On the Product of some posets: jump number, greediness

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Abstract. Through combinatorial analysis we study the jump number, greediness and optimality of the products of chains, the product of an (upward rooted) tree and a chain.

It is well known [1] that the dimension of products of n chains is n. We construct a minimum realizer $L_1, \ldots, L_n$ for the products of n chains such that $s(\bigcap_{i=1}^{j} L_i) \leq s(\bigcap_{i=1}^{j+1} L_i)$ where $j = 1, \ldots, n-1$.

1. Introduction

Suppose we are given a finite number of tasks to be sequenced subject to precedence constraints, that is, a task cannot be scheduled until all of its predecessors have been scheduled. If a task t is scheduled immediately after the task u, then there is a jump (or setup) resulting in a fixed cost if u is not one of t's precedecessors and there is no cost (no setup) if u is one of t's precedecessors. Since the cost of a jump does not depend on where it occurs, the cost of a schedule is completely defined by the structure of the underlying partial order which represents the precedence constraints. The problem is: *schedule the tasks to minimize the number of jumps.* This is the jump number problem of a poset.

Let P be a finite poset and let $|P|$ be the *number of vertices* in P. A *subposet* of P is a subset of P with the induced order. A *chain* C in P is a subposet of P which is a linear order. The *length* of the chain C is $|C| - 1$. A poset is *ranked* if every maximal chain has the same length. A *linear extension* of a poset P is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j$ in P implies $i < j$. Let $\mathcal{L}(P)$ be the set of all linear extensions of P. Szpilrajn [16] showed that $\mathcal{L}(P)$ is not empty. Algorithmically, a linear extension L of P can be defined as follows:

1. Choose a minimal element x_1 in P.
2. Given $x_1, x_2, \ldots, x_i$ choose a minimal element from $P \setminus \{x_1, \ldots, x_i\}$ and call this element x_{i+1}.

Let P, Q be two disjoint posets. The *disjoint sum* $P + Q$ of P and Q is the poset on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

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Throughout this section, L denotes an arbitrary linear extension of P. Let $a, b \in P$ with $a < b$. Then b *covers* a, denoted $a \prec b$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A *(P, L)-chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L.

A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* (or *setup*) of P in L if x_i is not comparable to x_{i+1} in P. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$ where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of P in L for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump number* of P. If $s(L, P) = s(P)$ then L is called an *optimal linear extension* of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$.

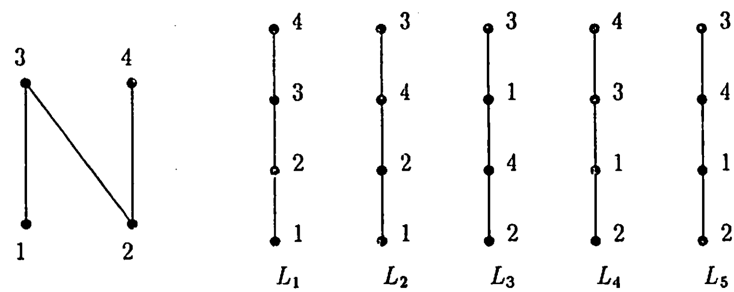


Figure 1: The poset N and its linear extensions.

In Figure 1 only L_3 is optimal.

The *width* $\omega(P)$ of P is the maximal number of elements of an *antichain* (mutually incomparable elements) of P. Dilworth [6] showed that $\omega(P)$ equals the minimum number of chains in a partition of P into chains. Since for any linear extension L of P the number of (P, L)-chains is at least as large as the minimum number of chains in a chain partition of P, it follows from Dilworth's theorem that

$$s(P) \geq \omega(P) - 1. \tag{1}$$

If equality holds in (1), then P is called a *Dilworth poset* or simply a *D-poset* [15].

A crown is a partially ordered set with diagram in Figure 2(a). In 1982, Duffus, Rival and Winkler [7] proved that every poset which contains no crown as a subposet is a D-poset.

A linear extension $L = x_1, x_2, \ldots, x_n$ of P is *greedy* if L can be obtained by applying the following algorithm:

1. Choose a minimal element x_1 of P.

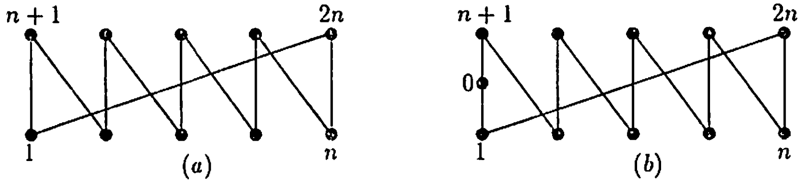


Figure 2: (a) Crown (b) C.

2. Suppose $x_1, ..., x_i$ have been chosen. If there is a minimal element of $P \setminus \{x_1, ..., x_i\}$ which is greater than x_i then choose x_{i+1} to be this minimal element. If not, choose x_{i+1} to be any minimal element of $P \setminus \{x_1, \dots, x_i\}$.

In words, L is obtained by *climbing as high as one can.* Let $\mathcal{G}(P)$ be the set of all greedy linear extensions of P. In Figure 1, L_1, L_2, L_3 are greedy linear extensions of the poset N, but L_4 is not greedy. So $\mathcal{O}(N) \subset \mathcal{G}(N)$. In fact, L_3 is a greedy optimal linear extension of N. Since the greedy algorithm above is a particular way of carrying out the algorithm for a linear extension, by induction we obtain [13] that every poset P has a greedy optimal linear extension.

A poset P is *greedy* if $\mathcal{G}(P) \subseteq \mathcal{O}(P)$, that is, every greedy extension is optimal.

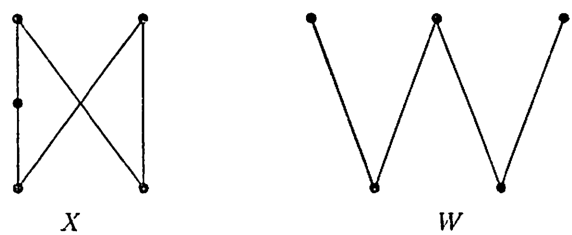


Figure 3: The poset X and W.

In Figure 3, $\mathcal{G}(X) \subset \mathcal{O}(X)$, $\mathcal{O}(W) = \mathcal{G}(W)$. In the above examples, X and W are greedy but N is not greedy.

A poset P is called *series parallel* if it can be constructed from singletons using the operations of disjoint sum (+) and linear sum ($\oplus$). For example, $(1 + 1) \oplus (1 + 1)$, a crown with 4 elements, is a series parallel poset.

In 1979, Cogis and Habib [5] proved that every series parallel poset is greedy. In 1982, Cogis [4] asked for a characterization of greedy posets. The problem remains open. A poset P is *N-free* if P contains no cover-preserving subposet isomorphic to the poset N in Figure 1. The next lemma [12] by Rival partially characterizes greedy posets.

Lemma 1.1. *Every N-free poset is greedy.*

In 1985, El-Zahar and Rival [9] showed that if P is a poset which contains no subposet isomorphic to C in Figure 2(b), then $\mathcal{O}(P) \subseteq \mathcal{G}(P)$.

Let P^d denote the *dual* of the poset P, that is, the poset obtained from P by reversing the order. If L is a linear extension of P, then its dual L^d is a linear extension of P^d.

A poset P is said to be *reversible* if $L^d \in \mathcal{G}(P^d)$ for every $L \in \mathcal{G}(P)$. In 1986, Rival and Zaguia [13] showed the following:

Lemma 1.2. *A poset P is reversible if and only if $\mathcal{O}(P) = \mathcal{G}(P)$.*

2. Products of chains

Let P, Q be two posets. The *direct product* $P \times Q$ of P and Q is the poset on $\{(p,q) : p \in P, q \in Q\}$ where $(a,b) \leq (c,d)$ if and only if $a \leq c$ in P and $b \leq d$ in Q. Let $\underline{k}$ be a *k-chain* and P^n be $P \times \cdots \times P$ (n times).

We consider the poset $\underline{a_1} \times \cdots \times \underline{a_n}$ where $a_1, \ldots, a_n$ are positive integers. We assume that $a_i \geq 2$ for $i = 1, \ldots, n$ and let $a^* = \max\{a_1, \ldots, a_n\}$. Without loss of generality, we assume that $a^* = a_1$.

Proposition 2.1. *Let*

$$L = C_1 \oplus C_2 \oplus \cdots \in \mathcal{L}(\underline{a_1} \times \cdots \times \underline{a_n}).$$

Then every $(\underline{a_1} \times \cdots \times \underline{a_n}, L)$-chain C_i is of the form

$$\{b_1 \ldots b_{i-1} p b_{i+1} \ldots b_n : c \leq p \leq d \text{ for some } c \geq 1, d \leq a_i\},$$

and hence has at most a^* elements.

Proof: Suppose there was a chain C_i containing two elements which differed in at least two coordinates, say $e = (e_1, e_2, \ldots, e_n)$ and $f = (f_1, f_2, \ldots, f_n)$ with $e_1 < f_1$, $e_2 < f_2$, $e_i \leq f_i$ ($i \geq 3$). The C_i contains at most one of $(e_1+1, e_2, e_3, \ldots, e_n)$ and $(e_1, e_2+1, e_3, \ldots, e_n)$. But each of these elements is between e and f in $\underline{a_1} \times \cdots \times \underline{a_n}$, and so cannot be in C_j for $j \neq i$, a contradiction. The proposition now follows.

Define $C(b_2, \ldots, b_n) = \{(i, b_2, \ldots, b_n) : 1 \leq i \leq a^*\}$. Then $C(b_2, \ldots, b_n)$ is a chain of length $a^* - 1$ for any $b_2, \ldots, b_n$ where $1 \leq b_j \leq a_j$ for $j = 2, \ldots, n$.

Theorem 2.2. *Let $L^* = \oplus C(b_2, \ldots, b_n)$ where the $(b_2, \ldots, b_n)$ are in lexicographic order. Then $L^* \in \mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_n})$ and*

$$s(\underline{a_1} \times \cdots \times \underline{a_n}) = (\textstyle\prod_{i=1}^{n} a_i)/a^* - 1.$$

Proof: Suppose $(b_2, \ldots, b_n) < (c_2, \ldots, c_n)$ (lexicographic order). Then for all $y \in C(c_2, \ldots, c_n)$ there does not exist $x \in C(b_2, \ldots, b_n)$ such that $y < x$. Hence $L^* \in \mathcal{L}(\underline{a_1} \times \cdots \times \underline{a_n})$. Thus we get

$$s(L^*, \underline{a_1} \times \cdots \times \underline{a_n}) = (\textstyle\prod_{i=1}^{n} a_i)/a^* - 1.$$

Now by the definition of jump number,

$$s(\underline{a_1} \times \cdots \times \underline{a_n}) \le (\prod_{i=1}^{n} a_i)/a^* - 1.$$

But Proposition 2.1 implies that

$$s(\underline{a_1} \times \cdots \times \underline{a_n}) \ge (\prod_{i=1}^{n} a_i)/a^* - 1.$$

Hence we get the result.

Theorem 2.2 implies that every $(\underline{a_1} \times \cdots \times \underline{a_n}, L)$-chain in an optimal linear extension L has length $a^* - 1$.

Corollary 2.3.

$$\mathcal{O}(\underline{a_1} \times \cdots \times \underline{a_n}) \subseteq \mathcal{G}(\underline{a_1} \times \cdots \times \underline{a_n}).$$

Proof: Use Proposition 2.1 and Theorem 2.2.

Corollary 2.4.

$$s(\underline{k}^n) = k^{n-1} - 1.$$

The Boolean algebra $B_A = (2^A, \subset)$ on a set A is the poset of all subsets of A ordered by inclusion. Let $[n]$ be $\{1, 2, ..., n\}$. For simplicity, we write B_n instead of $B_{[n]}$. It is well known that $\underline{2}^n = B_n$. Thus we get $s(B_n) = 2^{n-1} - 1$. For a subset $\{l_1, l_2, \ldots, l_t\}$ of $[n]$ with $l_1 < \cdots < l_t$ we define $B_n(l_1, \ldots, l_t)$ to be the subposet of B_n which is induced by restricting B_n to the sets of cardinalities $l_1, l_2, \ldots, l_t$. In [10] we can find

$$s(B_n(l_1, \ldots, l_t)) = -1 + \sum_{k=1}^{t} \binom{n}{l_k} - \sum_{k=1}^{t} \binom{n - l_{k+1} + l_k}{l_k}.$$

Theorem 2.5. *Let $n \ge 2$. Then $P = \underline{a_1} \times \cdots \times \underline{a_n}$ is greedy if and only if $n \le 3$ and all a_i equal 2.*

Proof: Without loss of generality we assume $a_1 \ge \cdots \ge a_n$. Suppose $n \le 3$ and all a_i equal 2. Then $n = 2$ or 3, and so P is B_2 or B_3. By direct construction, we can easily show that B_2 and B_3 are greedy.

Suppose that either $a_1 \ge 3$ or $n \ge 4$ and $a_i = 2$ for all i.

Case 1: $a_1 \ge 3$.
If $a_2 \ge 3$, then let $C_0 = \{(u, 1) : u = 1, \ldots, a_1\}$ and $C_i = \{(i, u) : u = 2, \ldots, a_2\}$ for $i = 1, \ldots, a_1$. Then $L_1 = C_0 \oplus C_1 \oplus \cdots \oplus C_{a_1}$ is a greedy linear extension of $a_1 \times a_2$ which is not optimal. If $a_2 = 2$, then let $C_i = \{(i, u) : u = 1, 2\}$. Then $L_2 = C_1 \oplus \cdots \oplus C_{a_1}$ is a greedy linear extension of $\underline{a_1} \times \underline{a_2}$ which is not optimal. Hence $\underline{a_1} \times \underline{a_2}$ is not greedy. Now we may identify $\underline{a_1} \times \underline{a_2}$ with

$\underline{a_1} \times \underline{a_2} \times \underline{1} \times \cdots \times \underline{1}$. Thus we regard L_1, L_2 as a greedy linear extension of $\underline{a_1} \times \underline{a_2} \times \underline{1} \times \cdots \times \underline{1}$.

Now let L be a linear extension of $\underline{a_1} \times \cdots \times \underline{a_n}$ such that $L = L_i \oplus l$ where $i = 1$ or 2 depending on the case above and l is a greedy linear extension of $\underline{a_1} \times \cdots \times \underline{a_n} \setminus \underline{a_1} \times \underline{a_2} \times \underline{1} \times \cdots \times \underline{1}$. Then L is greedy but not optimal, and so $\underline{a_1} \times \cdots \times \underline{a_n}$ is not greedy.

Case 2: $n \geq 4$ and $a_i = 2$ for all i.
Theorem 2.2 implies that for any optimal linear extension L_o of B_n every (B_n, L_o)-chain has length one. Thus it suffices to show that there exists a greedy linear extension L of B_n which has at least one one-element (B_n, L)-chain. Since B_4 is contained in B_n for $n \geq 4$, it is enough to show that B_4 has a greedy linear extension which is not optimal. Let $C_1 = \{\emptyset, \{1\}\}$, $C_2 = \{\{2\}, \{1,2\}\}$, $C_3 = \{\{3\}, \{1,3\}\}$, $C_4 = \{\{4\}, \{2,4\}\}$, $C_5 = \{\{3,4\}\}$, $l \in \mathcal{G}(B_4 \setminus \cup_{i=1}^{5} C_i)$. Then

$$L = C_1 \oplus C_2 \oplus C_3 \oplus C_4 \oplus C_5 \oplus l$$

is a greedy linear extension of B_4, but L is not optimal since C_5 is a one-element (B_4, L)-chain.

By Corollary 2.3 and Theorem 2.5, we get from Lemma 1.2 that B_n is reversible if and only if $n = 1, 2, 3$.

One may ask the following question: Let Q be a subposet of a greedy poset P. Is Q greedy?

This is false. For example, by Theorem 2.5 B_3 is greedy. Clearly, $\underline{2} \times \underline{3}$ is a subposet of B_3 but again by Theorem 2.5 not greedy.

3. Product of a tree and a chain

Throughout the section we let T denote an *upward rooted tree*, that is, a poset whose diagram is an upward rooted tree and we study the properties of posets $T \times \underline{k}$.

Proposition 3.1. *Let P, Q be posets, and $L_P = C_1 \oplus \cdots \oplus C_m \in \mathcal{L}(P)$, $L_Q = D_1 \oplus \cdots \oplus D_n \in \mathcal{L}(P)$, and $l_{ij} \in \mathcal{L}(C_i \times D_j)$. Let $l = \oplus_{i,j} l_{ij}$ where i, j is arranged by keeping the following order relation: $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. Then (i) $l \in \mathcal{L}(P \times Q)$, (ii) $l \in \mathcal{L}(L_P \times L_Q)$.*

Proof: Let $x = (x_P, x_Q) \in l_{i_1 j_1}$, $y = (y_P, y_Q) \in l_{i_2 j_2}$.

(i) If $x < y$ in $P \times Q$, then $x_P \leq y_P$ in P and $x_Q \leq y_Q$ in Q. Thus $i_1 \leq i_2$ and $j_1 \leq j_2$ and we get $(i_1, j_1) \leq (i_2, j_2)$.

(ii) If $x < y$ in $P \times Q$ then $(x_P, x_Q) < (y_P, y_Q)$ in $L_P \times L_Q$. This implies that $x_P \leq y_P$ in L_P and $x_Q \leq y_Q$ in L_Q, and thus $i_1 \leq i_2$ and $j_1 \leq j_2$. Hence $(i_1, j_1) \leq (i_2, j_2)$.

Lemma 3.2. $s(P\times \underline{k}) \le \sum_{i=1}^{n} \min\{|C_i|, k\} - 1$ *where* $C_1\oplus\cdots\oplus C_n \in \mathcal{O}(P)$.

Proof: Let $C_1\oplus\cdots\oplus C_n \in \mathcal{O}(P)$. Then for $l_i^o \in \mathcal{O}(C_i\times\underline{k})$, $c(l_i^o) = \min\{|C_i|, k\}$. By Proposition 3.1, $l_1^o\oplus\cdots\oplus l_n^o \in \mathcal{L}(P\times\underline{k})$. And also we get $c(l_1^o\oplus\cdots\oplus l_n^o) = \sum_{i=1}^{n}\min\{|C_i|, k\}$. Hence $s(P\times\underline{k}) \le \sum_{i=1}^{n}\min\{|C_i|, k\} - 1$.

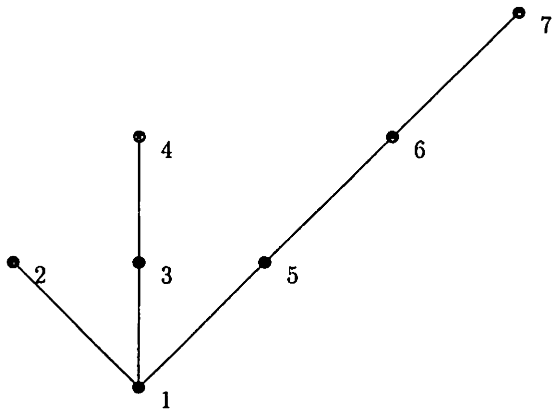


Figure 4: The tree T_e.

Consider the Figure 4. Let $C_1 = \{1,5,6,7\}$, $C_2 = \{3,4\}$, $C_3 = \{2\}$, and $D_1 = \{1,3,4\}$, $D_2 = \{5,6,7\}$, $D_3 = \{2\}$, and $E_1 = \{1,2\}$, $E_2 = \{3,4\}$, $E_3 = \{5,6,7\}$. Note that $L_1 = C_1\oplus C_2\oplus C_3 \in \mathcal{O}(T_e)$, $L_2 = D_1\oplus D_2\oplus D_3 \in \mathcal{O}(T_e)$, $L_3 = E_1\oplus E_2\oplus E_3 \in \mathcal{O}(T_e)$. Applying Lemma 3.2, we obtain $s(T_e\times\underline{3}) \le 5$. It follows from Theorem 3.5 that $s(T_e\times\underline{3}) = 5$.

To determine $s(T\times\underline{k})$, we need the following Lemma. A *k-family* in P is a subposet of P in which no $(k+1)$-chain exists.

Lemma 3.3 (Saks [14]). *The size of the largest k-family in a ranked poset P is the maximal size* $\omega(P\times\underline{k})$ *of a set of mutually incomparable elements in* $P\times\underline{k}$.

Proposition 3.4. *If the maximum size of a chain in a ranked poset P is at most k, then* $s(P\times\underline{k}) = |P| - 1$.

Proof: By the hypothesis, P itself is a k-family. By Saks' Lemma, $\omega(P\times\underline{k}) = |P|$. So $s(P\times\underline{k}) \ge |P| - 1$. Now let $C(a) = \{(a,i) : i = 1,\ldots,k\}$ for all $a\in P$. Choose a linear extension L of P, and let $L_* = \oplus C(a)$ where $C(a)$ is arranged just like the order of a in L. Then $s(L_*, P\times\underline{k}) = |P| - 1$. Hence $s(P\times\underline{k}) = |P| - 1$.

Since T is an upward rooted tree, T is N-free. By Lemma 1.1 T is greedy. Let C_1^* be the longest chain in T. For $j > 1$ let C_j^* be the longest chain in $T\setminus\bigcup_{i=1}^{j-1} C_i^*$. Then $C_1^*\oplus\cdots\oplus C_n^* \in \mathcal{G}(T)$ where $n = \omega(T)$. Since T is greedy, $C_1^*\oplus\cdots\oplus C_n^* \in \mathcal{O}(T)$.

Theorem 3.5. *Let $C_1^* \oplus \cdots \oplus C_n^* \in \mathcal{O}(T)$ be as above. Then*

$$l^* = l_1^* \oplus \cdots \oplus l_n^* \in \mathcal{O}(T \times \underline{k})$$

where $l_i^* \in \mathcal{O}(C_i^* \times \underline{k})$ $(i = 1, \ldots, n)$ and thus $s(T \times \underline{k}) = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1$.

Proof: By construction, $|C_1^*| \geq \cdots \geq |C_n^*|$. We use induction on the number of elements in T.

Case 1: $k \geq |C_1^*|$.
This follows by Proposition 3.4.

Case 2: $|C_n^*| > k$.
Let x be a leaf of T in C_n^* and T_o be $T \setminus \{x\}$. Define $D_i^* = C_i^*$ for $i = 1, \ldots, n-1$ and $D_n^* = C_n^* \setminus \{x\}$. Let $L_o^* = D_1^* \oplus \cdots \oplus D_n^*$. Then $L_o^* \in \mathcal{O}(T_o)$. The induction hypothesis implies that

$$s(T_o \times \underline{k}) = \sum_{i=1}^{n} \min\{|D_i^*|, k\} - 1 = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1 \geq s(T \times \underline{k}).$$

Let $l \in \mathcal{O}(T \times \underline{k})$ and l_r be the restriction of l to $T_0 \times \underline{k}$. Then $c(l_r) \leq c(l) = s(T \times \underline{k}) + 1$, and $s(T_0 \times \underline{k}) \leq c(l_r) - 1 \leq s(T \times \underline{k})$. Hence $s(T \times \underline{k}) = s(T_o \times \underline{k}) = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1$. So $l^* = l_1^* \oplus \cdots \oplus l_n^* \in \mathcal{O}(T \times \underline{k})$.

Case 3: $|C_1^*| > k \geq |C_n^*|$.
Let x be a leaf of T in C_n^*. Let $T_o = T \setminus \{x\}$. By the induction hypothesis, there exists $L \in \mathcal{O}(T_o \times \underline{k})$ such that $c(L) = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1$. Choose a $L^* \in \mathcal{O}(T \times \underline{k})$. Then $c(L^*) \geq c(L)$. Suppose that $c(L) = c(L^*)$. Let L_r be the restriction of L^* to $T_o \times \underline{k}$. Then $c(L^*) \geq c(L_r) \geq c(L)$. Since $c(L) = c(L^*)$, we have

$$c(L_r) = c(L). \tag{2}$$

Let y be an element in the tree covered by x. Then (2) implies that each (y, i) $(i = 1, \ldots, k)$ is the last element of some $(T_0 \times \underline{k}, L_r)$-chain in L_r. If $|C_n^*| < k$, then $c(L^*) = c(L_r)$ implies $c(L_r) > c(L)$. Contradiction! If $|C_n^*| = k$, then $\{x\} \times \underline{k}$ is a $(T \times \underline{k}, L^*)$-chain in L^*. Thus $c(L_r) = c(L) + 1$. Again contradiction! Hence $c(L) < c(L^*)$, and thus $s(T_o \times \underline{k}) \leq s(T \times \underline{k}) - 1$. Now let D_i^* be defined in the same way as in Case 2. Then

$$s(T_o \times \underline{k}) = \sum_{i=1}^{n} \min\{|D_i^*|, k\} - 1 = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1 - 1 \geq s(T \times \underline{k}) - 1.$$

Hence $s(T \times \underline{k}) = s(T_o \times \underline{k}) + 1 = \sum_{i=1}^{n} \min\{|C_i^*|, k\} - 1$.

Let V be a poset with three elements $\{x, y, z\}$ such that $x \geq y$ and $z \geq y$ hold, and x and z are incomparable.

Proposition 3.6. *If $T \times \underline{k}$ is greedy, then $|T| \leq 2$ or $k \leq 2$.*

Proof: Assume that $|T| > 2$ and $k > 2$. Let $C_1^* \oplus \cdots \oplus C_n^* \in \mathcal{O}(T)$. If $|C_1^*| > 2$ then $C_1^* \times \underline{k}$ is not greedy. Let l_1 be a greedy linear extension of $C_1^* \times \underline{k}$ which is not optimal. Choose any greedy linear extension l_2 of $T \times \underline{k} \setminus C_1^* \times \underline{k}$. Then $l_1 \oplus l_2$ is a greedy linear extension of $T \times \underline{k}$ but not optimal. Thus $T \times \underline{k}$ is not greedy. If $|C_1^*| \leq 2$ then T contains V. But $V \times \underline{k}$ is not greedy. So $T \times \underline{k}$ is not greedy.

4. Application

In 1930, Szpilrajn [16] also proved that any order relation is the intersection of its linear extensions. A set of linear extensions of P whose intersection is P is called a *realizer* of P. In 1941, Dushnik and Miller [8] defined the *dimension* of an ordered set P to be the minimum cardinality of a realizer of P. A *minimum realizer* of P is a realizer which achieves the dimension of P. Dushnik and Miller also showed that a poset has dimension at most 2 if and only if P has a conjugate order if and only if its incomparability graph is a comparability graph.

In 1962, Ore [11] proved that the dimension of a poset P is the least number of chains whose product contains P as a subposet.

The *greedy dimension* $\dim_g P$ of P is the least number of greedy linear extensions whose intersection is P. Bouchitte, Habib, and Jegou [2] showed that if P is N-free then $\dim_g P = \dim P$.

A poset is *bounded* if it has a least element and a greatest element. Baker [1] showed the following lemma:

Lemma 4.1. *Let P, Q be bounded posets. Then* $\dim P \times Q = \dim P + \dim Q$.

Let $P^0 = \underline{a_1} \times \underline{a_2} \times \cdots \times \underline{a_n}$. Without loss of generality assume that $a_1 \geq a_2 \geq \cdots \geq a_n$. A (reverse cyclic) *jth chain* $C_j(i_1, \ldots, i_{n-1})$ of P^0 is $\{(i_{j-1}, \ldots, i_1, l, i_{n-1}, \ldots, i_j) : l = 1, \ldots, a_j\}$ where $1 \leq i_{j-1} \leq a_1, \ldots, 1 \leq i_1 \leq a_{j-1}, 1 \leq i_{n-1} \leq a_{j+1}, \ldots, 1 \leq i_j \leq a_n$. For $j = 1, \ldots, n$, let

$$L_j = (\oplus C_j(i_1, \ldots, i_{n-1}) : (i_1, \ldots, i_{n-1}) \text{ lexicographic order}).$$

Then $L_j \in \mathcal{L}(P^0)$. We call above L_j the (reverse cyclic) *jth linear extension* of P^0. Let $P_k = L_1 \cap \cdots \cap L_k$ for $k = 1, \ldots, n$.

Proposition 4.2. *For $k = 1, \ldots, n$,*

$$P_k = \underline{a_1} \times \cdots \underline{a_{k-1}} \times L^k \tag{3}$$

where L^k is the first linear extension of $\underline{a_k} \times \cdots \times \underline{a_n}$.

Proof: We induct on k. If $k = 1$ then (3) is true. Suppose that (3) is true for $k = l$. Then $P_l = \underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l$ where L^l is a first linear extension of $\underline{a_l} \times \cdots \times \underline{a_n}$. It suffices to show that $P_{l+1} = \underline{a_1} \times \cdots \times \underline{a_l} \times L^{l+1}$, that is,

$$(\underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l) \cap L_{l+1} = \underline{a_1} \times \cdots \times \underline{a_l} \times L^{l+1}. \tag{4}$$

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Let any order relation be lexicographic order.

Suppose $x < y$ in $(\underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l) \cap L_{l+1}$. Since $x < y$ in L_{l+1},

$$(x_l, \ldots, x_1, x_n, \ldots, x_{l+1}) < (y_l, \ldots, y_1, y_n, \ldots, y_{l+1}). \tag{5}$$

Also, $x < y$ in $\underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l$. Thus we get

$$x_i \le y_i \text{ for } i = 1, \ldots, l-1, \tag{6}$$

and

$$(x_n, \ldots, x_l) < (y_n, \ldots, y_l). \tag{7}$$

Conversely, (5), (6), and (7) imply $x < y$ in $(\underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l) \cap L_{l+1}$. Hence (5), (6), and (7) are equivalent to $x < y$ in $(\underline{a_1} \times \cdots \times \underline{a_{l-1}} \times L^l) \cap L_{l+1}$.

Now suppose $x < y$ in $\underline{a_1} \times \cdots \times \underline{a_l} \times L^{l+1}$. Then

$$x_i \le y_i \text{ for } i = 1, \ldots, l, \tag{8}$$

and $(x_{l+1}, \ldots, x_n) < (y_{l+1}, \ldots, y_n)$ in L^{l+1}, that is,

$$(x_n, \ldots, x_{l+1}) < (y_n, \ldots, y_{l+1}). \tag{9}$$

Conversely, (8) and (9) imply $x < y$ in $\underline{a_1} \times \cdots \times \underline{a_l} \times L^{l+1}$. Hence (8) and (9) are equivalent to $x < y$ in $\underline{a_1} \times \cdots \times \underline{a_l} \times L^{l+1}$.

But we can easily show that (5), (6), and (7) are equivalent to (8) and (9). Therefore we get (4).

Thus Lemma 4.1 and Proposition 4.2 imply that reverse cyclic linear extensions $L_1, \ldots, L_n$ of P^0 are a minimum realizer. From Propostion 4.2 and Lemma 4.1, we also obtain that a minimum realizer of a poset P satisfies (3) if and only if P is a products of chains. But this does not imply any minimum realizer of a products of chains satisfies (3). For example, let L_1, L_2, L_3 be a minimum realizer of $B_3 = \underline{2}^3$ where

$$L_1 = \{\emptyset\} \oplus \{1\} \oplus \{2\} \oplus \{1,2\} \oplus \{3\} \oplus \{2,3\} \oplus \{1,3\} \oplus \{1,2,3\},$$

$$L_2 = \{\emptyset\} \oplus \{3\} \oplus \{1\} \oplus \{1,3\} \oplus \{2\} \oplus \{1,2\} \oplus \{2,3\} \oplus \{1,2,3\},$$

$$L_3 = \{\emptyset\} \oplus \{3\} \oplus \{2\} \oplus \{2,3\} \oplus \{1\} \oplus \{1,3\} \oplus \{1,2\} \oplus \{1,2,3\},$$

then $L_1 \cap L_2$ is B_3 with $\{2,3\} > \{1\}$.

Let $n \ge 3$ and $a_i \ge 2$ for $i = 1, \ldots, n$. Then $P^0 = \underline{a_1} \times \cdots \times \underline{a_n}$ always contains $\{(0,1,1,0,\ldots,0),(1,1,0,\ldots,0),(0,0,1,0,\ldots,0),(0,1,0,\ldots,0)\}$. So P^0 is not N-free. But since the reverse cyclic linear extensions $L_1, \ldots, L_n$ of P^0 are a greedy minimum realizer, P^0 satisfies $\dim_g P^0 = \dim P^0$.

Another interesting observation is that the reverse cyclic linear extensions $L_1, \ldots, L_n$ of $\underline{m}^n$ are an optimal greedy minimum realizer.

Now we get a good property of dimension and jump number for the products of n chains.

Theorem 4.3. *For $j = 1, \dots, n$, let L_j be the jth linear extension of $\underline{a_1} \times \cdots \times \underline{a_n}$. Then*

$$s(P_1) \leq s(P_2) \leq \cdots \leq s(P_n)$$

where $P_l = \bigcap_{i=1}^{l} L_i$. Furthermore, if all $a_i = m$, then we get strict inequality.

Proof: Without loss of generality, we may assume that $a_1 = \max\{a_i\}$. By Proposition 4.2, $P_k = \underline{a_1} \times \cdots \times \underline{a_{k-1}} \times L^k$ where L^k is is the first linear extension of $\underline{a_k} \times \cdots \times \underline{a_n}$. Since $|L^1| > |L^2| > \cdots > |L^n|$, we get $\max\{a_1, |L^i|\} \geq \max\{a_1, |L^{i+1}|\}$. Now by Theorem 2.2 $s(P_k) = (\prod_{i=1}^{n} a_i/c) - 1$ where $c = \max\{a_1, |L^k|\}$, and hence we have $s(P_i) \leq s(P_{i+1})$.

If all $a_i = m$, then we get $|L^i| = m^{n-i+1}$. Thus we get $\max\{a_1, |L^i|\} = m^{n-i+1}$. Again by Theorem 2.2 $s(P_k) = (\prod_{i=1}^{n} m/m^{n-i+1}) - 1 = m^{i-1} - 1$. Hence we get strict inequality.

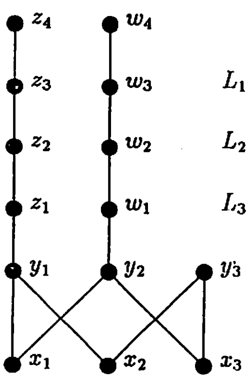
Suppose there are n jobs which are partially ordered, and there are a group of people who scheduled those jobs linearly without violating the given partial order. If consecutive jobs are not ordered then some extra cost is required. We want to interview some people about their schedule and get the best schedule as regards cost. Assume that after interviewing m people we can get the original partial order.

There are some questions about this.

"Find the minimum number of persons to get the original partial order on jobs" is equivalent to "find the dimension of the poset P obtained from n jobs".

Suppose we find the minimum number d of persons. After each interview we have a poset with the cumulative results. Finding the minimum cost after each interview is reformulated as follows: Let P be a poset and $L_1, \dots, L_d$ be any minimum realizer for P. Find $s(\bigcap_{i=1}^{j} L_i)$ for $j = 1, \dots, d$.

By observing some examples(e.g., standard poset), we may ask whether or not $s(\bigcap_{i=1}^{j} L_i) \leq s(\bigcap_{i=1}^{j+1} L_i)$ is true for $i = 1, \dots, d-1$.



$$L_1 = x_1x_2y_1x_3y_2w_1z_1w_2z_2w_3w_4z_3z_4y_3,$$

$$L_2 = x_2x_3y_3x_1y_2y_1w_1z_1z_2w_2w_3z_3w_4z_4,$$

$$L_3 = x_3x_2y_3x_1y_1y_2z_1z_2z_3z_4w_1w_2w_3w_4.$$

Figure 5: A poset and its minimum realizer L_1, L_2, L_3.

There is a counterexample [17] for this. In Figure 5, $s(L_1) = 0$, $s(L_1 \cap L_2) = 4$, and $s(L_1 \cap L_2 \cap L_3) = 3$. But we conjecture the following:

Conjecture 4.4. *Let P be a poset with dimension d. Then there exist a minimum realizer $L_1, \ldots, L_d$ for P such that $s(\bigcap_{i=1}^{j} L_i) \leq s(\bigcap_{i=1}^{j+1} L_i)$ for $j=1, \ldots, d-1$.*

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