Multiexponents of Tournament Matrices

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Abstract. In this paper we obtain some new relations on generalized exponents of primitive matrices. Hence the multiexponent of primitive tournament matrices are evaluated.

1. Introduction

The directed graph Γ defined by a $(0,1)$ matrix M consists of n vertices $1, 2, \ldots, n$ such that an arc $\vec{ij}$ goes from i to j if and only if the (i,j) entry of M is one. Let X, Y be two vertex sets. X is called a dominating set for Y if for every vertex j of Y there is a vertex i in X such that $\vec{ij}$. For some Y, its domonated set with minimum cardinality is called the minimum dominated set of Y.

A tournament T is a directed graph Γ_n such that each pair of distinct vertices i and j is joined by exactly one of the arcs $\vec{ij}$ or $\vec{ji}$ and no vertex is joined to itself by an arc. For convenience we also denote a matrix which defines a tournament by T_n.

A $(0,1)$ matrix M_n is primitive if there exists some positive integer k such that $M_n^k > 0$. The least k is called the exponent of M, denoted by $\exp(M_n)$. It is well known (see [1]) that a tournament T_n is primitive if and only if $n \geq 4$ and T_n is irreducible (i.e. strongly connected). For any tournament matrix T_n, it is known ([1]) that $\exp(T_n) \leq n+2$.

In [2], we introduced some new parameters related to the exponent as follows.

Let M be a primitive $n \times n$ matrix and k be an integer with $1 \leq k \leq n$. Then $\exp_M(k)$ is the smallest power of M for which there are k rows with no zero entry. The smallest power of M for which there are k rows having no column of all zeros is called the kth M for which no set of k rows has a column of all zeros is called the kth upper multiexponent of M, denoted by $F(M,k)$.

We let

$$exp(n,k) := \mathrm{MAX}_M(\exp_M(k))$$
$$f(n,k) := \mathrm{MAX}_M(f(M,k))$$
$$F(n,k) := \mathrm{MAX}_M(F(M,k))$$

where the maximum is taken over all primitive $n \times n$ matrices.

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For all primitive tournaments, corresponding multiexponents are denoted by $\exp^T(n,k)$, $f^T(n,k)$, $F^T(n,k)$ respectively. Clearly

$$\exp^T(n,n) = n+2 \qquad \text{(see [1])}$$

In [3] we have obtained

$$\exp^T(n,k) = k+2, n>6 \tag{1.1}$$

In this paper, we obtain some new relations on $f(n,k)$ and $F(n,k)$ for primitive matrices and derive $f^T(n,k)$ and $F^T(n,k)$.

2. The Relations Between $f(n,k)$ and $F(n,k)$

First of all, we give a derivation of multiexponents as follows.

Let k be an integer with $1 \leq k \leq n$ and let X be a set of k vertices of the primitive digraph Γ. We define the exponent of the set X to be $\exp_\Gamma(X)$:= the smallest integer p such that for each vertex i of Γ there exists a walk from at least one vertex in X to i of length p (and hence of each length greater than p).

Then

$$f(\Gamma, k) := \mathrm{MIN}_x\{\exp_\Gamma(X)\}$$

where the minimum is taken over all subsets X of k vertices of Γ, and

$$F(r, k) := \mathrm{MAX}_x\{\exp_\Gamma(X)\}$$

where the maximum is taken over all subset X of k vertices of Γ.

Let M be a matrix that defines a digraph Γ. Clearly,

$$f(M,k) = f(\Gamma,k)$$
$$F(M,k) = F(\Gamma,k)$$

Thus

$$f(n,k) = \mathrm{MAX}_M(f(M,k)) = \mathrm{MAX}_\Gamma(f(\Gamma,k))$$
$$F(n,k) = \mathrm{MAX}_M(f(M,k)) = \mathrm{MAX}_\Gamma(F(M,k))$$

where the maximum is taken over all primitive digraphs Γ with n vertices.

According to the definitions we establish the following lemmas easily.

Lemma 1. $f(n,1) = \exp(n,1)$.

Lemma 2. $f(n,1) \geq f(n,2) \geq \cdots \geq f(n,n) = 0$.

Lemma 3. $F(n,1) \geq F(n,2) \geq \cdots \geq F(n,n) = 0$.

Theorem 4. $F(n,k) \geq f(n,x)$ *where k and x satisfy*

$$\frac{1}{n}\binom{n}{x} > \binom{k-1}{x}. \tag{2.1}$$

Proof: Suppose that A is an $n \times n$ primitive matrix and that $F(A, k) = r$. Then A^r has no set of k rows which has a column of all zeros. In particular, the number of zeros in each column of A^r is less than or equal to $k-1$.

We now prove that there is a set of x rows to have no column of all zeros. If not, each set of x rows has a column of all zeros. Then the zero-columns appear $\binom{n}{x}$ times at least. There is a column to appear $\frac{1}{n}\binom{n}{x}$ times at least. Hence there is a column, the jth say, such that there are at least $\frac{1}{n}\binom{n}{x}$ sets of x rows of A^r having column j as an all zero column. But column j has at most $k-1$ zeros in it, so there are at most $\binom{k-1}{x}$ sets of x rows having column j as an all zero column. Thus

$$\frac{1}{n}\binom{n}{x} \leq \binom{k-1}{x}$$

contrary to the hypothesis. Hence there is a set of x rows to have no column of all zero. By the definitions of $F(n, k)$ and $f(n, k)$,

$$F(n, k) \geq f(n, x).$$

∎

It is difficult to obtain the explicit solution of (2.1). But we know that when $x \geq k-1$, except $k = n$ and $x = n-1$, $\frac{1}{n}\binom{n}{x} > \binom{k-1}{x}$. Thus we have

Corollary 4.1. *$F(n, k) \geq f(n, x)$ if $x \geq k-1$ except $k = n$ and $x = n-1$.*

Lemma 5. $F(n, k) \leq \exp(n, n-k+1)$.

Proof: Let $\exp_A(n-k+1) = e$. By the definition of $\exp_A(n-k+1)$, the matrix A^e has $n-k+1$ rows of all ones,the set of which is denoted by R_{n-k+1}. Take any k rows from A^e, denoted by R_k; then there is at least one row $r \in R_{n-k+1}$. Thus R_k has no column of all zeros. By the definitions,

$$F(A, k) \leq \exp_A(n-k+1)$$

Thus

$$F(n, k) \leq \exp(n, n-k+1)$$

∎

Lemma 6. ([2]) $\exp(n, k) = n^2 - 3n + k + 2$.

Here we obtain a bound on $F(n, k)$ depending only on k and n as follows.

Lemma 7. $F(n, k) \leq n^2 - 2n - k + 3$.

Proof: By Lemmas 5,6.

$$\begin{aligned} F(n, k) &\leq \exp(n, n-k+1) \\ &= n^2 - 3n + (n-k+1) + 2 \\ &= n^2 - 2n - k + 3. \end{aligned}$$

∎

Note that there is a digraph Γ such that $F(\Gamma, 1) = n^2 - 2n + 2$. (see [2]) From Theorem 7, we have

Corollary 7.1. ([2]) $F(n, 1) = n^2 - 2n + 2$.

There may be some room for improvement in the bound of Theorem 7, since in reference [2] it is shown that $F(n, n-1) = n$.

3. $f^T(n, k)$ for Primitive Tournament

It is easy to see that Lemma 1,2 also hold for primitive tournaments.

According to Lemmas 1,2 and formula (1.1), we have

Lemma 8.

$$f^T(n, k) \le f^T(n, 1) = \exp^T(n, 1) = 3, \quad n > 6.$$

Let $T_n = (V, E)$ be a primitive tournament on n vertices ($n > 6$) whose set of vertices is V and whose set of arcs is E. A vertex K is called a king if for every vertex $i \ne K$ in T_n, there is a path from K to i of length 1 or 2. We denote by $N_i(v)$ the set of vertices of T_n that can be reached by a path of length i that begins at v. Then v is a king if and only if $N_1(v) \cup N_2(v) \cup \{v\} = V$.

It is well known (see [4], Theorem 4.6) that a tournament has at least one king. If a tournament T_n has exactly one king then T_n is not strongly connected. Thus we have

Lemma 9. *If T_n is a irreducible tournament, then T has at least two kings.*

Lemma 10. $f^T(n, 2) = 2, n \ge 4$.

Proof: By Lemma 9, let K_1, K_2 be two kings of T_n and V be the vertex set of T_n. Without loss of generality, suppose $\vec{K_2 K_1}$. Then $K_2 \notin N_1(K_1)$. Since K_1 is a king of T_n, $K_2 \in N_2(K_1)$, i.e., $\exists x \in V' = V \backslash \{K_1, K_2\}$ such that $\vec{K_1 x}$ and $\vec{x K_2}$. V' can be partitioned into four sets, V_1, V_2, V_3, V_4, where

$$V_1 = \{v \mid v \in V' \text{ and } \vec{K_1 v}, \vec{v K_2}\}$$
$$V_2 = \{v \mid v \in V' \text{ and } \vec{v K_1}, \vec{v K_2}\}$$
$$V_3 = \{v \mid v \in V' \text{ and } \vec{K_1 v}, \vec{K_2 v}\}$$
$$V_4 = \{v \mid v \in V' \text{ and } \vec{v K_1}, \vec{K_2 v}\}$$

Since

$$x \in V_1, V_1 \ne \phi. \tag{3.1}$$

Case 1. $V_3 \cup V_4 = \phi$. In this case, $V_2 = \phi$. If not, suppose that there is $x' \in V_2$ such that $\vec{x' K_1}, \vec{x' K_2}$. Then there is no path of length 1 or 2 from K_2 to x'. This contradicts the fact that K_2 is a king.

Since $n \ge 4$, $\exists x_1, x_2 \in V_1$, without loss of generality, suppose $\vec{x_1 x_2}$. Then take $X = \{x_1, K_2\}$,

$$N_2(X) = N_2(x_1) \cup N_2(K_2) = \{K_1, K_2\} \cup V' = V.$$

Case 2. $V_3 \cup V_4 \neq \phi$.

Subcase 2.1. $V_4 \neq \phi$. Then $K_1 \in N_2(K_2)$. We take $X = \{K_1, K_2\}$,

$$N_2(X) = N_2(K_1) \cup N_2(K_2) = N_2(K_1) \cup \{K_1\} \cup N_1(K_1) = V. \text{ (since } \overrightarrow{K_2 K_1})$$

Subcase 2.2. $V_4 = \phi$. Then $V_3 \neq \phi$.

Subcase 2.2.1. $V_2 \neq \phi$. Then for any $y \in V_2$, $\overrightarrow{yK_1}, \overrightarrow{yK_2}$. Since K_2 is a king, $\exists x' \in V_3$ such that $\overrightarrow{x'y}$. Thus we take $X = \{x', K_2\}$.

$$N_2(X) = N_2(x') \cup N_2(K_2) = \{K_1, K_2\} \cup V' = V.$$

Subcase2.2.2. $V_2 = \phi$. By (3.1), $V_1 \neq \phi$. If $|V_1| > 1$, then $\exists x_1, x_2 \in V_1$, $\overrightarrow{K_1 x_1}, \overrightarrow{x_1 K_2}, \overrightarrow{K_1, x_2}, \overrightarrow{x_2 K_2}$. Without loss of generality, suppose $\overrightarrow{x_1 x_2}$. We take $X = \{K_2, x_1\}$, then

$$N_2(X) = N_2(X_1) \cup N_2(K_2) = \{K_1, K_2\} \cup V' = V.$$

If $|V_1| = 1$, by (3.1), $V_1 = \{x\}$, $\overrightarrow{K_1 x}, \overrightarrow{xK_2}$.
Since T_n is irreducible, this is an $x' \in V_3$ such that $\overrightarrow{x'x}$. We take $X = \{x, K_1\}$,

$$N_2(X) = N_2(K_1) \cup N_2(x) = \{K_2, x\} \cup V_3 \cup \{K_1\} = V.$$

Hence $\exp_{T_n}(X) = 2$.

$$f(n, 2) \leq 2.$$

We can show a primitive tournament as follows.

$$T'_n = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & 1 & 1 & \cdots & \cdots & 1 \\ 0 & 0 & 0 & \ddots & 1 & \cdots & 1 \\ 0 & \cdots & 0 & \ddots & \ddots & & \cdots \\ 0 & \cdots & & & & 0 & 1 \\ 1 & 0 & & \cdots & & & \end{bmatrix} \begin{matrix} 1 \\ 2 \\ \vdots \\ \vdots \\ \vdots \\ n \end{matrix}$$

It is easy to verify that $f(T'_n, 2) = 2$. Hence $f(n, 2) = 2$. ∎

By Lemmas 2,8,10, we have

Lemma 11.

$$f^T(n, k) \begin{cases} 3, & k = 1 \\ .2, & k = 2 \\ 1 \text{ or } 2, & 3 \leq k \leq n \end{cases}.$$

Lemma 11 shows that if a primitive tournament matrix M has the property that every set of k rows ($k \geq 3$) has a column of zeros then M^2 has k rows having no column of all zeros.

We have the following further result.

Lemma 12. *Let $T_n = (V, E)$ be a primitive tournament and let r be the maximal outdegree in T_n (the maximal row sum of matrix T_n). Then*

$$f(T_n, k) = 1 \text{ whenever } k \geq \left\lfloor \frac{n-r}{2} \right\rfloor + 2.$$

Proof: Let v be the vertex with maximal outdegree r in T_n, i.e. $d^+(v) = r$. We have known (see [4] Theorem 4.6) that v is a king of T_n. Let $V_1 = V \backslash (\{v\} \cup N_1(v))$. Then $|V_1| = n - 1 - r$. Let T_1 be the subtournament with vertex set V_1 and let v_1 be a vertex with the maximal outdegree r_1 in T_1, i.e. $d^+_{T_1}(v_1) = r_1$. Let $N^+_{T_1}(v_1) = u \mid v_1 \vec{u}, u \in T_1\}$

Clearly

$$r_1 \geq \binom{|V_1|}{2} / |V_1| = \binom{n-1-r}{2} / (n-1-r) = \frac{1}{2}(n-r-2) \tag{3.2}$$

(exactly $r_1 \geq \lceil \frac{1}{2}(n-r-2) \rceil$).

Since T_n is a strongly connected graph, there is a minimum dominated set V' of $V_1 \backslash N^+_{T_1}(v_1)$, $|V'| \leq |V_1| - r_1 = n - r - 1 - r_1$.

We take $X = \{v\} \cup \{v_1\} \cup V'$. Then

$$N(X) = N_1(v) \cup \{v\} \cup V_1 = V \tag{3.3}$$

$$\begin{aligned} |X| &\leq 1 + 1 + n - r - 1 - r_1 \\ &= n + 1 - r - r_1 \\ &\leq n + 1 - r - \frac{1}{2}(n - r - 2) \text{ (by (3.2))} \\ &= \frac{n-r}{2} + 2 . \end{aligned} \tag{3.4}$$

By Lemma 2 and (3.3),

$$f(T_n, k) = 1 \quad \text{whenever } k \geq \lfloor \frac{n-r}{2} \rfloor + 2 .$$

∎

Note that for any primitive tournament T_n, $r \geq \binom{n}{2}/n = \frac{1}{2}(n-1)$ (exactly $r \geq \lceil \frac{1}{2}(n-1) \rceil$). According to (3.4), we have

$$\begin{aligned} |X| &\leq \frac{1}{2}(n-r) + 2 \\ &\leq \frac{1}{2}\left(n - \frac{1}{2}(n-1)\right) + 2 \\ &= \frac{1}{4}(n+1) + 2 . \end{aligned}$$

Hence we have

Lemma 13. *$f^T(n,k) = 1$ whenever $k \geq 2 + \lfloor \frac{1}{4}(n+1) \rfloor$.*

By Lemmas 11,13, we have

Theorem 14.

$$f^T(n,k) = \begin{cases} 3 & k = 1 \\ 2 & k = 2 (n > 6) \\ 1 \text{ or } 2 & 3 \leq k \leq 2 + \lfloor \frac{1}{4}(n+1) \rfloor \\ 1 & k \geq 2 + \lfloor \frac{1}{4}(n+1) \rfloor . \end{cases}$$

We conjecture that Theorem 14 can be improved as follows.

$$f^T(n,k) = \begin{cases} 3 & k = 1 \\ 2 & k = 2 (n > 6) \\ 1 & k \geq 3 \end{cases}$$

4. $F^T(n,k)$ for Primitive Tournaments

Lemma 15.

$$F^T(n,k) \leq n - k + 3 \qquad n > 6 .$$

Proof: By Lemma 5

$$F^T(n,k) \leq \exp(n, n-k+1) .$$

For primitive tournaments, by (1.1),

$$\exp^T(n,k) = k + 2 . \qquad n > 6$$

Thus

$$\begin{aligned} F^T(n,k) &\leq \exp(n, n-k+1) \\ &= (n-k+1) + 2 \\ &= n - k + 3 . \end{aligned}$$

∎

Lemma 16. (see [1]) *Each vertex of a strongly connected tournament with n vertices is contained in a circuit of length r, for $r = 3, 4, \ldots, n$.*

Theorem 17.

$$F^T(n,k) = \begin{cases} n - k + 3 & k = 1,2 \\ & \qquad\qquad n > 6 \\ n - k + 2 & 3 \leq k \leq n - 1 \end{cases}$$

Proof: Consider the tournaments T_n^* defined on the nodes $1, 2, \ldots, n$ as follows. The arcs $\vec{1n}, \overrightarrow{n\, n-1}, \ldots, \vec{32}, \vec{21}$ are in T_n^* and the arcs not yet specified are all oriented toward the node with the larger subscript. This tournament contains a simple cycle of length n so it is irreducible and hence primitive.

Take $X = \{n, n-1, \ldots, n-k+1\} \setminus \{n-k+3\}$. It is easy to verify that there is no walk from any vertex in X to 1 of length $n-k+2$. (If the reader sketches the tournament T_n^*, the reason for this and subsequent statements should become apparent).

If $k = 1, 2$, then $|X| = k$ and since $F(T_n^*, k) \geq n-k+3$, we have

$$F^T(n, k) \geq n - k + 3.$$

By Lemma 15

$$F^T(n, k) = n - k + 3.$$

If $3 \leq k \leq n-1$, $|X| = k-1$, then

$$\begin{aligned} F^T(n, k-1) &\geq n - k + 3 \\ \text{i.e.} F^T(n, k) &\geq n - k + 2 \end{aligned} \tag{4.1}$$

We now prove $F^T(n, k) \leq n-k+2$ for primitive tournaments, $k \geq 3$. That is, we will show that for any set X of k vertices of a primitive tournament T_n and each vertex i of T_n there exists a walk from at least one vertex in X to i of length $n-k+2$.

Suppose that there is a vertex $j \in X$ such that the distance $d(j, i)$ from j to i is less than $n-k$, i.e.

$$d(j, i) < n - k. \tag{4.2}$$

According to Lemma 16 j is contained in a circuit of length r, $r = 3, 4, \ldots, n$. Thus there exists a walk from j to i of length $d(j, i) + r$, $r = 3, 4, \ldots, n$. It follow from (4.2) that there is a walk from j to i of length $n-k+2$.

Now suppose that for all $j \in X$, $d(j, i) \geq n-k$, then, by Lemma 16, all vertices of T_n are on a Hamilton cycle denoted by $\{v_1, v_2, \ldots v_n\}$, where $v_1\vec{v}_n$, $v_n\vec{v}_{n-1}, \ldots, v_3\vec{v}_2, v_2\vec{v}_1$. Without loss of generality, let $i = v_1$.

Since $d(j, i) \geq n-k$, $\forall j \in X$, $X = \{v_n, v_{n-1}, \ldots, v_{n-k+1}\}$ $k \geq 3$. Thus there is a walk from v_{n-k+3} to v_1 of length $n-k+2$. Hence, by the definition

$$F^T(n, k) \leq n - k + 2, k \geq 3 \tag{4.3}$$

By (4.1) and (4.3)

$$F^T(n, k) = n - k + 2, k \geq 3.$$

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