Tripacking of Pairs by Quintuples
The Case $v \equiv 2 \pmod 4$

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ABSTRACT. Let V be a finite set of order ν. A (ν, κ, λ) packing design of index λ and block size κ is a collection of k-element subsets, called blocks, such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks, $\sigma(\nu, \kappa, \lambda)$, in a packing design. It is well known that $\sigma(\nu, \kappa, \lambda) \leq \left[\frac{\nu}{\kappa}\left[\frac{\nu-1}{\kappa-1}\lambda\right]\right] = \psi(\nu, \kappa, \lambda)$, where $[x]$ is the largest integer satisfying $x \geq [x]$. It is shown here that if $\nu \equiv 2 \pmod 4$ and $\nu \geq 6$ then $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ with the possible exception of $\nu = 38$.

1. Introduction

A (ν, κ, λ) packing design of order ν, block size κ, and index λ is a collection β of κ-element subsets, of a ν-set V such that every 2-subset of V occurs in at most λ blocks. Let $\sigma(\nu, \kappa, \lambda)$ denote the maximum number of blocks in a (ν, κ, λ) packing design. A (ν, κ, λ) packing design with $|\beta| = \sigma(\nu, \kappa, \lambda)$ will be called a maximum packing design. It is well known, [15], that

$$\sigma(\nu, \kappa, \lambda) \leq \left[\frac{\nu}{\kappa}\left[\frac{\nu-1}{\kappa-1}\lambda\right]\right] = \psi(\nu, \kappa, \lambda)$$

where $[x]$ is the largest integer satisfying $x \geq [x]$.

When $\sigma(\nu, \kappa, \lambda) = \psi(\nu, \kappa, \lambda)$ the packing design is called an optimal packing design.

Many researchers have been involved in determining the packing numbers $\sigma(\nu, \kappa, \lambda)$ known to date (see bibliography). Our interest here is in the case $\kappa = 5$, $\lambda = 3$ and $\nu \equiv 2 \pmod 4$. Such packing is called tripacking of pairs by quintuples. Our goal is to prove the following.

Theorem 1.1. *For all positive integers $\nu \geq 6$ and $\nu \equiv 2 \pmod 4$, we have $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ with the possible exception of $\nu = 38$.*

2. Recursive Constructions of Packing Designs

In this section we require several other types of combinatorial configuration. A balanced incomplete block design, $B[\nu, \kappa, \lambda]$, is a (ν, κ, λ) packing design where every 2-subset of points is contained in exactly λ blocks. If a $B[\nu, \kappa, \lambda]$ exists then it is clear that $\sigma(\nu, \kappa, \lambda) = \lambda\nu(\nu-1)/\kappa(\kappa-1) = \psi(\nu, \kappa, \lambda)$, and Hanani [11] has proved the following existence theorem for $B[\nu, 5, \lambda]$.

Theorem 2.1. *Necessary and sufficient conditions for the existence of a $B[\nu, 5, \lambda]$ are that $\lambda(\nu - 1) \equiv 0 \pmod 4$ and $\lambda\nu(\nu - 1) \equiv 0 \pmod{20}$ and $(\nu, \lambda) \neq (15, 2)$.*

If from a $B[\nu, 5, 1]$ we delete a point and all the blocks containing this point we have the following.

Theorem 2.2. *If $\nu \equiv 0$ or $4 \pmod{20}$ then $\sigma(\nu, 5, 1) = \psi(\nu, 5, 1)$.*

A (ν, κ, λ) packing design with a hole of size h is a triple (V, H, β) where V is a ν-set, H is a subset of V of cardinality h; and β is a collection of κ-element subsets, called blocks, of V such that

1. no 2-subset of H appears in any block
2. every other 2-subset of V appears in at most λ blocks
3. $|\beta| = \psi(\nu, \kappa, \lambda) - \psi(h, \kappa, \lambda)$.

It is clear that if a (ν, κ, λ) packing design with a hole of size h exists and $\sigma(h, \kappa, \lambda) = \psi(h, \kappa, \lambda)$ then $\sigma(\nu, \kappa, \lambda) = \psi(\nu, \kappa, \lambda)$.

Let κ, λ, m and ν be positive integers. A group divisible design GD$[\kappa, \lambda, m, \nu]$ is a triple (V, β, γ) where V is a set of points with $|V| = v$ and $\gamma = \{G_1, \ldots, G_n\}$ is a partition of V into n sets of size m, called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties

1. $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;
2. every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

A GD$[\kappa, \lambda, m, \kappa m]$ is called a transversal design and denoted by T(κ, λ, m). It is well known that a T(κ, λ, m) is equivalent to $\kappa - 2$ orthogonal Latin squares of side m.

Let m, κ, λ and ν be positive integers. A modified group divisible design MGD$[\kappa, \lambda, m, \nu]$ is a triple (V, β, γ) where V is a set of points of size ν, and $\gamma = \{G_1, \ldots, G_n\}$ is a partition of V into n sets of size m, called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties

1. $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$,

2. every 2-subset $\{x, y\}$ of V such that x and y are neither in the same group nor in the same row is contained in exactly λ blocks of β. (We may look at the points of V as the points of an array of size $m \times n$ and then the groups of (V, β, γ) are precisely the columns of A).

3. a block can contain at most one element from any given row.

A resolvable modified group divisible design RMGD$[\kappa, \lambda, m, \nu]$ is a modified group divisible design where its blocks can be partitioned into parallel classes.

The following theorems are in the form most useful to us and may be found in [1].

Theorem 2.3. *There exists a RMGD$[5, 1, 5, 5m]$ for all $m \neq 2, 3, 4, 6$ and the possible exceptions of $m \in \{10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$.*

Theorem 2.4. *If there exists a (1) RMGD$[5, 1, 5, 5m]$ (2) a GD$[5, 1, \{4, s*\}, 4m + s]$ where $*$ means there is exactly one group of size s and (3) a $(20+h, 5, 3)$ packing design with a hole of size h, then there exists a $(20m+4u+h+s, 5, 3)$ packing design with a hole of size $4u+h+s$ where $0 \leq u \leq m-1$.*

To apply the previous theorem we require the existence of GD$[5, 1, \{4, s*\}, 4m + s]$. We observe that we may choose $s = 0$ if $m \equiv 1 \pmod 5$; $s = 4$ if $m \equiv 0$ or $4 \pmod 5$; and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$. For other cases of m the following theorem [10], is in the form most useful to us.

Theorem 2.5. *There exists a GD$[5, 1, \{4, 8*\}, 4m + 8]$ for all $m \equiv 0$ or $2 \pmod 5$, $m \geq 7$ with the possible exception of $m = 10$.*

The following is our last recursive construction.

Theorem 2.6. *If there exists a (1) GD$[6, 3, 5, 5n]$ (2) a $(20 + h, 5, 3)$ tri-packing design with a hole of size h (3) $\sigma(4u+h, 5, 3) = \psi(4u+h, 5, 3)$ where $0 \leq u \leq 5$ then $\sigma(20(n-1) + 4u + h, 5, 3) = \psi(20(n-1) + 4u + h, 5, 3)$.*

<u>Proof:</u> Take a GD$[6, 3, 5, 5n]$ and delete $5 - u$ points from the last group. Inflate this design by a factor of 4. On the blocks of size 5 and 6 construct

a GD[5, 1, 4, 20] and GD[5, 1, 4, 24] respectively. Add h points to the groups and on the first $(n-1)$ groups construct a $(20+h, 5, 3)$ tripacking design with a hole of size h. Then the h points with the last group are the hole of a $(20(n-1)+4u+h, 5, 3)$ tripacking with a hole of size $4u+h$. It is easily checked that if $\sigma(4u+h, 5, 3) = \psi(4u+h, 5, 3)$ then $\sigma(20(n-1)+4u+h, 5, 3) = \psi(20(n-1)+4u+h, 5, 3)$.

To apply the above theorem we require the existence of GD[6, 3, 5, 5n]. Our authority for that is the following.

Lemma 2.1. ([11]) *There exists a GD*[6, 3, 5, 35].

3. Tripacking of order $\nu \equiv 2 \pmod{20}$

The following construction combines other known designs to construct tripacking.

Theorem 3.1. *If $\nu \geq 22$ and $\nu \equiv 2 \pmod{20}$, then $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$. Furthermore these designs have a hole of size two.*

Proof: For all $\nu \equiv 2 \pmod{20}$, $\nu \geq 22$, a $(\nu, 5, 3)$ packing design with $\psi(\nu, 5, 3)$ blocks may be constructed as follows

1. take a $B[\nu-1, 5, 2]$;

2. take a $(\nu+2, 5, 1)$ optimal packing design. This design is constructed from a $B[\nu+3, 5, 1]$ by deleting one point and all the blocks containing this point. So without loss of generality we may assume that the pairs of $\{\nu-1, \nu, \nu+1, \nu+2\}$ do not appear in the blocks of the $(\nu+2, 5, 1)$ optimal packing design. Now change both $\nu+1$ and $\nu+2$ to ν. Then the blocks constructed in (1) and (2) yield the blocks of the required tripacking, and these designs have a hole of size 2.

4. Tripacking of order $\nu \equiv 14 \pmod{20}$

Before giving an induction proof of this case we require the following constructions of tripacking, some with holes. In our constructions the following notations are used: A block $\langle k, k+m, k+n, k+j, f(k)\rangle \pmod{\nu}$ where $f(k) = a$ if k is even and $f(k) = b$ if k is odd is denoted by $\langle 0, m, n, j\rangle \cup \{a, b\}$; and a block $\langle k, k+m, k+n, k+j, f(k)\rangle \pmod{\nu}$ where $f(k) = h_i$ if $k \equiv i \pmod{4}$ is denoted by $\langle 0, m, n, j\rangle \cup \{h_i\}_{i=1}^{4}$. Similarly a block $< (0,\kappa)(0,\kappa+m)(1,\kappa+n)(1,\kappa+j)f(\kappa) >$, $\kappa = 0, \ldots, \nu-1$ where $f(\kappa) = a$ if κ is even and $f(\kappa) = b$ if κ is odd is denoted by $\langle (0,0)(0,m)(1,n)(1,j)\rangle \cup \{a, b\} \pmod{-, \nu}$.

In the following lemma we give a table describing the constructions of a $(\nu, 5, 3)$ packing designs for $v = 14, 74, 94$. In general the constructions are as follows. Let $X = Z_2 \times Z_{\nu-n/2} \cup H_n$ or $X = Z_{\nu-n} \cup H_n$ where

$H_n = \{h_1, \ldots, h_n\}$. The blocks are constructed by taking the orbit of the tabulated base block, (mod $\frac{\nu-n}{2}$) or (mod $\nu - n$) respectively unless it is otherwise specified.

Lemma 4.1. *$\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ for $\nu = 14, 34, 54, 74, 94$.*

Proof: For $\nu = 14$ let $X = \{11, \ldots, 14\}$ the blocks are

⟨1 2 8 11 14⟩	⟨2 6 9 12 14⟩	⟨1 3 6 7 12⟩	⟨2 7 9 10 13⟩
⟨1 3 7 8 12⟩	⟨3 4 5 6 9⟩	⟨1 3 9 10 11⟩	⟨3 4 10 12 13⟩
⟨1 4 5 12 14⟩	⟨3 5 8 10 11⟩	⟨1 4 6 7 11⟩	⟨4 6 8 13 14⟩
⟨1 4 8 9 10⟩	⟨4 7 8 9 14⟩	⟨1 5 10 13 14⟩	⟨5 7 8 11 13⟩
⟨2 3 4 11 13⟩	⟨6 7 10 11 14⟩	⟨2 3 5 7 14⟩	⟨9 11 12 13 14⟩
⟨2 3 6 8 13⟩	⟨1 5 6 9 13⟩	⟨2 4 7 10 12⟩	⟨2 5 6 8 10⟩
⟨2 5 9 11 12⟩			

For $\nu = 34, 54$ the construction is as follows.

1. take a $(\nu - 1, 5, 2)$ optimal packing design [5]. This design has a hole of size 3, say, $\{\nu - 3, \nu - 2, \nu - 1\}$.

2. take a $(\nu+3, 5, 1)$ packing design with a hole of size 9 and assume the hole is $\{\nu - 5, \nu - 4, \ldots, \nu + 3\}$. These two designs exist by theorem 2.5. Delete the point $\nu + 3$ and all the blocks through this point. In all other blocks change $\nu + 1$ and $\nu + 2$ to ν.

3. Add the blocks $\langle \nu-4, \nu-3, \nu-2, \nu-1, \nu\rangle$ $\langle \nu-5, \nu-3, \nu-2, \nu-1, \nu\rangle$.

It is easily checked that the above construction yields a $(\nu, 5, 3)$ optimal packing design for $\nu = 34, 54$.

For $\nu = 74, 94$, in the table below we construct a $(\nu, 5, 3)$ tripacking design with a hole of size 14 and since $\sigma(14, 5, 3) = \psi(14, 5, 3)$ it follows that $\sigma(\nu, 5, 3) = (\nu, 5, 3)$ for $\nu = 74, 94$.

Theorem 4.1. *$\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ for all $\nu \equiv 14 \pmod{20}$.*

Proof: For $\nu \leq 94$, the result follows from lemma 4.1.

For $\nu \geq 114$, $\nu \neq 134$, simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h, and s are chosen so that the following 4 conditions hold

1. there exists a RMGD$[5, 1, 5, 5m]$

2. $4u + h + s \equiv 14 \pmod{20}$, $14 \leq 4u + h + s \leq 94$

3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 2$ or 6

4. there exists a GD$[5, 1, \{4, s*\}, 4m + s]$

ν	Point Set	Base Blocks
74	$Z_{60} \cup H_{14}$	On $Z_{60} \cup H_{13}$ construct a $(73, 5, 1)$ packing with a hole of size 13 and take the following blocks $\langle 0\ 15\ 30\ 45\ h_{14} \rangle + i$, $i \in Z_{15}$ $\langle 0\ 12\ 24\ 36\ 48 \rangle + i$, $i \in Z_{12}$, twice $\langle 0\ 2\ 8\ 18\ 34 \rangle$ $\langle 0\ 1\ 3\ 6 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 4\ 11\ 19 \rangle \cup \{h_3, h_4\} \langle 0\ 7\ 21\ 38 \rangle \cup \{h_5, h_6\}$ $\langle 0\ 9\ 18\ 37 \rangle \cup \{h_7, h_8\} \langle 0\ 10\ 23\ 43 \rangle \cup \{h_9, h_{10}\}$ $\langle 0\ 11\ 25\ 38 \rangle \cup \{h_{11}, h_{12}\} \langle 0\ 20\ 21\ 25 \rangle \cup \{h_{13}, h_{14}\}$
94	$Z_{80} \cup H_{14}$	On $Z_{80} \cup H_{13}$ construct a $(93, 5, 1)$ packing with a hole of size 13, [14], and take the following blocks $\langle 0\ 20\ 40\ 60\ h_{14} \rangle + i$, $i \in Z_{20}$ $\langle 0\ 16\ 32\ 48\ 64 \rangle + i$, $i \in Z_{16}$, twice $\langle 0\ 2\ 8\ 26\ 38 \rangle$ $\langle 0\ 3\ 7\ 21\ 34 \rangle$ $\langle 0\ 22\ 27\ 28\ 37 \rangle$ $\langle 0\ 22\ 25\ 39 \rangle \cup \{h_1, h_2\}$ $\langle 0\ 1\ 5\ 12 \rangle \cup \{h_3, h_4\}$ $\langle 0\ 2\ 11\ 45 \rangle \cup \{h_5, h_6\}$ $\langle 0\ 8\ 31\ 55 \rangle \cup \{h_7, h_8\}$ $\langle 0\ 10\ 29\ 57 \rangle \cup \{h_9, h_{10}\}$ $\langle 0\ 13\ 30\ 51 \rangle \cup \{h_{11}, h_{12}\}$ $\langle 0\ 15\ 35\ 54 \rangle \cup \{h_{13}, h_{14}\}$

Now apply theorem 2.4 and the result follows

For $\nu = 134$ apply theorem 2.6 with $h = 6$, $n = 7$, and $u = 2$.

See lemma 6.1 for a $(26, 5, 3)$ packing design with a hole of size 6.

5. Tripacking of order $\nu \equiv 18 \pmod{20}$

The following construction combines other designs to construct tripacking.

Lemma 5.1. *$\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ for $\nu = 18, 58, 78, 98$.*

For $\nu = 18$, [16], let $X = \{1, 2, \dots, 18\}$ then the required blocks are

⟨1 2 3 5 14⟩	⟨3 4 7 8 12⟩	⟨1 2 7 8 10⟩	⟨3 4 7 10 12⟩
⟨1 3 6 12 14⟩	⟨3 5 6 10 18⟩	⟨1 3 11 17 18⟩	⟨3 6 9 10 16⟩
⟨1 4 5 15 16⟩	⟨3 7 9 17 18⟩	⟨1 5 6 7 11⟩	⟨3 8 9 13 15⟩
⟨1 6 8 9 13⟩	⟨3 11 13 14 16⟩	⟨1 7 13 16 18⟩	⟨4 6 8 16 17⟩
⟨1 8 15 17 18⟩	⟨4 6 13 14 15⟩	⟨1 10 11 12 15⟩	⟨4 7 9 14 17⟩
⟨1 10 13 16 17⟩	⟨4 8 10 11 18⟩	⟨2 3 11 15 16⟩	⟨4 10 14 16 18⟩
⟨2 4 5 9 11⟩	⟨5 7 8 11 16⟩	⟨2 4 9 11 13⟩	⟨5 7 12 13 15⟩
⟨2 4 12 13 18⟩	⟨5 8 9 10 14⟩	⟨2 5 8 14 18⟩	⟨5 9 12 16 18⟩
⟨2 6 7 13 18⟩	⟨5 10 12 13 17⟩	⟨2 6 8 12 16⟩	⟨6 7 11 14 15⟩
⟨2 6 10 15 17⟩	⟨6 9 11 12 17⟩	⟨2 7 9 10 15⟩	⟨8 11 13 14 17⟩
⟨2 12 14 16 17⟩	⟨9 12 14 15 18⟩	⟨3 4 5 15 17⟩	

For $\nu = 58, 78, 98$ we first show that there exists a $(\nu - 1, 5, 2)$ packing with a hole of size 7.

For $\nu = 57$ see [13].

For $\nu = 77$ let $X = Z_2 \times Z_{35} \cup H_7$, then the required blocks are

$$\langle(0,0)\ (0,7)\ (0,14)\ (0,21)\ (0,28)\rangle + (-,i), i \in Z_7, \text{ twice}$$
$$\langle(1,0)\ (1,7)\ (1,14)\ (1,21)\ (1,28)\rangle + (-,i), i \in Z_7,$$
$$\langle(0,0)\ (0,2)\ (0,12)\ (0,20)\ (0,31)\rangle \pmod{-,35}$$
$$\langle(1,0)\ (1,2)\ (1,11)\ (1,19)\ (1,31)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,3)\ (0,13)\ (1,29)\ (1,34)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,2)\ (1,0)\ (1,1)\ (1,4)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,5)\ (0,9)\ (1,17)\ (1,33)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,8)\ (1,5)\ (1,14)\ (1,27)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,1)\ (0,16)\ (1,11)\ (1,23)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,12)\ (1,15)\ (1,25)\ (1,32)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,1)\ (1,1)\ (1,3)\ (1,6)\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,3)\ (1,14)\ (1,29) h_1\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,5)\ (1,23)\ (1,24) h_2\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,6)\ (1,18)\ (1,28) h_3\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,9)\ (1,17)\ (1,25) h_4\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,11)\ (1,15)\ (1,21) h_5\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,13)\ (1,9)\ (1,20) h_6\rangle \pmod{-,35}$$
$$\langle(0,0)\ (0,17)\ (1,9)\ (1,30) h_7\rangle \pmod{-,35}$$

For $\nu = 97$ take a RMGD[5, 1, 5, 45] and inflate this design by a factor of 2. To two parallel classes of quintuples add two points to each and replace their blocks by the blocks of a GD[5, 2, 2, 12], [11]. On the remaining parallel classes of quintuples construct a GD[5, 2, 2, 10], [11]. To the parallel class of block size 9 add two points and construct a GD[5, 2, 2, 20], [5]. Finally add to the groups a new point and on each group construct a B[11, 5, 2].

It is clear that this construction yields a $(97, 5, 2)$ packing design with a hole of size 7.

We now construct a $(\nu, 5, 3)$ optimal packing design for $\nu = 58, 78, 98$ as follows

1. take a $(\nu - 1, 5, 2)$ packing design with $\psi(\nu - 1, 5, 2) - 1$ blocks. For $\nu = 58, 78, 98$ there is a $(\nu - 1, 5, 2)$ packing design with a hole of size 7. But careful inspection of the $(7, 5, 2)$ packing design (notice that $\sigma(7, 5, 2) = \psi(7, 5, 2) - 1$) shows that there are four pairs, each appears only once, through the same point, say, $(1, 2)$ $(1, 3)$ $(1, 4)$ $(1, 5)$. Hence the $(\nu - 1, 5, 2)$ packing design, $\nu = 58, 78, 98$, has 4 pairs through the same point say $(1, 2)$ $(1, 3)$ $(1, 4)$ $(1, 5)$ such that each of these pair appears only once.

2. take a $(\nu + 2, 5, 1)$ optimal packing design, theorem 2.2, and assume

that $\{\nu, \nu+1, \nu+2\}$ and $\{2, 3, 4, 5\}$ are missing from this design. Now change both points $\nu + 1$ and $\nu + 2$ to ν.

3. add the block $\langle 1, 2, 3, 4, 5 \rangle$.

It is easily checked that the above three steps yield a $(\nu, 5, 3)$ optimal packing design for $\nu = 58, 78, 98$.

Theorem 5.1. *For all* $\nu \equiv 18 \pmod{20}$ *we have* $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ *with the possible exception of* $\nu = 38$.

<u>Proof:</u> For $\nu = 18$, 58, 78, 98 the result follows from lemma 5.1. For $\nu \geq 118$, $\nu \neq 138$, 178, 218 simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that the following 4 conditions hold

1. there exists a RMGD$[5, 1, 5, 5m]$
2. $4u + h + s \equiv 18 \pmod{20}$, $18 \leq 4u + h + s \leq 98$, $4u + h + s \neq 38$
3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 2$ or 6
4. there exists a GD$[5, 1, \{4, s*\}, 4m + s]$

Now apply theorem 2.4 and the result follows.

For $\nu = 138$ apply theorem 2.6 with $h = 6$, $n = 7$ and $u = 3$.

For $\nu = 178$ take a RMGD$[5, 1, 5, 40]$ and inflate this design by 4. To 3 parallel classes of quintuples add 4 points to each one and replace their blocks by the blocks of a GD$[5, 3, 4, 24]$. On the remaining parallel classes of quintuples construct a GD$[5, 3, 4, 20]$. To the parallel class of block size 8, after inflating by 4, add 4 points to the last group and construct a GD$[5, 3, \{4, 8*\}, 36]$. Finally add 2 points to the groups and on the first 7 groups construct a $(22, 5, 3)$ tripacking design with a hole of size 2 and on the last group construct a $(26, 5, 3)$ tripacking design with a hole of size 6. (See lemma 6.1 for the existence of this design). It is easy to check that the above construction yields a $(178, 5, 3)$ tripacking design with a hole of size 18. But $\sigma(18, 5, 3) = \psi(18, 5, 3)$ hence $\sigma(178, 5, 3) = \psi(178, 5, 3)$.

For $\nu = 218$ take a $T(6, 3, 10)$, [11], and delete 7 points from last group. Inflate the resultant design by a factor of 4, that is, replace all blocks of size 6 and 5 by the blocks of GD$[5, 1, 4, 24]$ and GD$[5, 1, 4, 20]$ respectively. To the groups add 6 new points and on the first 5 groups construct a $(46, 5, 3)$ tripacking design with a hole of size 6 (see lemma 6.1). Take these 6 points with the last group of size 12 to be the hole of a $(218, 5, 3)$ tripacking design with a hole of size 18. But $\sigma(18, 5, 3) = \psi(18, 5, 3)$ hence $\sigma(218, 5, 3) = \psi(218, 5, 3)$.

ν	Point Set	Base Blocks
6	Z_6	$\langle 0\ 1\ 2\ 3\ 4\rangle$ 3 times
10	Z_{10}	$\langle 0\ 2\ 4\ 6\ 8\rangle + i,\ i \in Z_2 \langle 0\ 1\ 2\ 4\ 5\rangle$
26	$Z_2 \times Z_{10} \cup H_6$	$\langle (0,0)(0,2)(0,3)(0,5)\rangle \cup \{h_1,h_2\}$ $\langle (1,0)(1,2)(1,3)(1,5)\rangle \cup \{h_1,h_2\}$ $\langle (0,0)(0,1)(1,0)(1,3)\rangle \cup \{h_1,h_2\}$ $\langle (0,0)(0,2)(1,4)(1,8)h_3\rangle$ $\langle (0,0)(0,4)(1,5)(1,9)h_4\rangle$ $\langle (0,0)(0,4)(1,5)(1,7)h_5\rangle$ $\langle (0,0)(0,4)(1,0)(1,4)h_6\rangle$ $\langle (0,0)(0,1)(1,7)(1,8)h_5\rangle \cup \{h_3,h_4\}$ $\langle (0,0)(0,3)(1,1)(1,2)\rangle \cup \{h_5,h_6\}$
30	Z_{30}	$\langle 0\ 6\ 12\ 18\ 24\rangle + i,\ i \in Z_6$ $\langle 0\ 1\ 2\ 3\ 7\rangle\langle 0\ 2\ 8\ 13\ 22\rangle\langle 0\ 3\ 10\ 17\ 21\rangle\langle 0\ 3\ 11\ 15\ 20\rangle$
46	$Z_{40} \cup H_6$	$\langle 0\ 5\ 20\ 25\rangle \cup \{h_1,h_2\}$ half orbit $\langle 0\ 1\ 3\ 15\ 19\rangle\langle 0\ 1\ 2\ 4\ 8\rangle\langle 0\ 5\ 11\ 19\ 33\rangle$ $\langle 0\ 8\ 17\ 27\rangle \cup \{h_1,h_2\}$ $\langle 0\ 3\ 13\ 30\rangle \cup \{h_3,h_4\}$ $\langle 0\ 7\ 16\ 31\rangle \cup \{h_5,h_6\}$ $\langle 0\ 5\ 11\ 22\rangle \cup \{h_i\}_{i=3}^{6}$
50	$Z_{48} \cup H_2$	$\langle 0\ 13\ 24\ 37\rangle \cup \{h_1,h_2\}$, half orbit $\langle 0\ 2\ 7\ 17\ 23\rangle\langle 0\ 8\ 9\ 12\ 22\rangle\langle 0\ 3\ 19\ 23\ 37\rangle$ $\langle 0\ 1\ 2\ 4\ 10\rangle\langle 0\ 5\ 16\ 28\ 33\rangle\langle 0\ 6\ 18\ 27\ 35\rangle$ $\langle 0\ 7\ 22\ 29\rangle \cup \{h_1,h_2\}$
66	$Z_{60} \cup H_6$	$\langle 0\ 13\ 30\ 43\rangle \cup \{h_1,h_2\}$, half orbit $\langle 0\ 1\ 5\ 12\ 26\rangle\langle 0\ 3\ 13\ 23\ 41\rangle$ $\langle 0\ 1\ 3\ 7\ 23\rangle\langle 0\ 5\ 16\ 34\ 48\rangle\langle 0\ 1\ 3\ 13\ 41\rangle\langle 0\ 2\ 8\ 33\ 44\rangle$ $\langle 0\ 8\ 27\ 35\rangle \cup \{h_1,h_2\}$ $\langle 0\ 9\ 24\ 45\rangle \cup \{h_3,h_4\}$ $\langle 0\ 4\ 9\ 43\rangle \cup \{h_5,h_6\}$ $\langle 0\ 6\ 15\ 29\rangle \cup \{h_i\}_{i=3}^{6}$
70	Z_{70}	$\langle 0\ 14\ 28\ 42\ 56\rangle + i,\ i \in Z_{14}$ $\langle 0\ 3\ 11\ 27\ 40\rangle\langle 0\ 1\ 5\ 23\ 43\rangle\langle 0\ 2\ 12\ 31\ 38\rangle\langle 0\ 6\ 21\ 37\ 46\rangle$ $\langle 0\ 1\ 4\ 9\ 26\rangle\langle 0\ 2\ 15\ 35\ 49\rangle\langle 0\ 6\ 17\ 24\ 36\rangle\langle 0\ 1\ 3\ 8\ 21\rangle$ $\langle 0\ 4\ 14\ 23\ 45\rangle\langle 0\ 6\ 16\ 44\ 59\rangle$
86	$Z_{80} \cup H_6$	$\langle \kappa, \kappa+14, \kappa+40, \kappa+54, f(\kappa)\rangle$ half orbit where $f(\kappa) = h_1$ if $\kappa \equiv 0$ or 1 (mod 4) and $f(\kappa) = h_2$ if $\kappa \equiv 2$ or 3 (mod 4). $\langle 0\ 3\ 11\ 27\ 41\rangle\langle 0\ 5\ 15\ 33\ 51\rangle\langle 0\ 2\ 9\ 22\ 41\rangle\langle 0\ 21\ 25\ 37\ 38\rangle$ $\langle 0\ 1\ 3\ 7\ 49\rangle\langle 0\ 5\ 15\ 35\ 59\rangle\langle 0\ 8\ 33\ 43\ 54\rangle\langle 0\ 9\ 24\ 37\ 60\rangle$ $\langle 0\ 1\ 3\ 7\ 19\rangle\langle 0\ 6\ 17\ 29\rangle \cup \{h_1,h_2\}$ $\langle 0\ 8\ 27\ 55\rangle \cup \{h_3,h_4\}$ $\langle 0\ 5\ 22\ 53\rangle \cup \{h_5,h_6\}$ $\langle 0\ 14\ 23\ 45\rangle \cup \{h_i\}_{i=3}^{6}$
90	Z_{90}	$\langle 0\ 18\ 36\ 54\ 72\rangle + i,\ i \in Z_{18}$ $\langle 0\ 1\ 3\ 12\ 50\rangle\langle 0\ 4\ 28\ 43\ 60\rangle\langle 0\ 5\ 34\ 40\ 59\rangle\langle 0\ 7\ 27\ 48\ 64\rangle$ $\langle 0\ 1\ 3\ 7\ 49\rangle\langle 0\ 5\ 15\ 26\ 38\rangle\langle 0\ 8\ 21\ 66\ 76\rangle\langle 0\ 9\ 25\ 53\ 72\rangle$ $\langle 0\ 8\ 22\ 39\ 52\rangle\langle 0\ 1\ 3\ 11\ 28\rangle\langle 0\ 4\ 16\ 23\ 36\rangle\langle 0\ 5\ 14\ 34\ 64\rangle$ $\langle 0\ 6\ 24\ 39\ 61\rangle$

6. Tripacking of orders $\nu \equiv 6$ or 10 (mod 20)

In this section we first require the existence of small tripacking designs.

Lemma 6.1. *For all* $6 \le \nu \le 90$, $\nu \equiv 6$ *or* 10 (mod 20), $\sigma(\nu,5,3) = \psi(\nu,5,3)$.

Proof: The required constructions are given in the following table. For $\nu = 26$, 46, 50, 86 we actually construct a $(\nu,5,3)$ packing design with a hole of size 2 or 6. But $\sigma(h,5,3) = \psi(h,5,3)$ for $h = 2$, 6 hence $\sigma(\nu,5,3) = \psi(\nu,5,3)$ for $\nu = 26$, 46, 50, 86.

Theorem 6.1. *For all positive integers* ν, $\nu \equiv 6$ *or* 10 (mod 20) *we have* $\sigma(\nu,5,3) = \psi(\nu,5,3)$.

F:or $6 \le \nu \le 90$, and $\nu \equiv 6$ or 10 (mod 20) the result follows from lemma 6.1. For $\nu \ge 106$, and $\nu \equiv 6$ or 10 (mod 20), $\nu \ne 130$, 146 simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that the following 4 conditions hold

1. there exists a RMGD$[5, 1, 5, 5m]$
2. $4u + h + s \equiv 6$ or $10 \pmod{20}$ and $6 \leq 4u + h + s \leq 90$
3. $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 2$ or 6
4. there exists a GD$[5, 1, \{4, s*\}, 4m + s]$

Now apply theorem 2.4 and the result follows.

For $\nu = 130$, 146 apply theorem 2.6 with $h = 6$, $n = 7$ and $u = 1, 5$ respectively.

7. Conclusion

We have shown that if $\nu \equiv 2 \pmod 4$, $\nu \geq 6$ then $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ with the possible exception of $\nu = 38$, which proves our theorem.

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