Optimal Completely Randomized Designs For A Cyclic Covariate Model With Even Replication Number

Stathis Chadjiconstantinidis
Department of Mathematics
Aristotle University of Thessaloniki
Thessaloniki 54006
Greece

and

Kiki Sotirakoglou
Science Department
Agriculture University of Athens
75 Iera Odos
Athens 11855
Greece

ABSTRACT. We consider a linear model for the comparison $V \geq 2$ treatments (or one treatment at V levels) in a completely randomized statistical setup, making r (the replication number) observations per treatment level in the presence of K continuous covariates with values on the K-cube. The main interest is restricted to cyclic designs characterized by the property that the allocation matrix of each treatment level is obtained through cyclic permutation of the columns of the allocation matrix of the first treatment level. The D-optimality criterion is used for estimating all the parameters of this model.

By studying the nonperiodic autocorrelation function of circulant matrices, we develop an exhaustive algorithm for constructing D-optimal cyclic designs with even replication number. We apply this algorithm for $r = 4$, $16 \leq V \leq 24$, $N = rV \equiv 0 \mod 4$, for $r = 6$, $12 \leq V \leq 24$, $N = rV \equiv 0 \mod 4$, for $r = 6$, $V = m \cdot n$, m is a prime, $N = rV \equiv 2 \mod 4$ and the corresponding cyclic designs are given.

1 Introduction

Let $D(r, V)$ be the set of all $rV \times 2V$ "design matrices" X, where

$$X = \begin{bmatrix} I_v & B_1^* \\ I_v & B_2^* \\ \vdots & \vdots \\ I_v & B_r^* \end{bmatrix}$$

(see (2.21) in J.L.Troya [11]), and $B_1^*, B_2^*, \ldots, B_r^*$ be $V \times V$ circulant matrices with entries in $\{1, +1\}$. Here I_n is the identity matrix of order n.

Such matrices X arise in linear cyclic covariate models, as defined by J.L. Troya [12], for situations where our interest is in the joint estimation of regression coefficients (covariates) in addition to the estimation of treatment contrasts. Also, we consider the case where the values of the covariates are not fixed but are to be chosen by the experimenter. Situations in which the values of covariates are not under experimenter's control (i.e. are fixed) are considered by Haggstrom [4], Harville [5] and Wu [17].

More precisely, consider a linear model for comparing $V \geq 2$ treatments (or one treatment at V levels) in a completely randomized statistical setup, making r observations per treatment level (equireplicate design) and it is known that in each experimental unit K continuous covariates can be observed, each of them assuming values on a bounded interval. For example, the score of a subject in a preliminary psychological test can be used as a covariate; characteristics such as sex or previous surgery or diseases in medical studies, might be considered as binary covariates, if the effect of a certain drug or a comparison among different drugs to treat the same disease is under study. Also, in the same medical study age, for instance, might be considered a continuous covariate. For this model see J.L. Troya [11].

When $K = V$, J.L. Troya [12] defined a subclass of these equireplicated designs, the "cyclic" ones, characterized by the property that the allocation matrix of each treatment level is obtained through cyclic permutation of the columns of the allocation matrix of the first treatment level. For more information regarding covariate models we refer to Harville [6], J.L. Troya [11], [12], Kurotska and Wierich [10], Kurotska [7], [8], [9], Wierich [13], [14], [15], [16], Chadjiconstantinidis and Moyssiadis [3], Chadjiconstantinidis and Chadjipadelis [1], [2].

Next, suppose rank $(X) = 2V$. Then, the vector of regression coefficients and treatment contrasts is estimable, with variance proportional to $(X^T X)^{-1}$, where A^T denotes the transpose of the matrix A. In this paper, we consider the D-optimality criterion for estimating this vector, i.e. we are interested in finding a $X^* \in D(r, V)$ which minimizes logdet

(M^{-1}), or equivalently maximizes the determinant of the "information matrix" $M = X^T X$ over all $X \in D(r, V)$. Such a design matrix X is called D-optimal cyclic design.

Note that,

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}.$$

where

$$M_{11} = rI_v, \quad M_{12} = \sum_{j=1}^{r} B_j^*, \quad M_{22} = \sum_{j=1}^{r} B_j^{*T} B_j^* = \sum_{j=1}^{r} B_j^* B_j^{*T}$$

(since B_j^* is circulant).

J.L. Troya [11] has shown that for a D-optimal design (for the estimation of all parameters), the values of the covariates must be ± 1. For this reason we take the B_j^*'s to have entries in $\{-1, +1\}$, since these matrices contain the values of the covariates. She also gave the maximum value for the determinant of the information matrix M for all the $2V$ parameters for an equireplicate design. From table 1 in J.L. Troya [11] and for $K = V$ we have

$$det(M) \leq \begin{cases} r^v \cdot N^v, & \text{if } N \equiv 0 \mod 4, r \equiv 0 \mod 2 \quad (1) \\ r^v (N-2)^{v-1}(N + 2V - 2), & \text{if } N \equiv 2 \mod 4, r \equiv 2 \mod 4 \quad (2) \end{cases}$$

Here $N = rV$ is the total number of observations and r is the common replication number.

In section 2, by studying the nonperiodic autocorellation function of circulant matrices we give a construction method of D-optimal cyclic designs for the case $N = rV \equiv 0 \mod 4$, $r \equiv 0 \mod 2$.

In section 3, using again the nonperiodic autocorellation function of circulant matrices, we give a construction method of D-optimal cyclic designs when $N = rV \equiv 2 \mod 4, r \equiv 2 \mod 4$ and V is not a prime, i.e. $V = mn$, $m, n > 1$.

In section 4, we describe an exhaustive algorithm, based on the results of sections 2 and 3, for constructing D-optimal cyclic designs with even replication number. We apply this algorithm for $r = 4$, $16 \leq V \leq 24$, $N = rV \equiv 0 \mod 4$; $r = 6$, $12 \leq V \leq 24$, $N = rV \equiv 0 \mod 4$ and $r = 6$, $V = 9, 15$. The corresponding cyclic designs are also given in Tables 1, 2 and 3.

2 D-optimal Cyclic Designs For $N = rV \equiv 0 \mod 4$, $r \equiv 0 \mod 2$

The equality in (1) is attained, for instance, by designs satisfying

$$\sum_{j=1}^{r} B_j^* B_j^{*T} = rVI_v \tag{3}$$

and

$$\sum_{j=1}^{r} B_j^* = Q, \tag{4}$$

where Q denotes the $V \times V$ matrix of zeros.

Designs satisfying (3) and (4) are referred to as designs attaining the "Hadamard upper bound" (H.u.b).

Let the sequence $B_j = \{b_{j1}, \ldots, b_{jv}\}$ of length V, consisting of the elements of the first row of the $V \times V$ circulant matrix B_j^*, $j = 1, \ldots, r$, with $b_{ji} = \pm 1$, $i = 1, \ldots, V$.

The nonperiodic autocorrelation function $N_{B_j}(s)$ is defined as

$$N_{B_j}(s) = \sum_{i=1}^{v-s} b_{ji} b_{j,i+s}, \qquad s = 0, 1, \ldots, V-1; \quad j = 1, 2, \ldots, r$$

and the polynomial associated with the sequence B_j is

$$B_j(z) = b_{j1} + b_{j2}z + \ldots b_{jv}z^{v-1}, \qquad j = 1, 2, \ldots, r.$$

Then, for every $z \neq 0$

$$B_j(z)B_j(z^{-1}) = \sum_{j=1}^{r} N_{B_j}(O) + \sum_{s=1}^{v-1} \left[\sum_{j=1}^{r} N_{B_j}(s) + \left(\sum_{j=1}^{r} N_{B_j}(V-s) \right) z^{-v} \right] z^s. \tag{5}$$

Theorem 1. *Suppose there exists a cyclic design attaining the H.u.b. Then*

$$\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) = rv, \qquad \forall z^v = 1 \tag{6}$$

and

$$\sum_{j=1}^{r} B_j(z) = 0, \qquad \forall z. \tag{7}$$

Proof: Let there exists a cyclic design attaining the H.u.b. Then, the relations (3) and (4) are equivalent to

$$\sum_{j=1}^{r} N_{B_j}(0) = rV, \qquad \text{if } s = 0 \tag{8}$$

$$\sum_{j=1}^{r} (N_{B_j}(s) + N_{B_j}(V-s)) = 0, \qquad \text{if } 1 \le s \le V-1 \tag{9}$$

and

$$\sum_{j=1}^{r} B_j = \{0, 0, \ldots, 0\}. \tag{10}$$

Let $z^v = 1$. Then, from (5) we have

$$\begin{aligned}\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) &= \sum_{j=1}^{r} N_{B_j}(0) + \sum_{s=1}^{v-1}\left[\sum_{j=1}^{r}(N_{B_j}(s) + N_{B_j}(V-s))\right] z^s \\ &= rV \qquad \text{(from (8) and (9)).}\end{aligned}$$

Also

$$\sum_{j=1}^{r} B_j(z) = \sum_{j=1}^{r}\left(\sum_{i=1}^{v} b_{ji}\right) z^{i-1} = 0 \qquad \text{(from (10)).}$$

□

For every given sequence $B_j = \{b_{j1}, \ldots, b_{jv}\}$, $j = 1, 2, \ldots, r$ and for some $m \in \{2, 3, \ldots, v\}$ we define the m subsequences B_{ji}, $i = 1, 2, \ldots, m$, where

$$B_{ji} = \{b_{ji}, b_{j,i+m}, \ldots, b_{j,i+s_i m}\}, \qquad s_i = \left\lceil \frac{V-i}{m} \right\rceil, \quad i = 1, 2, \ldots, m$$

with associated polynomials

$$B_{ji}(z) = \sum_{p \equiv i \mod m}^{s_i} b_{j,i+pm} z^p, \qquad j = 1, \ldots, r; \quad i = 1, \ldots, m.$$

Then

$$B_j(z) = B_{j1}(z^m) + zB_{j2}(z^m) + \cdots + z^{m-1}B_{jm}(z^m)$$

or

$$B_j(z) = \sum_{i=1}^{m} z^{i-1} B_{ji}(z^m), \qquad j = 1, \ldots, r \tag{11}$$

and for every $z \neq 0$

$$\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) = \sum_{i=1}^{m}\sum_{j=1}^{r} B_{ji}(z^m)B_{ji}(z^{-m}) + \sum_{\mu=1}^{m-1}\left\{\sum_{i=1}^{m-s}\sum_{j=1}^{r} B_{ji}(z^m)B_{j,i+s}(z^{-m}) + z^m \sum_{i=1}^{s}\sum_{j=1}^{r} B_{j,i+m-s}(z^m)B_{ji}(z^{-m})\right\} z^{-\mu}. \quad (12)$$

From theorem 1 and relation (11) we obtain the following:

Theorem 2. *Suppose there exists a cyclic design attaining the H.u.b. Then,*

$$\sum_{i=1}^{m}\sum_{j=1}^{r} B_{ji}(z^m)B_{ji}(z^{-m}) = rV, \qquad \forall m \in \{2,3,\dots,V\}, \quad z^v = 1 \quad (13)$$

$$\sum_{i=1}^{m-s}\sum_{j=1}^{r} B_{ji}(z^m)B_{j,i+s}(z^{-m}) + z^m \sum_{i=1}^{s}\sum_{j=1}^{r} B_{j,i+m-s}(z^m)B_{ji}(z^{-m}) = 0 \quad (14)$$

with $s = 1,2,\dots,[m/2]$, $z^v = 1$, *and*

$$\sum_{j=1}^{r} B_{ji}(z^m) = 0, \qquad \forall i = 1,2,\dots,m. \quad (15)$$

Proof: Writing $B_j(z), j = 1,2,\dots,r$ as in (11), and equating all coefficients of z^t in (6), where $t \equiv s \mod m$, we find the above relations (13) and (14) for $s = 1,\dots,m-1$. By taking complex conjugates (i.e. by setting z^{-1} instead of z) in (13) and (14) we see that it is enough to take $s = 1,2,\dots,[m/2]$. Also, from (7) and (11) we can easily obtain the relation (15). □

Theorem 3. *Suppose there exists a cyclic design attaining the H.u.b., and let $B_1, B_2, \dots, B_r$ be the corresponding sequences of length V. Let also*

(a)

$$c_{jim} = \sum_{p \equiv i \mod m} b_{jp}, \qquad j = 1,2,\dots,r$$

$$C_{jm} = \{c_{j1m}, c_{j2m}, \dots, c_{jmm}\}$$

for any given $m \in \{2,3,\dots,V\}$.

(b) $N_{c_{jm}}(s) = \sum_{i=1}^{m-s} c_{jim}c_{j,i+s,m}, \qquad j = 1,\dots,r.$

Then, for the given $m \in \{2,3,\dots,V\}$,

$$\sum_{j=1}^{r} N_{c_{jm}}(0) = \sum_{j=1}^{r}\sum_{i=1}^{m} c_{jim}^2 = rV, \tag{16}$$

$$\sum_{j=1}^{r}(N_{c_{jm}}(s) + N_{c_{jm}}(m-s)) = 0, \qquad s = 1,2,\dots,[m/2], \tag{17}$$

$$\sum_{j=1}^{r} c_{jim} = 0, \qquad \forall i = 1,2,\dots,m. \tag{18}$$

Proof: Let there exists a cyclic design attaining the H.u.b. Then, the relations (13)-(15) are valid. By setting $z = 1$, then $B_{ji}(1) = c_{jim}$ and (13)-(15) become

$$\sum_{i=1}^{m}\sum_{j=1}^{r} c_{jim}^2 = rv, \qquad m \in \{2,3,\dots,V\}$$

$$\sum_{i=1}^{m-s}\sum_{j=1}^{r} c_{jim}c_{j,i+s,m} + \sum_{i=1}^{s}\sum_{j=1}^{r} c_{j,i+m-s,m}c_{jim} = 0, \qquad s = 1,2,\dots,[m/2]$$

$$\sum_{j=1}^{r} c_{jim} = 0, \qquad \forall i = 1,2,\dots,m$$

which are the relations (16), (17) and (18) respectively. □

Also, it holds

$$c_{jim} = c_{ji,2m} + c_{j,i+m,2m}, \qquad i = 1,2,\dots,m. \tag{19}$$

Using theorem 3 we developed an algorithm, given in section 4, for constructing D-optimal cyclic designs attaining the H.u.b. We applied this algorithm for $r = 4$, $16 \le V \le 24$, $V \not\equiv 7 \mod 8$ and for all decompositions of $4V$ into four squares having zero sum (see table 1), as well as for $r = 6$, $12 \le V \le 24$, $V \not\equiv 2 \mod 4$ and for all decompositions of $6V$ into six squares having zero sum (see table 2). Chadjiconstantinidis and Chadjipadelis [1], [2] have proved the non-existence of cyclic designs attaining the H.u.b. when $r = 4$, $V \equiv 7 \mod 8$ and $r = 6$, $V \equiv 2 \mod 4$.

Remark 1: Let

$$X^* = \begin{bmatrix} I_v & I_v & I_v & I_v \\ B_1^{*T} & B_2^{*T} & B_3^{*T} & B_4^{*T} \end{bmatrix}^T \tag{20}$$

be a cyclic design matrix attaining the H.u.b. Then, for $r \equiv 0 \mod 4$, i.e. $r = 4\lambda$, $\lambda \in Z$, $\lambda > 1$, we can construct a cyclic design matrix X^{**} attaining the H.u.b. by taking

$$X^{**} = [\underbrace{X^{*T} X^{*T} \dots X^{*T}}_{\leftarrow \lambda - \text{times} \rightarrow}]^T$$

Hence, from table 1 we can construct cyclic designs attaining the H.u.b. for $r \equiv 0 \mod 4$.

Chadjiconstantinidis and Chadjipadelis [1] constructed cyclic designs attaining the H.u.b. for (i) $r \equiv 0 \mod 4$, $V < 15$ (ii) $r \equiv 0 \mod 4$, $V = 2^a \cdot 10^b \cdot 26^c$, a, b, c are non-negative integers, (iii) $N = 4r$ for every $r \equiv 0 \mod 2$. J. Lopes Troya [12] constructed cyclic designs attaining the H.u.b. for $r = 4$ and $V = 4, 5, 6$.

Remark 2: Let X_1^* be a cyclic design matrix attaining the H.u.b. with replication number equals to 6. For $r = 4\lambda + 2$, $\lambda \in Z$, $\lambda > 1$, we can construct a cyclic design matrix X_1^{**} attaining the H.u.b. by taking

$$X_1^{**} = [X_1^{*T} \underbrace{X^{*T} \dots X^{*T}}_{\leftarrow (\lambda - 1) - \text{times} \rightarrow}]^T$$

where X^* is given by (20). Hence, from table 2, and using table 1, we can construct cyclic designs attaining the H.u.b. for $r \equiv 2 \mod 4$.

The case $r = 2$, $N = 2V \equiv 0 \mod 4$ has examined extensively by J.L Troya [12] and Chadjiconstantinidis and Chadjipadelis [2]. They also constructed cyclic designs attaining the H.u.b. for $r \equiv 2 \mod 4$, $r \geq 6$ and $V = 4, 8, 12$.

3 *D*-optimal Cyclic Designs For $N = rV \equiv 2 \mod 4$, $r \equiv 2 \mod 4$

The equality in (2) is attained, for instance, by designs

$$\sum_{j=1}^{r} B_j^* B_j^{*T} = (rV - 2) I_v + 2 J_v \tag{21}$$

and

$$\sum_{j=1}^{r} B_j^* = Q. \tag{22}$$

where J_n denotes the $n \times n$ matrix of ones.

Designs satisfying (21) and (22) are referred to as cyclic designs attaining the "r even upper bound" (r.e.u.b.)

Theorem 4. *Suppose there exists a cyclic design attaining the r.e.u.b. Then*

$$\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) = \begin{cases} V(r+2)-2, & \text{if } z=1 \qquad (23a) \\ rV-2, & \text{if } z^v=1, \quad z\neq 1 \qquad (23b) \end{cases}$$

and

$$\sum_{j=1}^{r} B_j(z) = 0, \qquad \forall z. \tag{24}$$

Proof: Let there exists a cyclic design attaining the r.e.u.b. Then, the relations (21) and (22) are equivalent to

$$\sum_{j=1}^{r} N_{B_j}(0) = rV, \qquad \text{if } s=0 \tag{25}$$

$$\sum_{j=1}^{r}(N_{B_j}(s)+N_{B_j}(V-s)) = 2, \qquad \text{if } 1\leq s\leq V-1 \tag{26}$$

and

$$\sum_{j=1}^{r} B_j = \{0,0,\ldots,0\}. \tag{27}$$

Let $z=1$. Then, from (5) we have

$$\begin{aligned}\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) &= \sum_{j=1}^{r} N_{B_j}(0) + \sum_{s=1}^{v-1}\sum_{j=1}^{r}(N_{B_j}(s)+N_{B_j}(V-s)) \\ &= rV + 2(V-1) \qquad \text{(from (25) and (26))}.\end{aligned}$$

Let $z\neq 1$ and $z^v=1$. Then, again from (5) we have

$$\begin{aligned}\sum_{j=1}^{r} B_j(z)B_j(z^{-1}) &= \sum_{j=1}^{r} N_{B_j}(0) + \sum_{s=1}^{v-1}\left[\sum_{j=1}^{r}(N_{B_j}(s)+N_{B_j}(V-s))\right] z^s \\ &= rV + 2\sum_{s=1}^{v-1} z^s \qquad \text{(from (25) and (26))} \\ &= rV-2.\end{aligned}$$

Finally, the relation (24) follows directly from (27). □

Using the relation (12) and the previous theorem we obtain the following:

Theorem 5. *Suppose there exists a cyclic design attaining the r.e.u.b., and let $B_1, B_2, \ldots, B_r$ be the corresponding sequences of length $V = m \cdot n \equiv 1 \bmod 2$, where $m, n > 1$ and m is a prime. Let also*

(a)

$$c_{jim} = \sum_{p \equiv i \mod m} b_{jp}, \qquad j = 1, 2, \ldots, r$$

$$C_{jm} = \{c_{j1m}, c_{j2m}, \ldots, c_{jmm}\}$$

for some given m.

(b) $N_{C_{jm}}(s) = \sum_{i=1}^{m-s} c_{jim} c_{j,i+s,m}, \qquad j = 1, 2, \ldots, r.$

Then, for the given m,

$$\sum_{j=1}^{r} N_{C_{jm}}(0) = \sum_{j=1}^{r} \sum_{i=1}^{m} c_{jim}^2 = rV + 2n - 2, \qquad n = \frac{v}{m} \tag{28}$$

$$\sum_{j=1}^{r} (N_{C_{jm}}(s) + N_{C_{jm}}(m-s)) = 2n, \qquad s = 1, 2, \ldots, \left(\frac{m-1}{2}\right) \tag{29}$$

$$\sum_{j=1}^{r} c_{jim} = 0, \qquad \forall i = 1, 2, \ldots, m. \tag{30}$$

Proof: For every given sequence $B_j = \{b_{j1}, \ldots, b_{jv}\}$, $j = 1, 2, \ldots, r$ and for a given m we define the m subsequences B_{ji}, $i = 1, 2, \ldots, m$, where

$$B_{ji} = \{b_{ji}, b_{j,i+m}, \ldots, b_{j,i+s_i m}\}, \qquad s_i = \left[\frac{V-i}{m}\right], \quad i = 1, 2, \ldots, m.$$

Suppose there exists a cyclic design attaining the r.e.u.b. Then, the relations (23a), (23b) and (24) are valid.

Let $z^m = 1$, $z \neq 1$ for the given m. Then $z^v = 1$ and since $B_{ji}(1) = c_{jim}$, $z^{-\mu} = z^{m-\mu}$, the relation (23b) with the help of (12) becomes

$$\sum_{i=1}^{m} \sum_{j=1}^{r} c_{jim}^2 + \sum_{\mu=1}^{m-1} \left[\sum_{i=1}^{m-s} \sum_{j=1}^{r} c_{jim} c_{j,i+s,m} + \sum_{i=1}^{s} \sum_{j=1}^{r} c_{jim} c_{j,i+m-s,m} \right] z^{m-\mu}$$
$$= rV - 2, \qquad s = 1, 2, \ldots, m-1$$

or

$$\sum_{j=1}^{r} N_{C_{jm}}(0) + \sum_{\mu=1}^{m-1} \left[\sum_{j=1}^{r} (N_{C_{jm}}(s) + N_{C_{jm}}(m-s)) \right] z^{\mu}$$
$$= rV - 2, \qquad s = 1,2,\ldots,m-1$$

and since $z^m = 1$ we take

$$\sum_{j=1}^{r} N_{C_{jm}}(0) - \sum_{j=1}^{r} (N_{C_{jm}}(s) + N_{C_{jm}}(m-s))$$
$$= rV - 2, \qquad s = 1,2,\ldots,m-1. \tag{31}$$

Let $z = 1$. Then, the relation (23a) with the help of (12) becomes

$$\sum_{i=1}^{m}\sum_{j=1}^{r} c_{jim}^2 + \sum_{\mu=1}^{m-1} \left[\sum_{i=1}^{m-s}\sum_{j=1}^{r} c_{jim}c_{j,i+s,m} + \sum_{i=1}^{s}\sum_{j=1}^{r} c_{jim}c_{j,i+m-s,m} \right].$$
$$= V(r+2) - 2, \qquad s = 1,2,\ldots,m-1$$

or

$$\sum_{j=1}^{r} N_{C_{jm}}(0) + (m-1)\sum_{j=1}^{r} (N_{C_{jm}}(s) + N_{C_{jm}}(m-s))$$
$$= V(r+2) - 2, \qquad s = 1,2,\ldots,m-1. \tag{32}$$

Solving the system of equations (31) and (32) we obtain the relations (28), (29). By taking complex conjugates in (12) and (23a), (23b) we see that it is enough to take $s = 1,2,\ldots,(m-1)/2$.

From the definition of subsequences B_{ji}, it follows that relation (24) with the help of (7) becomes

$$\sum_{j=1}^{r} B_{ji}(z^m) = 0, \qquad \forall z, \quad \forall i = 1,2,\ldots,m.$$

If we set $z^m = 1$ we find the relation (30). □

Using theorem 5 we developed an algorithm, given in section 4, for constructing cyclic designs attaining the r.e.u.b. We applied this algorithm for $r = 6$, $V = 9$, 15 and for all decompositions of $8V - 2$ into six odd squares having zero sum (see table 3).

J.L. Troya [12] constructed cyclic designs attaining the r.e.u.b. (see Theorem 5.1) for $r = 2V$ or $r = 2(V-4)$ and V is a prime. Chadjiconstantinidis and Chadjipadelis [2] constructed cyclic designs attaining the r.e.u.b. for $r = 6$, $V = 3,5,7,9,11,13,15,19,23,27,31,43$ and for $r = 6$, $V = q^2+q+1$, q is a prime or a prime power. They also examined extensively the case $r = 2$, $N = 2V \equiv 2 \mod 4$.

4 The Algorithm

In this section we describe an algorithm, based on Theorems 3 and 5, for the construction of cyclic designs attaining the H.u.b. and the r.e.u.b.

Since it is difficult to find directly the values $b_{11},\ldots,b_{1v}; b_{21},\ldots,b_{2v};\ldots; b_{r1},\ldots,b_{rv}$ of the sequences $B_1,B_2,\ldots,B_r$ respectively, we find the values of integers

$$c_{11m},\ldots,c_{1mm}; c_{21m},\ldots,c_{2mm};\ldots; c_{r1m},\ldots,c_{rmm}$$

as defined from relations (a) in Theorems 3 and 5.

Step 1. Given r and V find all the integer solutions of the system (33):

$$\begin{aligned}
rV &= \sum_{j=1}^{r} c_{j11}^2, &&\text{if } N = rV \equiv 0 \mod 4, \quad r \equiv 0 \mod 2\\
V(r+2)-2 &= \sum_{j=1}^{r} c_{j11}^2, &&\text{if } N = rV \equiv 2 \mod 4, \quad r \equiv 2 \mod 4\\
&\sum_{j=1}^{r} c_{j11} = 0\\
&c_{j11} \equiv V \mod 2, \quad \forall j = 1,\ldots,r\\
&c_{111} \geq c_{211} \geq \cdots \geq c_{r11}
\end{aligned} \tag{33}$$

If the system (33) has no solution, then stop; otherwise go to step 2.

Step 2. For every solution $c_{111}, c_{211},\ldots,c_{r11}$ and a given m, with

$$\begin{aligned}
m &\in \{2,3,\ldots,V\}, && \text{if } N = rV \equiv 0 \mod 4, \quad r \equiv 0 \mod 2\\
m &= V/n, \quad m \text{ is a prime}, && \text{if } N = rV \equiv 2 \mod 4, \quad r \equiv 2 \mod 4
\end{aligned}$$

find $c_{11m},\ldots,c_{1mm}; c_{21m},\ldots,c_{2mm};\ldots; c_{r1m},\ldots,c_{rmm}$ satisfying

(i) $c_{j11} = c_{j1m} + \cdots + c_{jmm}, \quad \forall j = 1,\ldots,r$

(ii) $c_{1im}, c_{2im},\ldots,c_{rim}$ are all odd (even) if $[\frac{V-i}{m}]+1$ is odd (even), $i = 1,\ldots,m$

(iii) $|c_{jim}| \leq [\frac{V-i}{m}]+1, \quad \forall j = 1,\ldots,r; \quad i = 1,\ldots,m$

(iv) $\sum_{j=1}^{r}\sum_{i=1}^{m} c_{jim}^2 = \begin{cases} rV, & \text{if } N = rV \equiv 0 \mod 4, \quad r \equiv 0 \mod 2\\ rV+2n-2, & \text{if } N = rV \equiv 2 \mod 4, \quad r \equiv 2 \mod 4,\end{cases}$

(v) $\sum_{j=1}^{r} c_{jim} = 0, \quad \forall i = 1,\ldots,m$

(vi) $\sum_{j=1}^{r}(N_{C_{jm}}(s)+N_{C_{jm}}(m-s))$
$$= \begin{cases} 0, & \text{if } N=rV\equiv 0 \mod 4, \quad r\equiv 0 \mod 2 \\ 2n, & \text{if } N=rV\equiv 2 \mod 4, \quad r\equiv 2 \mod 4, \end{cases} \qquad s=1,\dots,[m/2]$$

where

$$N_{C_{jm}}(s)=\sum_{i=1}^{m-s} c_{jim}c_{j,i+s,m} \text{ and } C_{jm}=\{c_{j1m},\dots,c_{jmm}\}, \qquad j=1,\dots,r.$$

If for this given m, the requirements (i)-(vi) do not satisfied, then stop; otherwise go to step 3.

Step 3.

(a) For $N=rV\equiv 0 \mod 4$, $r\equiv 0 \mod 2$, let $m_t=2m_{t-1}$, $t=2,3,\dots$ with $m_1=m$ (i.e. $m_t=2^{t-1}m$).

Then for $t=2$ and for every

$$c_{11m_{t-1}},\dots,c_{1m_{t-1}m_{t-1}};c_{21m_{t-1}},\dots,$$
$$c_{2m_{t-1}m_{t-1}};\dots;c_{r1m_{t-1}},\dots,c_{rm_{t-1}m_{t-1}}$$

found in step 2, find

(i) $c_{11m_t},\dots,c_{1m_tm_t};c_{21m_t},\dots,c_{2m_tm_t};\dots;c_{r1m_t},\dots,c_{rm_tm_t}$ satisfying:

(i1) relations (ii), (iii) in step 2 (setting m_t instead of m)

(i2) $c_{jim_{t-1}}=c_{jim_t}+c_{j,i+m_{t-1},m_t}, \qquad j=1,\dots,r; \quad i=1,\dots,m_{t-1}$

(i3) relations (iv)-(vi) in step 2 (setting m_t instead of m).

If there are no such solutions c_{jim_t}, then stop; otherwise for every solution c_{jim_t} repeat the steps (il)-(i3) in step 3 (a) (i) by augmenting t, i.e. taking $t=3,4,\dots$ until $m_t<V$. If for each t, such that $m_t<V$, there are no solutions c_{jim_t}, then stop; otherwise go to step 4(a).

(b) If $N=rV\equiv 2 \mod 4$, $r\equiv 2 \mod 4$, then for every

$$c_{11m},\dots,c_{1mm};c_{21m},\dots,c_{2mm};\dots;c_{r1m},\dots,c_{rmm}$$

found in step 2, find

$$c_{11,nm},\dots,c_{1,nm,nm};c_{21,nm},\dots,$$
$$c_{2,nm,nm};\dots;c_{r1,nm},\dots,c_{r,nm,nm}, \qquad n=v/m$$

satisfying relations (ii)-(vi) in step 2, setting nm instead of m.

If there are no such solutions $c_{ji,nm}$, then stop; otherwise go to step 4(b).

Step 4. Stop when

(a) $m_t \geq V$ if $N = rV \equiv 0 \mod 4$, $r \equiv 0 \mod 2$ and examine if

$$\sum_{j=1}^{r} N_{C_{jm_t}}(0) = rV,$$

and

$$\sum_{j=1}^{r}(N_{C_{jm_t}}(s) + N_{C_{jm_t}}(m_t - s)) = 0, \qquad s = 1, 2, \ldots, m_t - 1$$

since for $m_t \geq v$, it holds $c_{jim_t} = 0, 1, -1$.

(b) If $N = rV \equiv 2 \mod 4$, $r \equiv 2 \mod 4$ and examine if

$$\sum_{j=1}^{r} N_{C_{j,nm}}(0) = rV$$

$$\sum_{j=1}^{r}(N_{C_{j,nm}}(s) + N_{C_{j,nm}}(m - s)) = 2, \qquad s = 1, 2, \ldots, nm - 1.$$

We applied this algorithm for $r = 4$, $16 \leq V \leq 24$, for $r = 6$, $12 \leq V \leq 24$ and for $r = 6$, $V = 9, 15$.

Example: The case $r = 4$, $V = 21 (N \equiv 0 \mod 4)$.

There is one solution for the system (33) in step 1:
$c_{111} = 7$, $c_{211} = 1$, $c_{311} = -3$, $c_{411} = -5$

Let $m = 2$. Then, there are 8 solutions in step 2:

c_{112}	c_{122}	c_{212}	c_{222}	c_{312}	c_{322}	c_{412}	c_{422}
7	0	1	0	-3	0	-5	0
7	0	-1	2	-1	-2	-5	0
5	2	3	-2	-5	2	-3	-2
5	2	-3	4	1	-4	-3	-2
3	4	3	-2	-5	2	-1	-4
3	4	-3	4	1	-4	-1	-4
1	6	1	0	-3	0	1	-6
1	6	-1	2	-1	-2	1	-6

There are no solutions c_{ji4}, $j = 1, 2, 3, 4$; $i = 1, 2, 3, 4$ for $t = 2$ in step 3(a) ($m_2 = 2m_1 = 2m = 4$). Since for $m_2 = 4$ there no solutions, it follows that there are no cyclic designs attaining the H.u.b. for $r = 4$, $V = 21$; otherwise, if there exist B_1, B_2, B_3, B_4 attaining the H.u.b., then Theorem 3 must be valid for any $m \in \{2, 3, \ldots, 21\}$, which is not true for $m = 4$.

Table 1

D-optimal cyclic designs attaining the H.u.b. for $r \equiv 0 \mod 4$, $16 \leq V \leq 24$. The sequences B_1, B_2, B_3, B_4 are given (+ stands for +1 and − stands for −1).

V = 16 $4V = 4^2 + 4^2 + (-4)^2 + (-4)^2$

(i) {+--++-+-+--+++++} {+++---+++-+-++-+}
{-++--+---+-+--+-} {---+++-+-++-----}

(ii) {+---+-+++-++++-+} {++++--+-+++-+-++}
{-++--+---++-+--} {--++++-+-+----+-}

(iii) {+-+++--+-+++++-+} {+-++-+++++--+-+-}
{-+--++-+++----} {-+--+--+----+++}

V = 17 $4V = 7^2 + (-1)^2 + (-3)^2 + (-3)^2$

(i) {-++-+++++++--+-+++} {+---++-+++++-+--}
{-++++---+-+--+---} {+--+---+---+-+-++}

(ii) {+++-++++-+++++--+-} {-+++--+-+---++-+}
{+--++-+----++-+} {--+--+-++++--+-}

V = 17 $4V = 5^2 + 3^2 + (-3)^2 + (-5)^2$

(i) {+--+++++++--++-+-+} {+--++++++-++--+-+- }
{-++-----+-+--+++} {-++----+-+-++---}

(ii) {+-++++++-+---++-++} {-+--+++--++++++-+-}
{-+-----+-+++--+-+} {+-++---++-----+--}

V = 18 $4V = 6^2 + 0 + 0 + (-6)^2$

{+++-++-+++--+++-+-} {-+-+++++-+----++--}
{+--+--+---++-+-+++} {--+-----+-+++----+}

V = 18 $4V = 6^2 + 2^2 + (-4)^2 + (-4)^2$

(i) {++--++++++++-+---++} {+-+-+-++-+--+++++--}
{-+++-+-----+---++-} {---+----+-++-++--+}

(ii) {++-+++++-+-+-++--++} {+++---++++++-+--+--}
{----++-+-+-+--++--} {--++-------+-++-++}

V = 19 $4V = 7^2 + (-1)^2 + (-1)^2 + (-5)^2$

(i) {+++-++--++-++-+-++++} {---++++++-++--+-+---}
{---+--++---++-+-+++} {+++-----+-+--+-+---}

(ii) {+++-++--++++++-+-++-} {---+++++-+---+-+--+}
{---+--++--+++-+-++-} {+++-----+----+-+--+}

(iii) {+++++-+-++--+++-++-} {-+-+++-+--++-+----+}
{------+++-+++--+-++} {+-+--+---+----+++--}

(iv) {+++-+-+-+++++++--++-} {---+-+-+--++++--++-}
{++++-+--+-----++--+} {----+-++-+----++--+}

V = 20 $4V = 6^2 + 2^2 + (-2)^2 + (-6)^2$

(i) {+-+-+++--++++-+++-++} {+-++-+++++++-----++--}
{-+--++---+++--+--+-+} {-+-+-------++++--+-}

(ii) {+--++-++++-++++++++---} {+-++++--+-+--++--+-+}
{-+---+++-+--+----+++} {-++------+-+---++-+-}

(iii) {+--++--++++-+-+++-++} {--+-++-+++++++----++-}
{++++--+---+-+-+-+--} {-+---++-----++-+--+}

(iv) {+-+-++--+-+++--++++++} {++--+-+++++++-+--+---}
{-+++-++-----+++---+} {---+---+-+--+-+--++-}

(v) {+-+-+--++++-+++++++--+} {++-+----++-++-+-+++-}
{---+-+++--+--+---+++} {-++-+++----+---+----}

V = 21 $4V = 7^2 + 1^2 + (-3)^2 + (-5)^2$ No Solution

V = 22	$4V = 8^2 + (-2)^2 + (-2)^2 + (-4)^2$	
(i)	{++-+++--+++++-++++--+-}	{-++--+++-----+-+-+-++-}
	{+-++--+-+++-------++-+}	{----+--+---++++-+-+--+}
(ii)	{+++-++--+--++-++++-+++}	{-+----++-+---+-+-++++-}
	{+-++-+-++++---+---+---}	{---++-+---++++--+----+}
(iii)	{++-+++--+--++++-+++++-}	{-+-++-++-+++-+-+------}
	{+-+----++-+-+----+++-+}	{--+--++--+----+++---++}
(iv)	{+-+++++-++-+++--+--+++}	{-+-----+--++-+-+-++++-}
	{---++--++++---+-+-+--+}	{+++--++-----+-++-+----}

V = 23	No Solution	
V = 24	$4V = 8^2 + 0 + (-4)^2 + (-4)^2$	
(i)	{+---+++-+-+++-+++-++++-+}	{-++-+--+-+++-+++--+--+--}
	{++++-----+---+--++----++}	{---+-++++---+----+-++-+-}
(ii)	{+-+-++-++-+++-+++--++++-}	{--+-++---+++-+++-++----+}
	{++-+--++-+---+--+++-----}	{-+-+--+-+---+------+++++}
(iii)	{+-+++++++-+++-+-+---++-+}	{-+++-------+-+++-++-+++-}
	{-+--+-+--+--++---+++---+}	{+----+-++++----++--+--+-}
(iv)	{++--+-+++-+-++-++-+++-++}	{-++--+-+-+----+---++++++}
	{--++++---+-++++--+------}	{+--+--+-+-++---+++---+--}.

Table 2

D-optimal cyclic designs attaining the H.u.b. for $r \equiv 2 \mod 4$, $r \geq 6$, $12 \leq V \leq 24$. The sequences $B_1, B_2, B_3, B_4, B_5, B_6$ are given (+ stands for $+1$ and $-$ stands for -1).

V = 12	$6V = 6^2 + 0 + 0 + 0 + 0 + (-6)^2$	
(i)	{++-++-++-+++}	{+----++-+++-}
	{--++---+-+++}	{-+-++++----+}
	{+-+-++-++---}	{-++-----+---}
(ii)	{++-++-++-+++}	{+---+-+-++-+}
	{-+++++---+--}	{+--+-+-+--++}
	{--+--++++-+-}	{-++-----+---}
(iii)	{++-++-++-+++}	{+--++++-+---}
	{-+++-+---++-}	{+----+-+-+++}
	{--+-+-+++--+}	{-++-----+---}
(iv)	{++-+++++-++-}	{+---+-+-++-+}
	{--++---+-+++}	{++-+-+----++}
	{-++-+++-+---}	{--+----++---}
(v)	{++-++-++-+++}	{+-+++++-----}
	{--++---+-+++}	{-++-+-+-++--}
	{++---+-++--+}	{-----+--+-+-}

V = 12	$6V = 6^2 + 2^2 + 0 + 0 + (-4)^2 + (-4)^2$	
(i)	{++-++-++-+++}	{+---+-++++-+}
	{--++-+-+-++-}	{--+++++---+-}
	{+++-----+---}	{-+---+--+--+}
(ii)	{++-+++++-++-}	{++++--+---++}
	{-+-++---++-+}	{+-+-++---++-}
	{--+---+++---}	{-----+-++--+}
(iii)	{++-++-++-+++}	{++-++-+-+--+}
	{-+++-+---++-}	{+----+-+-+++}
	{--+---+++---}	{--+-++--+---}
(iv)	{++-+++++-++-}	{+---+-++++-+}
	{-+++-+---++-}	{-+-++++----+}
	{+-+----++---}	{--+-----+-++}
(v)	{++-++-++-+++}	{++-++++-+---}
	{--++-+-+-++-}	{--+---+++++-}
	{++---+-----+}	{--+-+---+--+}
V = 12	$6V = 4^2 + 4^2 + 2^2 + (-2)^2 + (-4)^2 + (-4)^2$	
(i)	{++-+--++-+++}	{+--++++--+++}
	{--+--++++++-}	{+++-+---+---}
	{-+-+-+--+---}	{--+-+--+---+}
(ii)	{++-+-+++-++-}	{+--++++--+++}
	{--+--++++++-}	{+-+-+--++---}
	{-+++-------+}	{-+--+---+--+}
(iii)	{++-++-++-+-+}	{+--++++--+++}
	{++++----+-++}	{--+-+-+++---}
	{-+---+--++--}	{--+--+-+--+-}
(iv)	{++-+--++-+++}	{+--++++-+++-}
	{-+++-+--+++-}	{----+-+++--+}
	{+-+-+------+}	{-++--+-+----}
(v)	{++-++-++-+-+}	{+--++-+-++++}
	{--++---+++++}	{-++-+++-----}
	{+-+--+----+-}	{-+---+-++---}
V = 14	No Solution	
V = 16	$6V = 8^2 + 0 + 0 + 0 + (-4)^2 + (-4)^2$	
(i)	{+-+++-+++-+++-++}	{-+----++-+--++++}
	{-----++--+++++-+}	{++++---++-+--+--}
	{-+-+++--+-----+-}	{+-+-++---+-+----}
(ii)	{+-+++-+++-+++-++}	{-+----++-+--++++}
	{-+++-++-----++-+}	{+-+----+++++-+--}
	{+--+++---+----+-}	{-+--++--+-++----}
(iii)	{+-+++-+++-+++-++}	{-+--+-++-+---+++}
	{-+++-+-+----+++-}	{+++-----+-++-+-+}
	{+--+-++--+--+---}	{----++--++++----}
V = 16	$6V = 8^2 + 2^2 + (-2)^2 + (-2)^2 + (-2)^2 + (-4)^2$	
(i)	{+-+++-+++-+++++-}	{+---++-++---++++}
	{-+----+--++++--+}	{-+-+-+--+-++--+-}
	{+-+++----+---+-+}	{-++--+++-+------}
(ii)	{+-+++-+++-+++++-}	{+---+++++---++-+}
	{-+-+-----++-+-++}	{-+++-+--+--+--+-}
	{+---+----+++-+-+}	{-++--+++-+------}

V = 16	$6V = 6^2 + 4^2 + 0 + (-2)^2 + (-2)^2 + (-6)^2$	
(i)	{+-+++--++-+++-++}	{+-+--+-++-+++++-}
	{----+++--+-++-++}	{-+-+--++++---+--}
	{+++---+--+---+-+}	{-+-+++----+-----}
(ii)	{+-+++--++-+++-++}	{+-+-++--+-++-+++}
	{-+-++-++----+++-}	{++----+--+-+-+-+}
	{-++--+++++------}	{---+-+---++-+----}
(iii)	{+-+++-+++-+++--+}	{+-+--+-++-+++++-}
	{-+-++-++----+++-}	{++-+-++--+-----+}
	{-+------+++--+++}	{--+-++---+-+----}
V = 16	$6V = 4^2 + 4^2 + 4^2 + (-4)^2 + (-4)^2 + (-4)^2$	
(i)	{+-+++---+-+++-++}	{++---++-++--++++}
	{-++-++++-++++---}	{---++-+--+---+-+}
	{---+-+-++-+--+--}	{+++----+---+--+-}
(ii)	{+-+++---+-+++-++}	{++--+++-++---+++}
	{-++-++-+-++++-+-}	{-+----+----+++-+}
	{+--+-+-+--+--+--}	{--++--++++------}
V = 18	No Solution	
V = 20	$6V = 10^2 + (-2)^2 + (-2)^2 + (-2)^2 + (-2)^2 + (-2)^2$	
(i)	{++++-+++--++++-++++-}	{+-----+-+-+----+++++}
	{---+--++-++-++++----}	{-+--++-+--++-+----++}
	{+-+++---++-+--+-+---}	{-++-++--++---+---+-+}
(ii)	{++++-+++--++++-++++-}	{+---++-+-+----+++++}
	{+-++++---+---+--++--}	{-+--+--+---++-++--++}
	{-+-++-+-+++-+-+-----}	{--+----+++-+-+++---+}
V = 20	$6V = 8^2 + 2^2 + 0 + 0 + (-4)^2 + (-6)^2$	
(i)	{+++--+++--+++--+++++}	{+--+-+-+--+-+-++++++}
	{----+++-++--++--++-+}	{-++-+--+++-+-+++----}
	{--+++-+-+-++--+-----}	{++-+-----+--+-+---+-}
(ii)	{+++--+++--+++--+++++}	{+--+-+-++--+-++++++-}
	{--++-++-++----+-++-+}	{-++++--+-+-+--++--+-}
	{----+-+-+-+++++-----}	{++-+---+---++------+}
V = 20	$6V = 8^2 + 2^2 + 2^2 + (-4)^2 + (-4)^2 + (-4)^2$	
(i)	{+++--+++--+++--+++++}	{+--+--+-+++--+-++++-}
	{---+-+--++-++-++++-+}	{-++++-+-+-+---+-----}
	{-++-++-+-----++---+-}	{+---+--+-+-+++-----+}
(ii)	{+++--+++--+++--+++++}	{+--+-++-+++--+--+++-}
	{--+--+-+-+-+-++++-++}	{---++-+-+-+++-+-----}
	{-++-+--++----+++----}	{++-++----+--+----+-+}
V = 20	$6V = 6^2 + 6^2 + (-2)^2 + (-2)^2 + (-2)^2 + (-6)^2$	
(i)	{+++--++-+-+++--++++-}	{+--+-+++-++--+-+++++}
	{---+++-++-+++------+}	{+-++----++-++-+-+---}
	{-+--+-+------+++-+++}	{-++-+--+-+---++-----}
(ii)	{+++--++-+-+++--++++-}	{+--+-+++-++--+-+++++}
	{+------++--+++++---+}	{-++++-+--++-+-----+-}
	{-+-+++-++-----+-+--+}	{--+-+----+-+-++--+--}

V = 20		$6V = 6^2 + 6^2 + 0 + (-4)^2 + (-4)^2 + (-4)^2$	
	(i)	{+++--++-+-+++--++++-}	{+--+-+++--+--+++++++}
		{---+-++--+-++++++-+--}	{+-+-+--+---+-+----++}
		{-++-+--+++++---+-----}	{-+-++---++--+---+--+}
	(ii)	{+++--++-+-+++--++++-}	{+--+-++++-++--++++++-}
		{-+++++-------+-++++-+}	{+-++--+++-+--+-----}
		{----+--+--+++-+---++}	{-+--+-+-++--++------+}
V = 22		No Solution	
V = 24		$6V = 10^2 + 2^2 + (-2)^2 + (-2)^2 + (-4)^2 + (-4)^2$	
	(i)	{++-+-+++-++-++-+-+++-+++}	{++----+---++++---++++++-}
		{---+-+-+---+--+-+-++++++-}	{--+++-+-++---++-++-----+}
		{-++-+---++-++-++----+---}	{+-+-++-++-+---++-------+}
	(ii)	{++-+-+++-++-++-+-+++-+++}	{++----+---++++---++++++-}
		{--+++--++-+------+++-+-+}	{-++-+++-++-+--+++-------}
		{+--++---++-+-++-+---+---}	{--+--+-+----+-+++----+-++}
V = 24		$6V = 8^2 + 2^2 + 2^2 + (-2)^2 + (-2)^2 + (-8)^2$	
	(i)	{++-+--++-++++++-+--++-+++}	{--+++---+-----+++-++++++}
		{--+---++-+++-+--++++-++-+}	{++-+-+-+++----+-++----+-}
		{+---+++-+-+-+++--+-+----}	{-++-++-----++--+-----+---}
	(ii)	{++-+--++-++++++-+--++-+++}	{--+++---+-----+++-++++++}
		{--+--+++-+-+--+-+-+++++}	{++----+-+++---++++-+----}
		{+-+-+++++--+-++---+--+---}	{-+-++---+--++-+--+------}
V = 24		$6V = 8^2 + 2^2 + 2^2 + 0 + (-6)^2 + (-6)^2$	
	(i)	{++-+--++-++++++-+--++-+++}	{--+++-----++--+++-+-++++}
		{+-+--+++--+-+++++++--+---}	{-+-++++-+---+----+-++++-}
		{-+---+--+++-+--+-+-++----}	{+-+-+--+++----+----+----+}
	(ii)	{++-+--++-++++++-+--++-+++}	{--+++-----++--+++-+-++++}
		{+-+-+++---+-+++++-+-++---}	{---+-+---++--++--++++++-+-}
		{-++-+-+++++-------+------+}	{++---+-++--+--+-+-----+--}
V = 24		$6V = 6^2 + 6^2 + 0 + 0 + (-6)^2 + (-6)^2$	
	(i)	{+--++-+--++++--++-+-+++++}	{++-----+++--++--++++++++}
		{-+---++-+--+-+-++++-+-++-}	{--+++++---+--+++---++--+}
		{--++-+-++-+-+-+---+-----}	{+++-+--+-+-+--+--+------}
	(ii)	{+--++-+--++++--++-+-++++}	{++-----+++--++--++++++++}
		{-+-+-++++-++++-+--++--+---}	{--++++-++-+--+++--+----+}
		{+-+---+-+-----++-+-+--+-}	{-++-++-----++-+----+-+--}
V = 24		$6V = 6^2 + 6^2 + 2^2 + (-4)^2 + (-4)^2 + (-6)^2$	
	(i)	{++-+--++-+-++++-+--++-+++}	{+++---+---+-+++--+++++++}
		{--+++-+-++----++++-+--++}	{---+++--+--+-+--+-+-++--}
		{-++--+-+-+++++---+---+---}	{+---++-++-+---++-+------}
	(ii)	{++-+--++-+-++++-+--++-+++}	{+++---+---+-+++--+++++++}
		{--+++-+-++++--++++-+----}	{-+-+++--++-+----+-+-+---}
		{--+--+-++---++--+----+++}	{+---++-+--+---++-+--+---}

Table 3

D-optimal cyclic designs attaining the r.e.u.b. for $r = 6$, $V = 9, 15$. The sequences $B_1, B_2, B_3, B_4, B_5, B_6$ are given (+ stands for +1 and − stands for −1).

V = 9 $\quad 8V-2 = 7^2 + 1^2 + (-1)^2 + (-1)^2 + (-3)^2 + (-3)^2$

(i)	{++++++++-}	{++-++-+--}
	{+++--+---}	{---+-+-++}
	{------+++}	{--+-+---+}
(ii)	{++++++++-}	{++-++-+--}
	{--+++---+}	{-+---++-+}
	{+------++}	{--+--+-+-}
(iii)	{++++++++-}	{++-++-+--}
	{-++---+-+}	{---+++--+}
	{+------++}	{--+--+-+-}

V = 9 $\quad 8V-2 = 5^2 + 3^2 + 1^2 + (-1)^2 + (-3)^2 + (-5)^2$

(i)	{+++++-++-}	{+++++-+--}
	{++-+-+--+}	{--+--+++-}
	{-----+-++}	{---+----+}
(ii)	{+++++-++-}	{+++++-+--}
	{--++-+++-}	{+----+-++}
	{-+---+--+}	{---+----+}
(iii)	{+++++-++-}	{++-++++--}
	{+-+---+++}	{---+++--+}
	{--+--+-+-}	{-+------+}

V = 15 $\quad 8V-2 = 9^2 + 1^2 + (-1)^2 + (-1)^2 + (-3)^2 + (-5)^2$

(i)	{++++-++++-++++-}	{+++++++---+----}
	{++----+-++--+-+}	{---+++--+--++-+}
	{--+----+-++--++}	{----+--+-+-+-+-}
(ii)	{++++-++++-++++-}	{+++++++---+----}
	{++--+--+-+-+-+-}	{---+-++--+--+++}
	{--+----++++---+}	{----+---+--++-+}

V = 15 $\quad 8V-2 = 7^2 + 3^2 + 1^2 + (-1)^2 + (-3)^2 + (-7)^2$

(i)	{++++-++++-+++--}	{+++++++-+-+----}
	{+----+-+-+-++++}	{-++-+--+-++--+-}
	{---++---+--++-+}	{------+--+---++}
(ii)	{++++-++++-+++--}	{++++-++-+++----}
	{++-+++-+----+-+}	{--+-+--+--++-++}
	{------+-++--+++}	{----+----+-+-+-}

V = 15 $\quad 8V-2 = 7^2 + 5^2 + (-1)^2 + (-3)^2 + (-3)^2 + (-5)^2$

(i)	{++++-++++-+++--}	{+++++++++-+----}
	{+----+-+-+-+-++}	{--++---+-++---+}
	{----+-+--+--+++}	{-+--+----+--++-}
(ii)	{++++-++++-+++--}	{++++-++++++----}
	{+---++-++--+--+}	{-+-+------+-+++}
	{--+-+----+-+-++}	{----+-+--+--++-}

$V = 15 \quad 8V-2 = 5^2 + 5^2 + 3^2 + (-3)^2 + (-5)^2 + (-5)^2$

(i)	{++++-+++–+++--}	{+++++++-+-+--+-}
	{+-+--+++++---++}	{–-+–+--++-++}
	{-+-+----+--+-+}	{–-+---++-++--}
(ii)	{++++-+++–+++--}	{+++++++-+-+--+-}
	{+-+--++-++--+++}	{–-+---+++++-}
	{--+---+-+-+--+}	{-+--+--++-----+}

References

[1] S. Chadjiconstantinidis and T. Chadjipadelis, *D*-optimal cyclic complex linear designs and supplementary difference sets with association vector, Submitted (1992).

[2] S. Chadjiconstantinidis and T. Chadjipadelis, Further results on supplementary difference sets and optimal cyclic complex linear designs, *Utilitas Math.* **44**(1993), 17–40.

[3] S. Chadjiconstantinidis and C. Moyssiadis, Some *D*-optimal odd-equireplicated designs for a covariate model, *J. Statist. Plann. Inference* **28** (1991), 83–93.

[4] G. Haggstrom, The pitfalls of manpower experimentation, Rand Corp., Santa Monica, CA (1975).

[5] D.A. Harville, Optimal allocation of experimental material, Techn. Rep. No 73-0112, Aerospace Res. lab., Wright-Patterson AFB, OH (1973).

[6] D.A. Harville, Computing optimum designs for covariance models, In: J.N. Srivastava, Ed., *A Survey of Statistical Design and Linear Models*, North-Holland, Amsterdam (1975).

[7] V.G. Kurotschka, Optimal designs of complex experiments with qualitative factors of influence, *Commun. Statist. - Theor. Meth.* **A7** (1978), 1363–1378.

[8] V.G. Kurotschka, A general approach to optimum design of experiments with qualitative and quantitative factors, In: *Proc. of I.S.I. Golden Jubilee Intern. Conf. on Stat.: Appl. and New Directions*, Calcutta (1981), 353–368.

[9] V.G. Kurotschka, Characterizations and examples of optimal experiments with qualitative and quantitative factors, *Lecture Notes in Economics and Mathematical Systems* No **297** Springer, Berlin (1987).

[10] V.G. Kurotschka and W. Wierich, Optimale Planung eines kovarianzanalyse und eines Intraclass Regressions-Experiments, *Metrika* **31** (1984), 361–378.

[11] J.L. Troya, Optimal designs for covariates models, *J. Statist. Plann. Inference* **6** (1982), 373–419.

[12] J.L. Troya, Cyclic designs for a covariate model, *J. Statist. Plann. Inference* **7** (1982), 49–75.

[13] W. Wierich, Konkrete optimale Versuchsplane fur ein lineares Modell mit einem qualitativen und zwei quantitativen Einflussfaktoren, *Metrika* **31** (1984), 285–301.

[14] W. Weirich, On optimal designs and complete class theorems for experiments with continuous and discrete factors of influence, *J. Statist. Plann. Inference* **15** (1986), 19–27.

[15] W.Weirich, The D-and A-optimality of product design measures for linear model with discrete and continuous factors of influence, *Habilitationsschrift*, Freie Universitat Berlin, (1987).

[16] W.Weirich, A-optimal design measures for one-way layouts with additive regression, *J. Statist. Plann. Inference* **18**(1988), 57–68.

[17] C.-F. Wu, Iterative construction of nearly balanced assignments I: Categorical covariates, *Technometrics* **23** (1981), 37–44.