Symmetric and Skew Equivalence of Hadamard Matrices of Order 28

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ABSTRACT. In this paper, we consider symmetric and skew equivalence of Hadamard matrices of order 28 and present some computational results and some applications.

1 Introduction

An *Hadamard matrix* of order n is an n by n matrix H with all elements in $\{1, -1\}$, satisfying $HH^T = nI$. It is well-known that if there is an Hadamard matrix of order n, then $n = 1$ or $n = 2$ or n is a multiple of 4.

Two Hadamard matrices are called *Hadamard equivalent* (or simply H-equivalent) if one can be obtained from the other by a sequences of row and column permutations and negations.

If H is an Hadamard matrix of order n, then $HH^T = nI$ implies that H is a nonsingular and has an inverse $n^{-1}H^T$, whence $H^TH = nI$, so H^T is also an Hadamard matrix. However it is not necessary for H^T to be Hadamard equivalent to H.

An Hadamard matrix H is *symmetric* if $H^T = H$, and *skew* if $H = S - I$ where $S^T = -S$. We refer to this as a skew Hadamard matrix of type I. Some authors use $H = S + I$, which we shall call a skew Hadamard matrix of type II. (Clearly, type II can be obtained by negating type I, and conversely.)

In [6], the following problem was considered: is a given Hada- mard matrix equivalent to a symmetric or skew Hadamard matrix? We say such an

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Hadamard matrix is *symmetric equivalent* or *skew equivalent* respectively. This is often useful information because of the importance of symmetric and skew Hadamard matrices in the construction of Hadamard matrices of larger orders, of other designs, and of Hadamard tournaments (see, for example, [8] and [1]). Also, symmetric and skew matrices require about half the storage space of arbitrary Hadamard matrices.

In this paper, we consider symmetric and skew equivalence of Hadamard matrices of order 28 and present some computational results and some applications in the construction of weighing matrices and Hadamard tournament matrices. Finally we consider the conjectures, presented in [6], on symmetric and skew equivalence of Hadamard matrices.

2 Symmetric, skew and transpose equivalence

Usually, we consider the equivalence classes of Hadamard matrices under Hadamard equivalence. A class is called symmetric if its members are symmetric-equivalent. This implies that the class contains a member which is symmetric. Skew classes are defined analogously.

The *transpose class* of an equivalence class is the set of transposes of members of the class. Clearly this will be an equivalence class; if it equals the original class, then the class is called *transpose equivalent*. Symmetric classes are transpose-equivalent, but the converse is not obviously true (and in fact it is sometimes false). H^T is a symmetric equivalent Hadamard matrix if and only if H is a symmetric equivalent Hadamard matrix, and H^T is a skew equivalent Hadamard matrix if and only if H is a skew equivalent Hadamard matrix. Thus we do not need to determine symmetric and skew equivalence of both a class and its transpose class.

In [7] Longyear gave three criteria to determine skew equivalence of a given Hadamard matrix and computed the skew equivalence of Hadamard matrices of order 16 and 20. In [6], we presented two theorems and corresponding algorithms which improve the results in [7] on skew equivalence and presented new result on symmetric equivalence, including computational results on symmetric and skew equivalence of Hadamard matrices of orders 16, 20 and 24.

3 Hadamard matrices of order 28

Hall sets (see, for example, [3]) have been used to construct Hadamard matrices of order 28, and 487 equivalence classes of Hadamard matrices of order 28 have been presented in [2], [3], [4] and [5]. This is currently best known partial result on constructing and classifying the equivalence classes of Hadamard matrices of order 28. It is known that the Paley type Hadamard matrix of order 28 does not contain Hall sets, and it was conjectured by H. Kimura that this is the only one containing no Hall set. If this conjecture is true, then the classification of equivalence classes of Hadamard matrices of order 28 will be complete. It is known [6] that the Paley type Hadamard matrices of order $(p^r + 1)$ with p prime are both symmetric and skew equivalent. The 487 equivalence classes of Hadamard matrices of order 28 with Hall sets plus the Paley type Hadamard matrix of order 28 provide a larger set of data on symmetric and skew equivalence of Hadamard matrices than those of orders 2, 4, 8, 12, 16, 20 and 24, so it is interesting and useful to investigate symmetric and skew equivalence of Hadamard matrices of order 28.

We carried out a computer search on a PC with 386 processor and 387 coprocessor running at 25 MHz. It took from about 1 minute to about 8 minutes to test an Hadamard matrix of order 28 for skew equivalence. It took from about 3 minutes to about 60 minutes to test an Hadamard matrix of order 28 for symmetric equivalence. We will follow the notation for equivalence classes of Hadamard matrices of order 28 in [2], [3], [4] and [5], so that $H001$ through H476 are from [4] and [5]. H00A through H00C are from [2] and H00I through $H0VI$ are from [3] where $H00A^T \neq H00A$, $H00B^T = H00B$, $H00C^T \neq H00C$, $H0VI = H0II^T$ and $H0IV = HIII^T$. We denote the Paley type Hadamard matrix of order 28 by $H00P$. The results of computation are as follows.

Skew equivalence classes of order 28:
{H002, H028, H042, H043, H113, H177, H178, H179, H180, H181, H182, H201, H204, H205, H224, H225, H265, H266, H303, H304, H311, H312, H326, H327, H328, H329, H355, H356, H359, H371, H372, H391, H392, H396, H397, H401, H405, H409, H410, H412, H419, H440, H441, H444, H452, H454, H458, H461, H466, H471, H0II, H00V, H0VI, H00P}.

Symmetric equivalence classes of order 28:
{H001, H002, H010, H023, H024, H027, H028, H033, H050, H061, H062, H077, H096, H113, H114, H137, H146, H149, H150, H153, H154, H157, H158, H159, H201, H206, H207, H208, H209, H210, H213, H220, H221, H226, H251, H260, H287, H300, H301, H302, H305, H306, H316, H317, H320, H321, H352, H353, H359, H368, H375, H388, H393, H401, H404, H405, H406, H411, H412, H413, H418, H419, H421, H422, H423, H435, H436, H437, H440, H441, H444, H445, H450, H451, H452, H454, H457, H458, H459, H460, H461, H462, H463, H466, H470, H471, H476, H00B, H00V, H00P}.

Therefore equivalence classes of Hadamard matrices of order 28 from [2], [3], [4] and [5] can be classified as four sets as follows.

Both symmetric and skew equivalence classes:
{H002, H028, H113, H201, H359, H401, H405, H412, H419, H440, H441, H444, H452, H454, H458, H461, H466, H471, H00V, H00P};

Symmetric but not skew equivalence classes:
{H001, H010, H023, H024, H027, H033, H050, H061, H062, H077, H096, H114, H137, H146, H149, H150, H153, H154, H157, H158, H159, H206, H207, H208, H209, H210, H213, H220, H221, H226, H251, H260, H287, H300, H301, H302, H305, H306, H316, H317, H320, H321, H352, H353, H368, H375, H388, H393, H404, H406, H411, H413, H418, H421, H422, H423, H435, H436, H437, H445, H450, H451, H457, H459, H460, H462, H463, H470, H476, H00B};

Skew but not symmetric equivalence classes:
{H042, H043, H177, H178, H179, H180, H181, H182, H204, H205, H224, H225, H265, H266, H303, H304, H311, H312, H326, H327, H328, H329, H355, H356, H371, H372, H391, H392, H396, H397, H409, H410, H0II, H0VI};

Neither symmetric nor skew equivalence classes:
{H003, H004, H005, H006, H007, H008, H009, H011, H012, H013, H014, H015, H016, H017, H018, H019, H020, H021, H022, H025, H026, H029, H030, H031, H032, H034, H035, H036, H037, H038, H039, H040, H041, H044, H045, H046, H047, H048, H049, H051, H052, H053, H054, H055, H056, H057, H058, H059, H060, H063, H064, H065, H066, H067, H068, H069, H070, H071, H072, H073, H074, H075, H076, H078, H079, H080, H081, H082, H083, H084, H085, H086, H087, H088, H089, H090, H091, H092, H093, H094, H095, H097, H098, H099, H100, H101, H102, H103, H104, H105, H106, H107, H108, H109, H110, H111, H112, H115, H116, H117, H118, H119, H120, H121, H122, H123, H124, H125, H126, H127, H128, H129, H130, H131, H132, H133, H134, H135, H136, H138, H139, H140, H141, H142, H143, H144, H145, H147, H148, H151, H152, H155, H156, H160, H161, H162, H163, H164, H165, H166, H167, H168, H169, H170, H171, H172, H173, H174, H175, H176, H183, H184, H185, H186, H187, H188, H189, H190, H191, H192, H193, H194, H195, H196, H197, H198, H199, H200, H202, H203, H211, H212, H214, H215, H216, H217, H218, H219, H222, H223, H227, H228, H229, H230, H231, H232, H233, H234, H235, H236, H237, H238, H239, H240, H241, H242, H243, H244, H245, H246, H247, H248, H249, H250, H252, H253, H254, H255, H256, H257, H258, H259, H261, H262, H263, H264, H267, H268, H269, H270, H271, H272, H273, H274, H275, H276, H277, H278, H279, H280, H281, H282, H283, H284, H285, H286, H288, H289, H290, H291, H292, H293, H294, H295, H296, H297, H298, H299, H307, H308, H309, H310, H313, H314, H315, H318, H319, H322, H323, H324, H325, H330, H331, H332, H333, H334, H335, H336, H337, H338, H339, H340, H341, H342, H343, H344, H345, H346, H347, H348, H349, H350, H351, H354, H357, H358, H360, H361, H362, H363, H364, H365, H366, H367, H369, H370, H373, H374, H376, H377, H378, H379, H380, H381, H382, H383, H384, H385, H386, H387, H389, H390, H394, H395, H398, H399, H400, H402, H403,

H407, H408, H414, H415, H416, H417, H420, H424, H425, H426, H427, H428, H429, H430, H431, H432, H433, H434, H438, H439, H442, H443, H446, H447, H448, H449, H453, H455, H456, H464, H465, H467, H468, H469, H472, H473, H474, H475, H00A, H00A^T, H00C, H00C^T, H001, HIII, H0IV}.

Obviously, the matrices previously listed as being symmetric and transpose-equivalent.

The following are transpose-equivalent but not symmetric:
{H007, H162, H301, H313, H354, H398, H414, H415, H420, H426, H453, H469, H00I}.

We have conjectured [6] that for every order (except order 16) there is an Hadamard matrix which is both symmetric and skew equivalent. This was shown in [6] to be true for orders up to 24. It was also known [6] that the Paley matrix has this property, but we have now shown it for several other matrices of order 28. It was also conjectured [6] that, for large n, the majority of equivalence classes are neither symmetric nor skew. This is true for orders 24 and 28.

4 Some results using symmetric and skew equivalence

¿From [8], we know various ways in which we can use symmetric and skew Hadamard matrices to construct symmetric and skew Hadamard matrices of larger orders.

A weighing matrix $W(n, n-1)$ of order n is an n by n matrix with all elements in {-1,0,1}, satisfying $WW^T = (n-1)I$. It is easy to show that there exists a weighing matrix $W(n, n-1)$ if there is a skew Hadamard matrix of order n by taking the matrix S from the skew Hadamard matrix $H = S - I$ of order n where $S^T = -S$. Therefore, there is exactly one class of H-equivalent weighing matrices $W(2,1)$, $W(4,3)$, $W(8,7)$, and $W(12,11)$, respectively. There are exactly 2 classes of H-equivalent weighing matrices $W(16,15)$ and $(20,19)$, respectively. There are exactly 16 classes of H-equivalent weighing matrices $W(24,23)$. There are at least 56 classes of H-equivalent weighing matrices $W(28,27)$.

In [1] Ito considers the Hadamard tournament matrix A which is a (0,1) matrix of order $4\lambda+3$ such that

$$AA^T = (\lambda+1)I + \lambda J \quad \textit{and} \quad A + A^T + I = J$$

where J is the matrix of order $4\lambda+3$ with all elements +1. An Hadamard matrix is called normalized if every element in its first row and column is +1. An Hadamard matrix is called skew-normalized if every element in its first column is -1 and every element but the first in its first row is +1. Obviously, every Hadamard matrix is equivalent to one which is normalized and skew normalized. Thus if H is a skew-normalized skew Hadamard matrix $H = S - I$ of order $n = 4t$ where $S^T = -S$, then we remove the first row and first column of S, change the elements from -1 to 1 and from 1 to 0 and get a (0,1) matrix B of order $4t - 1$. It is easy to verify that B is an Hadamard tournament matrix of order $4t-1$. Therefore, there is exactly one class of H-equivalent Hadamard tournament matrix of orders 2,4,8 and 12 respectively. There are exactly 2 classes of H-equivalent Hadamard tournament matrices of orders 16 and 20, respectively. There are exactly 16 classes of H-equivalent Hadamard tournament matrices of order 24. There are at least 56 classes of H-equivalent Hadamard tournament matrices of order 28.

5 Remark

The paper [6] contains an incomplete proof. For the purpose of completeness, here we restate two theorems in [6] and give a complete proof for Theorem 1, the proof for Theorem 2 is similar.

In the following two theorems, A is the set of permutation matrices and U is the set of diagonal monomial matrices.

Theorem 1: A semi-normalized Hadamard matrix H is normalized symmetric equivalent if and only if there exists a normalized symmetric Hadamard matrix Q such that $PHN = Q$, where $P \in A$, $N \in U$ and $N(1,1) = 1$.

Proof: The sufficiency is obvious, so we only prove the necessity. Assume there is normalized Hadamard matrix K equivalent to H. It suffices to prove there is a normalized symmetric matrix $Q = PN_1HN$ where $P \in A$ and N_1, $N \in U$. Without loss of generality, $N(1,1) = 1$ (otherwise negate both N_1 and N); since H is semi-normalized, HN has first column all 1's, so $N_1 = I$ and we have the result.

To construct Q_1, observe that $K = P_2N_2HN_3P_3$ for some $P_2, P_3 \in A$ and N_2, $N_3 \in U$. So $Q_1 = P_3KP_3^T = P_3P_2N_2HN_3$ is a symmetric Hadamard matrix.

If $Q_1(1,1) = 1$, find N_4 in U such that $Q = N_4Q_1N_4$ is normalized. Then $Q = (N_4P_3P_2)\,N_2HN_3N_4 = (P^*N^*)\,N_2HN_3N_4$ for some $P^* \in A$ and $N^* \in U$, by Lemma 4 in [6]; putting $P = P^*$, $N_1 = N^*N_2$, $N = N_3N_4$ we have the required form. If $Q_1(1,1) = -1$, we proceed similarly after finding N_4 such that $Q = N_4\,(-Q_1)\,N_4$ is normalized.

Theorem 2: A semi-skew-normalized Hadamard matrix H is skew-normalized skew equivalent if and only if there exists a skew-normalized skew Hadamard matrix $Q = S - I$ with $S^T = -S$ such that $PHN = Q$, where $P \in A$, $N \in U$ and $N(1,1) = 1$.

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