Properties of Minimal Dominating Functions of Graphs

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ABSTRACT. A dominating function for a graph is a function from its vertex set into the unit interval so that the sum of function values taken over the closed neighbourhood of each vertex is at least one. We prove that any graph has a positive minimal dominating function and begin an investigation of the question: When are convex combinations of minimal dominating functions themselves minimal dominating?

1 Introduction

A dominating function (DF) of a graph $G = (V, E)$ is a function $f : V \to [0, 1]$ such that for each vertex $v \in V$, $f[v] = \sum_{u \in N[v]} f(u) \geq 1$, where $N[v]$ denotes the closed neighbourhood of the vertex v. The concept extends the idea of a dominating set of a graph: For subsets A and B of V, we say A *dominates* B and write $A \succ B$ if every vertex in $B - A$ is adjacent to a vertex in A. If $A \succ V$, then A is a *dominating set* of G. If $A \succ B = \{u\}$, we also write $A \succ u$ for simplicity. If each value $f(u)$ of a DF f is integral (i.e. $f(u) \in \{0, 1\}$), then $S = \{v \mid f(v) = 1\}$ is a dominating set of G.

Dominating functions also arise as solutions to the linear programming relaxation of the following $0 - 1$ integer program (P):

$$\begin{aligned} & Min \textstyle\sum_{i=1}^{n} x_i \\ \text{s.t.} \quad & Nx \geq 1 \\ & x_i \in \{0,1\}, \quad x = (x_1, \ldots, x_n). \end{aligned}$$

Here, N denotes a neighbourhood matrix of the n-vertex graph G, *i.e.* $N = A + I$ where A is an adjacency matrix of G and I is the $n \times n$

identity matrix. The question of when a solution to the linear programming relaxation of (P) where $x_i \geq 0$, yields an optimal solution to the 0–1 integer program (P) defined above, was first studied by Farber in 1984 [8]. However, the first definition and study of the fractional dominating number $\gamma_f(G)$, (i.e. the minimum value of $\sum_{i=1}^{n} x_i$ in a solution x to the linear programming relaxation) was done by Hedetniemi, Hedetniemi and Wimer in 1987 [17]. Since then a considerable amount of work has been done on dominating functions and fractional domination numbers of graphs (*cf* [1-7, 9-16, 18]).

In this paper there are two principal topics. In Section 2 we prove that every graph has a minimal dominating function (MDF), all of whose values are positive and in the following section we begin an investigation of the following question: When are convex combinations of MDFs themselves MDFs?

2 Existence of Positive Minimal Dominating Functions

For DFs f, g of G, we write $f \leq g$ if for all $v \in V$, $f(v) \leq g(v)$. Further, we write $f < g$ if $f \leq g$ and for some $v \in V$, $f(v) < g(v)$. A DF g of G is *minimal* if for all functions $f : V \to [0,1]$ such that $f < g$, f is not dominating. For a DF f, we define the *boundary of* f, denoted by B_f, to equal $\{v \mid f[v] = 1\}$ and the *positive set of* f, denoted by P_f, to equal $\{v \mid f(v) > 0\}$.

Proposition 1. *A $DF f$ of G is an MDF if and only if $B_f \succ P_f$.*

Proof: Let f be an MDF of G with $v \in P_f$ and suppose, to the contrary, that $B_f \not\succ v$. Then $N[v] \cap B_f = \phi$ which implies that $f[u] > 1$ for each $u \in N[v]$. Choose $\epsilon > 0$ in such a way that $\epsilon \leq \min_{u \in N[v]}\{f[u] - 1\}$ and define $g : V \to [0,1]$ by

$$g(v) = f(v) - \epsilon$$

and

$$g(x) = f(x) \text{ for } x \in V - \{v\}.$$

For each $u \in N[v]$, $g[u] = f[u] - \epsilon \geq 1$ and for $x \in V - N[v]$, $g[x] = f[x] \geq 1$; hence g is a DF. But $g < f$, contradicting the minimality of f.

Conversely, suppose $B_f \succ P_f$ and let $g : V \to [0,1]$ satisfy $g < f$; say $g(v) < f(v)$ for some $v \in V$. Then $v \in P_f$ and so, by assumption, $u \in B_f$ for some $u \in N[v]$. Now

$$
\begin{aligned}
g[u] &= \sum_{w \in N[u]} g(w) \\
&= g(v) + \sum_{w \in N[u]-\{v\}} g(w) \\
&< f(v) + \sum_{w \in N[u]-\{v\}} f(w) \\
&= f[u] \\
&= 1.
\end{aligned}
$$

Hence g is not a DF and it follows that f is an MDF. □

The main result of this section asserts, for any graph G, the existence of an MDF whose function values are positive on all vertices of G. This result is often used in subsequent work (see [3]) and sometimes the specific function as defined in the proof is needed. Notice that while the statement of the theorem might be as expected, a similar result does not hold for total dominating functions (which are defined similar to DFs by considering open neighbourhoods instead of closed - see [4]).

Theorem 2. *Any graph $G = (V, E)$ has an $MDF f$ satisfying $P_f = V$.*

Proof: Among all maximum independent sets of G, choose S such that the sum of the degrees of vertices in S is a minimum. For each $s \in S$, define

$$Z_s = \{z \in V - S \mid N(z) \cap S = \{s\}\}$$

and let

$$
\begin{aligned}
Z &= \cup_{s \in S} Z_s \\
T &= \{t \in V - S \mid \; |N(t) \cap S| \geq 2\}.
\end{aligned}
$$

Since S is maximal independent, every vertex in V–S is adjacent to at least one vertex in S and hence $V = S \cup T \cup Z$ (disjoint union). Note that if $z \in Z_s$ for some $s \in S$, then $(S - \{s\}) \cup \{z\}$ is a maximum independent set of G so that, by the choice of S, $\deg(z) \geq \deg(s)$.

Let $|V| = n$, set $\epsilon = 1/2n$ and define $f : V \to [0, 1]$ as follows: For $v \in V - S$, define $f(v) = \epsilon$ and for $s \in S$ define $f(s)$ such that $f[s] = 1$. Note that

$$1 = f[s] = f(s) + \sum_{u \in N(s)} f(u) \leq f(s) + (n-1)/2n$$

and hence $f(s) > 0$. Therefore $P_f = V$.

We show that f is a DF. Firstly, let $z \in Z_s$ for some $s \in S$. Then $\deg(z) \geq \deg(s)$ and z is not adjacent to any vertex in $S - \{s\}$. Also, $N(s) \subseteq V - S$. Hence

$$\begin{aligned} f[z] &= f(s) + \sum_{v \in N[z]-\{s\}} f(v) \\ &\geq f(s) + \sum_{v \in N(s)} f(v) \\ &= f[s] \\ &= 1. \end{aligned}$$

Now let $t \in T$ and recall that t is adjacent to at least two vertices, say s and s', of S. Then

$$f[t] \geq f(s) + f(s') + f(t) > 1 .$$

Thus f is a DF. Since $B_f \supseteq S \succ P_f = V$, f is an MDF. □

We observe that the maximum property of S is not essential for the assignment of function values performed in Theorem 2, which constructed a positive MDF. The same construction may be performed provided that S satisfies the following conditions:

(i) S is maximal independent.

(ii) If $X_s = \{x \in Z_s | \deg(x) < \deg(s)\}$, then $\cup_{s \in S} X_s = \phi$.

A set S with these two properties can easily be constructed in polynomial time and thus, for any graph, a positive MDF may be found in polynomial time. A linear algorithm for producing a positive MDF for an arbitrary tree is given in [2].

3 Convexity of Minimal Dominating Functions

The *aggregate* of a DF f of G is defined to be $\sum_{u \in V} f(u)$. The work of this section was motivated by the following interpolation question: Given MDFs f, g of G with aggregates t_1, t_2 respectively and any t satisfying $t_1 < t < t_2$, does there exist an MDF of G with aggregate t? A. Majumdar [19] noticed that in some cases convex combinations are suitable.

Suppose that f and g are DFs of G. For $\lambda \in (0, 1)$, define $h_\lambda : V \to [0, 1]$ by

$$h_\lambda(v) = \lambda f(v) + (1 - \lambda)g(v) \text{ for each } v \in V. \tag{5}$$

Using (5) it is elementary to show that h_λ is a DF of G and that if f, g have aggregates t_1, t_2 respectively and $t_1 < t < t_2$, by a suitable choice of

$\lambda \in (0,1)$, h_λ has aggregate t. However, if f, g are MDFs, h_λ is not always minimal. We are thus led to study the relation $\mathcal{R}$ on the set $\mathcal{F}$ of MDFs of G, defined by: $f\mathcal{R}g$ if and only if h_λ is an MDF for all $\lambda \in (0,1)$. The remainder of the section begins this investigation.

Theorem 3. *For MDFs f, g of G, $f\,\mathcal{R}g$ if and only if $B_f \cap B_g \succ P_f \cup P_g$.*

Proof: We prove that $B_{h_\lambda} = B_f \cap B_g$ and $P_{h_\lambda} = P_f \cup P_g$. The result is then immediate from Proposition 1. If $v \notin P_f \cup P_g$, then $f(v) = g(v) = h_\lambda(v) = 0$. If, say, $v \in P_f$, then $h_\lambda(v) \geq \lambda f(v) > 0$. Thus $P_{h_\lambda} = P_f \cup P_g$.

Suppose $v \in B_f \cap B_g$. Then

$$\begin{aligned} h_\lambda[v] &= \lambda f[v] + (1-\lambda)g[v] \\ &= \lambda + (1-\lambda) = 1. \end{aligned}$$

A similar calculation shows $h_\lambda[v] > 1$ for $v \notin B_f \cap B_g$ and hence $B_{h_\lambda} = B_f \cap B_g$. □

We now consider the question of existence of MDFs which relate in $\mathcal{R}$ to every other MDF. The MDF g of G is called a *universal* MDF if for all $f \in \mathcal{F}$ and all $\lambda \in (0,1)$, $h_\lambda \in \mathcal{F}$. The following proposition enables us to obtain classes of graphs which have universal MDFs.

Proposition 4. *If the MDF g satisfies $B_g = V$ and for all $f \in \mathcal{F}$, $B_f \succ V$, then g is a universal MDF.*

Proof: For $f \in \mathcal{F}$, $B_g \cap B_f = B_f$ which dominates $V \supseteq P_f \cup P_g$. The result follows from Theorem 3. □

Theorem 5. *The path P_n $(n \geq 1)$, the cycle $C_n (n \geq 3)$, the complete bipartite graph $K_{m,n}$ $(m, n \geq 1)$, the n-vertex wheel W_n $(n \geq 4)$ and the complete graph K_n $(n \geq 5)$ all have universal MDFs.*

Proof: It is easy to verify that for any n, the path P_n has an MDF g with $B_g = V$ by assigning to consecutive vertices in the path, suitable consecutive elements from the sequence $100100100\cdots$ and it remains to prove that for all $f \in \mathcal{F}$, $B_f \succ V$. Suppose this is false and for $f \in \mathcal{F}$, $B_f \not\succ v$ where $v \in V$. It is obvious that an end-vertex of a tree T is in the boundary of any MDF of T. Therefore end-vertices and their neighbours are dominated by the boundary of any MDF. It follows that P_n contains a subpath with vertex sequence v_2, v_1, v, v_3, v_4. Since $B_f \succ P_f$, $f(v) = 0$ and by the dominating property, say, $f(v_1) > 0$. Vertex $v_1 \notin B_f$ (since v is undominated), hence $f(v_1) + f(v_2) > 1$. But $B_f \succ v_1$ $(\in P_f)$ and so $v_2 \in B_f$, which implies the contradiction $f(v_1) + f(v_2) \leq 1$. Therefore g is a universal MDF by Proposition 4.

The function which assigns $\frac{1}{3}$ to each vertex is an MDF g of C_n satisfying $B_g = V$. A similar argument to that used for paths shows for all $f \in \mathcal{F}$, $B_f \succ V$ and hence g is a universal MDF.

Suppose that $K_{m,n}$ has defining independent sets $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_n\}$. Define the MDF g by $g(a_i) = (n-1)/(mn-1)$ for $i = 1, \ldots, m$ and $g(b_j) = (m-1)/(mn-1)$ for $j = 1, \ldots, n$. This function satisfies $B_g = V$. Without losing generality suppose $f \in \mathcal{F}$ is such that $B_f \not\succ b_1$. Then $f(b_1) = 0$ and $f[b_1] = \sum_{i=1}^{m} f(a_i) > 1$. Moreover, no $a_i \in B_f$ and hence some $b_j \in B_f$. This implies $\sum_{i=1}^{m} f(a_i) \leq 1$, a contradiction. Therefore $B_f \succ V$ and g is a universal MDF by Proposition 4.

The proofs for W_n and K_n are omitted. □

For $S \subseteq V$, let f_S be defined by $f_S(u) = 1$ if $u \in S$ and $f_S(u) = 0$ otherwise.

Proposition 6. *Let g be an MDF such that $B_g \not\succ v$ and let S be any minimal dominating set containing v. Then $g \not\mathcal{R} f_S$.*

Proof: Since $v \in S$, $f_S(v) = 1$, *i.e.* $v \in P_{f_S} \subseteq P_{f_S} \cup P_g$, $B_g \cap B_{f_S} \subseteq B_g \not\succ v$. Therefore , by Theorem 3, $g \not\mathcal{R} f_S$. □

Corollary. *If g is a universal MDF, then B_g dominates V.*

The next result enables us to demonstrate the existence of universal MDFs whose boundaries do not contain all vertices (as required by Proposition 4) and graphs, all of whose MDFs are universal, *i.e.* $\mathcal{R} = \mathcal{F} \times \mathcal{F}$.

Let u be a vertex of graph H. By a *complete addition to H at u* we mean the identification of u and a vertex of some complete graph with at least two vertices.

Proposition 7. *Let H be any graph. Form G from H by making one or more complete additions to H at u, for each vertex u of H. Then each MDF of G is universal.*

Proof: If $a_1 \in V(G)-V(H)$, then $N[a_1] = \{a_1, a_2, \ldots, a_n, u\}$ where $u \in V(H)$ and $n \geq 1$. Suppose $f \in \mathcal{F}$ and $a_1 \notin B_f$. Then $\sum_{v \in N[a_1]} f(v) = \sum_{i=1}^{n} f(a_i) + f(u) > 1$ and hence for some i, $f(a_i) > 0$. But $N[a_j] = N[a_1]$ for each $j \in 1, \ldots, n$ and so $a_j \notin B_f$. Further, $\sum_{v \in N[u]} f(v) \geq \sum_{v \in N[a_1]} f(v) > 1$ and so $u \notin B_f$. Therefore $B_f \not\succ a_i$ which implies (using Proposition 1) that f is not minimal. We conclude $V(G) - V(H) \subseteq B_f$. Thus, for any pair f, $g \in \mathcal{F}$, $B_f \cap B_g$ contains $V(G) - V(H)$ which is a dominating set of G and hence $B_f \cap B_g \succ P_f \cup P_g$. By Theorem 3, $f \mathcal{R} g$, therefore each $f \in \mathcal{F}$ is universal. □

Corollary. *Any MDF of a tree whose end-vertices form a dominating set, is universal.*

The final result concerns non-existence of universal MDFs.

Proposition 8. *Let G be a graph which contains a vertex v such that for every $u \in N[v]$ there exists an MDF f_u such that $B_{f_u} \not\succ u$. Then G does not have a universal MDF.*

Proof: Suppose g is a universal MDF of G and let $v \in V(G)$ satisfy the hypothesis above. $B_g \cap B_{f_v} \not\succ v$ and hence $g(v) = 0$ (for otherwise $v \in P_g \cup P_{f_v}$). Similarly, for each $u \in N[v]$, $B_h \cap B_{f_u} \not\succ u$ and hence $g(u) = 0$. But then

$$\sum_{u \in N[v]} g(u) = 0 ,$$

so that g is not a DF, a contradiction. □

Corollary. *If G is vertex-transitive, then G has a universal iff for every MDF f of G, $B_f \succ V$.*

Proof: Let G be vertex-transitive; suppose G is r-regular. If $B_f \succ V(G)$ for every MDF f, then $g = \frac{1}{r+1}$ is universal, for $B_g = V$, hence $B_g \cap B_f \succ V(G) = P_g \cup P_f$ (Theorem 3).

Conversely, let f be an MDF of G such that $B_f \not\succ V(G)$; say $B_f \not\succ v$. Since G is vertex-transitive, there exists, for every $u \in V(G)$ and in particular for every $u \in N[v]$, an MDF f_u of G such that $B_{f_u} \not\succ u$. By Proposition 8, G does not have a universal MDF. □

Let G be the circulant formed by adding edges $\{i, i+5\}$ for $i = 1, \dots, 5$ to the cycle with vertex sequence $1, \dots, 10$. Then for example the function f which is 1 on $\{1, 3, 6, 8\}$ and 0 elsewhere is an MDF with $B_f = \{4, 5, 9, 10\}$ which does not dominate V. By the corollary G has no universal MDF.

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