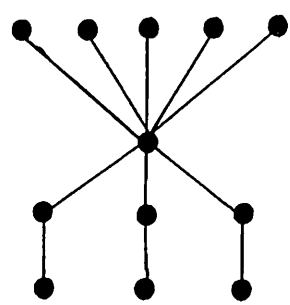
Definition. *A tree T is called k-symmetric if there is a vertex v of T in which every branch B_i appears α_i times in v and $k \mid \alpha_i$.*

The *k-closure* of a k-symmetric tree T, denoted by $T(k)$, is the tree obtained from T by adding $k-1$ copies of B_i for every type of branches at v.

Example.



a 2 – symmetric tree



$T(2)$

Theorem 2.5. *Let T be a k-symmetric tree on m edges. Let G be a graph such that $\delta(G) \geq \frac{(2k-1)m}{k}$. Then, in every Z_k-coloring of $E(G)$ there is a zero-sum (mod k) copy of T.*

Proof: Observe first that $k \mid m$ since T is k-symmetric. Next consider $T(k)$. Every branch B_i at v appears α_i-times in T hence $\alpha_i + k - 1$ times in $T(k)$.

Hence,

$$
\begin{aligned}
e(T(k)) &= \sum_i (\alpha_i + k - 1)e(B_i) \\
&= \sum_i \alpha_i e(B_i) + (k-1)\sum_i e(B_i) \le e(T) + \frac{(k-1)e(T)}{k} \\
&= \frac{(2k-1)e(T)}{k} = \frac{(2k-1)m}{k} \le \delta(G).
\end{aligned}
$$

Hence, by Observation 1.2 G contains a copy of $T(k)$. By Theorem A (applied to the branches at v), $T(k)$ contains a zero-sum copy $\pmod{k}$ of T in every Z_k-coloring. ∎

Final Remarks

(1) Using Fact 2, it follows that if T_m is k-simple then $T(n, T_m, Z_k) \le (m + k - 2)n$, which is much better than the bound $k(m-1)n$ given in statement (4).

(2) It follows from Theorem 2.5 that if $\delta(G) \ge 2m-1$ then for every $k \mid m$ and in every Z_k-coloring of its edges, G contains a zero-sum copy $\pmod{k}$ of every k-symmetric tree T on m edges. In particular for a k-symmetric tree T on m edges we have, (using observations 1.2 and 1.3),

$$T(n, T_m, Z_k) \le \left(\frac{(2k-1)m}{k} - 1\right) n \le 2(m-1)n,$$

again much better than the bound $k(m-1)n$ given in statement (4).

(3) The construction given below gives a lower bound for $T(n, T_m, Z_m)$ namely,

$$n\left(\frac{3m-4}{4}\right) \le T(n, T_m, Z_m) \text{ for } 2m \mid n.$$

Take $\frac{n}{2m}$ pairs of the complete graph K_m. Each component of K_m is colored by 0. In each pair color by 1 the two vertex disjoint complete bipartite graphs $K_{\lceil m/2\rceil, \lfloor m/2\rfloor}$. It is obvious that there is no zero-sum $\pmod{m}$ copy of T_m.

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