Observability of Complete Multipartite Graphs with Equipotent Parts

Mirko Horňák and Roman Soták

Department of Geometry and Algebra
P. J. Šafárik University
Jesenná 5
041 54 Košice
Slovakia
email: hornak@turing.upjs.sk, sotak@turing.upjs.sk

Abstract. Observability of a graph is the least k admitting a proper colouring of its edges by k colours in such a way that each vertex is identifiable by the set of colours of its incident edges. It is shown that for $p \geq 3$ and $q \geq 2$ the complete p-partite graph with all parts of cardinality q has observability $(p-1)q+2$.

For integers m, n denote by $[m, n]$ the interval of all integers between m and n (inclusively) and by $[m, \infty)$ the interval of all integers not less than m; for $n \in [1, \infty)$ the representative of the residue class modulo n in the set $[1, m]$ will be $(m)_n$.

Let G be a finite undirected graph without loops or multiple edges (for basic notions and notations see e.g. Harary [4]), φ a k-edge-colouring of G—a map from $[1, k]^{E(G)}$—and $\mathrm{Im}_x(\varphi)$ the *colour set* of x induced by φ—the set consisting of $\varphi(e)$ for all e incident to $x \in V(G)$. If G does not have a component K_2 and has at most one component K_1, then any colouring $\varphi \in [1, |E(G)|]^{E(G)}$ using each colour exactly once distinguishes vertices of G by their colour sets: $\mathrm{Im}_x(\varphi) \neq \mathrm{Im}_y(\varphi)$ whenever $x, y \in V(G)$, $x \neq y$; moreover, φ is proper: $\varphi(e_1) \neq \varphi(e_2)$ for e_1 adjacent to e_2. Let $\mathrm{Obs}_k(G)$ be the set of all proper k-edge-colourings of G distinguishing its vertices by colour sets. *Observability* of G, denoted by $\mathrm{obs}(G)$, is the minimum k with $\mathrm{Obs}_k(G) \neq \emptyset$—see Černý, Horňák and Soták [1]. For the sake of completeness observability of graphs having a component K_2 or at least two components K_1 is defined to be ∞. This graph invariant has been inspired by the point-distinguishing chromatic index, the line-distinguishing chromatic number and the harmonious chromatic number of a graph introduced by Harary and Plantholt [5], Frank, Harary and Plantholt [3] and Miller and Pritikin [6], respectively.

Let $v_d(G)$ be the number of vertices of degree d in G and $\Delta(G)$ the maximum degree of a vertex in G. Corollary 1.2 of [1] states that

$$\mathrm{obs}(G) \geq \min \left\{ k \in [0, \infty) : \ \forall d \in [0, \Delta(G)] \ v_d(G) \leq \binom{k}{d} \right\}.$$

The aim of the present paper is to determine observability of $K(p \times q)$—the p-partite graph with all parts of cardinality q. For $K(p\times 1) = K_p$ and $K(2\times q) = K_{q,q}$ the values of this invariant have been found in [1]: $\mathrm{obs}(K_1) = 0$, $\mathrm{obs}(K_2) = \infty$, $\mathrm{obs}(K_p) = p$ for p odd, $p \geq 3$, $\mathrm{obs}(K_p) = p+1$ for p even, $p \geq 4$, $\mathrm{obs}(K_{1,1}) = \infty$ and $\mathrm{obs}(K_{q,q}) = q+2$ for $q \geq 2$. Moreover, $K(1\times q) = qK_1$ for $q \in [2,\infty)$ has more than one component K_1 and its observability is ∞. Thus here we can restrict ourselves to $p \in [3,\infty)$ and $q \in [2,\infty)$.

Theorem. *If $p \in [3,\infty)$ and $q \in [2,\infty)$, then* $\mathrm{obs}(K(p\times q)) = (p-1)q+2$.

Proof: The graph $G = K(p\times q)$ consists of pq vertices of degree $d = (p-1)q$, hence $\mathrm{obs}(G) \geq \min\{k \in [0,\infty): v_d(G) \leq \binom{k}{d}\} = d+2$:

$$\binom{d+1}{d} = d+1 = pq - q + 1 < pq,$$

$$\binom{d+2}{d} = \frac{(d+2)(d+1)}{2} \geq (q+1)((p-1)q+1) > (p-1)q+q+1 > pq.$$

Our task will be accomplished by determining a map $\varphi \in \mathrm{Obs}_{d+2}(G)$. We shall suppose $V(G) = [1,pq]$ with parts $[(i-1)q+1, iq]$ for $i \in [1,p]$. The analysis is divided into 9 cases and depends mainly on the parity of p and/or q.

1. First let us treat the case p even, $p \in [6,\infty)$, and $q \in [3,\infty)$. According to Dénes and Keedwell [2] there exists an idempotent quasiqroup (a Latin square) Q_q of order q. Without loss of generality the carrier of Q_q is $[1,q]$. Denoting $i * j$ the value of the binary operation of Q_q applied to the pair $(i,j) \in [1,q]^2$ we have

$$i * i = i \quad \text{and} \quad \bigcup_{j=1}^{q}\{i*j\} = [1,q] = \bigcup_{j=1}^{q}\{j*i\} \text{ for each } i \in [1,q].$$

Let ψ be a proper $(p-1)$-edge-colouring of K_p with $V(K_p) = [1,p]$ and colour an edge of G joining vertices $(i-1)q+j$ and $(k-1)q+l$ by $(\psi\{i,k\}-1)q+(j*l)$ for $i,k \in [1,p], i \neq k$, and $j,l \in [1,q]$.

The resulting d-edge-colouring ψ' of G is a proper one: Consider two adjacent edges of G joining $(i_m-1)q+j_m$ and $(k-1)q+l$, $m = 1,2$. If $i_1 \neq i_2$, then corresponding colours are different since the absolute value of their difference is

$$\begin{aligned}&|(\psi\{i_1,k\}-\psi\{i_2,k\})q + (j_1 * l) - (j_2 * l)|\\ \geq &|\psi\{i_1,k\}-\psi\{i_2,k\}|q - |(j_1*l)-(j_2*l)| \geq q-(q-1) = 1;\end{aligned}$$

we have used $\psi\{i_1,k\} \neq \psi\{i_2,k\}$ – ψ is proper. On the other hand $i_1 = i_2$ implies $j_1 \neq j_2$ and then the difference of corresponding colours is $(j_1*l) - (j_2*l) \neq 0$ (which follows from properties of Q_q).

However, colour sets induced by ψ' are $[1, d]$ for all vertices of G. Two additional colours, $d+1$ and $d+2$, are therefore used to recolour certain edges of G in order to transform ψ' to a map $\varphi \in \mathrm{Obs}_{d+2}(G)$; in this procedure parameters j and l of vertices of recoloured edges are equal which enables (employing the idempotence of $Q_q : j * l = j * j = j$) to control the structure of colour sets induced by φ.

The situation is for $p = 6$ and $q = 3$ illustrated by means of two square symmetric matrices of order 18 consisting of square blocks of order 3. The involved proper 5-edge-colouring ψ of K_6 is determined by the symmetric matrix $B = (b_{ik})$ of order 6 with $b_{ik} = \psi\{i, k\}$ for $i \neq k$ and with diagonal 0's representing "non-loops" in K_6 and the idempotent quasigroup Q_3 of order 3 is given (in this case uniquely) by the matrix $M = (m_{jl})$ with $m_{jl} = j * l$.

$$B = \begin{pmatrix} 0&2&3&4&5&1 \\ 2&0&4&5&1&3 \\ 3&4&0&1&2&5 \\ 4&5&1&0&3&2 \\ 5&1&2&3&0&4 \\ 1&3&5&2&4&0 \end{pmatrix} \qquad M = \begin{pmatrix} 1&3&2 \\ 3&2&1 \\ 2&1&3 \end{pmatrix}$$

In the left matrix (describing ψ') a block with coordinates i, k corresponds to colours of edges between i-th and k-th part of G – they are taken from the set $[(\psi\{i,k\}-1)\cdot 3+1, \psi\{i,k\}\cdot 3]$ and the colour of an edge with internal (in a block) coordinates j, l is congruent with $j * l$ modulo 3; diagonal zero blocks represent "non-edges" between vertices of same part. The right matrix describes the final state after introducing new colours 16,17. Differing entries in both matrices are bold and $\bar{k}$ stands instead of $1k$ for $k \in [0, 7]$.

$$\begin{pmatrix}
0&0&0&\mathbf{4}&6&5&7&9&8&\bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&1&3&2 \\
0&0&0&6&\mathbf{5}&4&9&8&7&\bar{2}&\bar{1}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&2&1 \\
0&0&0&5&4&\mathbf{6}&8&7&9&\bar{1}&\bar{0}&\bar{2}&\bar{4}&\bar{3}&\bar{5}&2&1&3 \\
\mathbf{4}&6&5&0&0&0&\mathbf{\bar{0}}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&1&3&2&7&\mathbf{9}&\mathbf{8} \\
6&\mathbf{5}&4&0&0&0&\bar{2}&\mathbf{\bar{1}}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&2&1&9&8&7 \\
5&4&\mathbf{6}&0&0&0&\bar{1}&\bar{0}&\mathbf{\bar{2}}&\bar{4}&\bar{3}&\bar{5}&2&1&3&8&7&9 \\
7&9&8&\mathbf{\bar{0}}&\bar{2}&\bar{1}&0&0&0&\mathbf{1}&3&2&4&6&5&\bar{3}&\bar{5}&\bar{4} \\
9&8&7&\bar{2}&\mathbf{\bar{1}}&\bar{0}&0&0&0&3&\mathbf{2}&1&6&5&4&\mathbf{\bar{6}}&\bar{4}&\bar{3} \\
8&7&9&\bar{1}&\bar{0}&\mathbf{\bar{2}}&0&0&0&2&1&\mathbf{3}&5&4&6&\bar{4}&\bar{3}&\bar{5} \\
\bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&\mathbf{1}&3&2&0&0&0&\mathbf{7}&9&8&4&6&5 \\
\bar{2}&\bar{1}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&\mathbf{2}&1&0&0&0&9&\mathbf{8}&7&6&5&4 \\
\bar{1}&\bar{0}&\bar{2}&\bar{4}&\bar{3}&\bar{5}&2&1&\mathbf{3}&0&0&0&8&7&\mathbf{9}&5&4&6 \\
\bar{3}&\bar{5}&\bar{4}&1&3&2&4&6&5&\mathbf{7}&9&8&0&0&0&\mathbf{\bar{0}}&\bar{2}&\bar{1} \\
\bar{5}&\bar{4}&\bar{3}&3&2&1&6&5&4&9&\mathbf{8}&7&0&0&0&\bar{2}&\mathbf{\bar{1}}&\bar{0} \\
\bar{4}&\bar{3}&\bar{5}&2&1&3&5&4&6&8&7&\mathbf{9}&0&0&0&\bar{1}&\bar{0}&\mathbf{\bar{2}} \\
1&3&2&7&9&8&\bar{3}&\bar{5}&\bar{4}&4&6&5&\mathbf{\bar{0}}&\bar{2}&\bar{1}&0&0&0 \\
3&2&1&9&8&7&\bar{5}&\bar{4}&\bar{3}&6&5&4&\bar{2}&\mathbf{\bar{1}}&\bar{0}&0&0&0 \\
2&1&3&8&7&9&\bar{4}&\bar{3}&\bar{5}&5&4&6&\bar{1}&\bar{0}&\mathbf{\bar{2}}&0&0&0
\end{pmatrix}
\begin{pmatrix}
0&0&0&\mathbf{\bar{6}}&6&5&7&9&8&\bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&1&3&2 \\
0&0&0&6&\mathbf{\bar{6}}&4&9&8&7&\bar{2}&\bar{1}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&2&1 \\
0&0&0&5&4&\mathbf{\bar{6}}&8&7&9&\bar{1}&\bar{0}&\bar{2}&\bar{4}&\bar{3}&\bar{5}&2&1&3 \\
\mathbf{\bar{6}}&6&5&0&0&0&\mathbf{\bar{7}}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&1&3&2&7&\mathbf{8}&\mathbf{9} \\
6&\mathbf{\bar{6}}&4&0&0&0&\bar{2}&\mathbf{\bar{7}}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&2&1&9&8&7 \\
5&4&\mathbf{\bar{6}}&0&0&0&\bar{1}&\bar{0}&\mathbf{\bar{7}}&\bar{4}&\bar{3}&\bar{5}&2&1&3&8&7&9 \\
7&9&8&\mathbf{\bar{7}}&\bar{2}&\bar{1}&0&0&0&\mathbf{\bar{6}}&3&2&4&6&5&\bar{3}&\bar{5}&\bar{4} \\
9&8&7&\bar{2}&\mathbf{\bar{7}}&\bar{0}&0&0&0&3&\mathbf{\bar{6}}&1&6&5&4&\mathbf{\bar{5}}&\bar{4}&\bar{3} \\
8&7&9&\bar{1}&\bar{0}&\mathbf{\bar{7}}&0&0&0&2&1&\mathbf{\bar{6}}&5&4&6&\bar{4}&\bar{3}&\bar{5} \\
\bar{0}&\bar{2}&\bar{1}&\bar{3}&\bar{5}&\bar{4}&\mathbf{\bar{6}}&3&2&0&0&0&\mathbf{\bar{7}}&9&8&4&6&5 \\
\bar{2}&\bar{1}&\bar{0}&\bar{5}&\bar{4}&\bar{3}&3&\mathbf{\bar{6}}&1&0&0&0&9&\mathbf{\bar{7}}&7&6&5&4 \\
\bar{1}&\bar{0}&\bar{2}&\bar{4}&\bar{3}&\bar{5}&2&1&\mathbf{\bar{6}}&0&0&0&8&7&\mathbf{\bar{7}}&5&4&6 \\
\bar{3}&\bar{5}&\bar{4}&1&3&2&4&6&5&\mathbf{\bar{7}}&9&8&0&0&0&\mathbf{\bar{6}}&\bar{2}&\bar{1} \\
\bar{5}&\bar{4}&\bar{3}&3&2&1&6&5&4&9&\mathbf{\bar{7}}&7&0&0&0&\bar{2}&\mathbf{\bar{6}}&\bar{0} \\
\bar{4}&\bar{3}&\bar{5}&2&1&3&5&4&6&8&7&\mathbf{\bar{7}}&0&0&0&\bar{1}&\bar{0}&\mathbf{\bar{6}} \\
1&3&2&7&9&8&\bar{3}&\bar{5}&\bar{4}&4&6&5&\mathbf{\bar{6}}&\bar{2}&\bar{1}&0&0&0 \\
3&2&1&9&8&7&\bar{5}&\bar{4}&\bar{3}&6&5&4&\bar{2}&\mathbf{\bar{6}}&\bar{0}&0&0&0 \\
2&1&3&8&7&9&\bar{4}&\bar{3}&\bar{5}&5&4&6&\bar{1}&\bar{0}&\mathbf{\bar{6}}&0&0&0
\end{pmatrix}$$

To define φ in a general case we have to determine a colour for each edge of G. The position of an edge $\{(i-1)q+j, (k-1)q+l\}$ (without loss of generality

$k \in [i+1, p]$) can be coded using the quadruple i, j, k, l and, analogously, its colour can be expressed as $(\alpha - 1)q + \beta$ for some $\alpha \in [1, p]$ and $\beta \in [1, q]$ (where $\alpha = p$ implies $\beta \in [1, 2]$) and coded by α, β. Concentrating this all a code

$$i\,,\, j\,;\, k\,,\, l\,:\, \alpha\,,\, \beta$$

will mean that

$$\varphi\{(i-1)q + j, (k-1)q + l\} = (\alpha - 1)q + \beta.$$

Codes used in the case 1 are

$$\begin{array}{llllll}
i, & j;k, & l : (i+k-1)_{p-1}, & j*l & \text{for } k \in [i+2, p-1], & (1)\\
i, & j;p, & l : (2i-1)_{p-1}, & j*l & i \in [1, p-2], & (2)\\
i, & j;i+1, & l : (2i)_{p-1}, & j*l & i \in [1, p-2], j \neq l, & (3)\\
p-1, & j;p, & l : p-2, & j*l & j \neq l, & (4)\\
i, & j;i+1, & j : p, & (i)_2. & & (5)
\end{array}$$

Conditions accompanying the codes $(1)-(4)$ are only additional requirements with respect to general conditions concerning i, j, k, l, i.e. $i \in [1, p]$, $k \in [i+1, p]$, $j, l \in [1, q]$.

Of course, $\mathrm{Im}_x(\varphi) \neq \mathrm{Im}_y(\varphi)$ for $x, y \in [1, pq], x \neq y$, is equivalent with $\mathrm{Om}_x(\varphi) \neq \mathrm{Om}_y(\varphi)$, where

$$\mathrm{Om}_v(\varphi) := [1, d+2] - \mathrm{Im}_v(\varphi)$$

is the set of colours omitted by φ at a vertex $v \in V(G)$. To check that φ is proper it suffices to show that $\mathrm{Om}_v(\varphi)$ has two elements for each $v \in V(G)$.

Now we will be interested in the structure of $\mathrm{Im}_v(\varphi)$ for $v = (i-1)q + j$, $i \in [2, p-2]$. The code (1) gives colours of edges $\{(i-1)q + j, (k-1)q + l\}$ for $k \in [1, i-2] \cup [i+2, p-1]$. If $k \geq i+2$ is fixed and l runs over $[1, q]$, colours of corresponding edges form a subset of $\mathrm{Im}_v(\varphi)$, namely

$$\bigcup_{l=1}^{q}\{((i+k-1)_{p-1} - 1)q + (j*l)\} = [((i+k-1)_{p-1} - 1)q + 1, (i+k-1)_{p-1}q];$$

in other words it is the set $I(i+k-1)$ of q consecutive integers terminated by the product of q and the integer in $[1, p-1]$ congruent with $i+k-1$ modulo $p-1$. Analogously for $k \leq i-2$ the involved subset of $\mathrm{Im}_v(\varphi)$ can be formally expressed in the same manner:

$$\bigcup_{l=1}^{q}\{((k+i-1)_{p-1} - 1)q + (l*j)\} = I(i+k-1).$$

The code (2) with respect to $i \le p-2$ yields the subset $I(2i-1)$ of $\mathrm{Im}_v(\varphi)$.
The code (3) corresponds to the sets of colours

$$\bigcup_{\substack{l=1\\ l\neq j}}^{q}\{((2i)_{p-1}-1)q+(j*l)\} = I(2i) - \{((2i)_{p-1}-1)q+j\},$$

$$\bigcup_{\substack{l=1\\ l\neq j}}^{q}\{((2i-2)_{p-1}-1)q+(l*j)\} = I(2i-2) - \{((2i-2)_{p-1}-1)q+j\}.$$

The code (4) does not touch the vertex v while the code (5) brings the colours $(p-1)q+(i)_2$ and $(p-1)q+(i-1)_2$.
Since

$$\bigcup_{k=1}^{i-2}\{i+k-1\} \cup \{2i-2, 2i-1, 2i\} \cup \bigcup_{k=i+2}^{p-1}\{i+k-1\} = [i, i+p-2],$$

which is a set of $p-1$ consecutive integers and corresponding representatives modulo $p-1$ in $[1, p-1]$ fill in completely this set, the set $\mathrm{Im}_v(\varphi)$ does not contain only two colours from $[1, d+2]$, one from $I(2i-2)$ and one from $I(2i)$. Thus finally

$$\mathrm{Om}_v(\varphi) = \{((2i-2)_{p-1}-1)q+j, ((2i)_{p-1}-1)q+j\}.$$

In general, if

$$\mathrm{Om}_{(i-1)q+j}(\varphi) = \{(\alpha-1)q+\beta, (\gamma-1)q+\delta\}$$

for some $i, \alpha, \gamma \in [1, p]$ and $j, \beta, \delta \in [1, q]$ (as before, $\alpha = p$ implies $\beta \in [1, 2]$ and $\gamma = p$ implies $\delta \in [1, 2]$), this fact will be coded by

$$i\,,\,j\;:\;\alpha\,,\,\beta\,;\,\gamma\,,\,\delta\,.$$

Proceeding as above it is easy to derive from "colouring" codes (1) – (5) "omitting" codes (6) – (9) :

$$\begin{array}{llll} 1, & j:2, & j;p, & 2, & & (6)\\ i, & j:(2i-2)_{p-1}, & j;(2i)_{p-1}, & j & \text{for } i \in [2, p-2], & (7)\\ p-1, & j:p-3, & j;p-2, & j, & & (8)\\ p, & j:p-2, & j;p, & 2. & & (9) \end{array}$$

Now it is sufficient to check that all omitted pairs of colours are pairwise different.

Of course, each of codes (6), (8), (9) yields q different omitted pairs (j runs over $[1,q]$).

The code (7) leads to $(p-3)q$ different omitted pairs; indeed, the assumption

$$\mathrm{Om}_{(i_1-1)q+j_1}(\varphi) = \mathrm{Om}_{(i_2-1)q+j_2}(\varphi)$$

for omitted pairs coded by (7) implies $j_1 = j_2$ (a consequence of $j_1, j_2 \in [1,q]$) and either $2i_1 \equiv 2i_2 \pmod{p-1}$, $i_1 = i_2$ ($p-1$ is odd), or

$$2i_1 \equiv 2i_2 - 2 \pmod{p-1}, \qquad 2i_2 \equiv 2i_1 - 2 \pmod{p-1},$$

$$2(i_1 + i_2) \equiv 2(i_1 + i_2) - 4 \pmod{p-1},$$

which is impossible since 4 is not divisible by $p-1$.

Thus we shall be done by proving that the sets of omitted colours coded by different codes are pairwise disjoint. As only codes (6) and (9) force the colour $(p-1)q+2$ to be omitted, it suffices to compare omissions coded by (6) and (9) or by (7) and (8), respectively.

Pairs omitted according to (6) and (9) are different due to $p-2 \not\equiv 2 \pmod{p-1}$.

For the codes (7) and (8) consider that $2i-2 \equiv p-2 \pmod{p-1}$ implies $i = \frac{p}{2}$ and $(2i)_{p-1} = 1 \neq p-3$ (here we see that $p = 4$ must be tackled differently), while from $2i \equiv p-2 \pmod{p-1}$ it follows $i = \frac{p-2}{2}$ and $(2i-2)_{p-1} = p-4 \neq p-3$.

In the rest of our analysis (except the case 6 where a different approach is used) proofs of correctness of φ are left to the reader and only codes describing φ and pairs omitted by φ are presented.

2. For an even $p \in [6,\infty)$ and $q = 2$ we can not proceed as in the case 1— there is no idempotent quasigroup of order 2. However, an appropriate map φ can be taken as follows:

$$\begin{array}{llll}
i, \; j; k, & l : (i+k-1)_{p-1}, & (j+l-1)_2 & \text{for } k \in [i+2, p-1], \\
i, \; j; p, & l : (2i-1)_{p-1}, & (j+l-1)_2 & i \in [1, p-2], \\
2i-1, j; 2i, & 2 : (4i-2)_{p-1}, & 3-j & i \in [1, r-1], \\
p-1, \; j; p, & 2 : p-2, & 3-j, & \\
2i, \; 1; 2i+1, l : (4i)_{p-1}, & & l, & \\
2i-1, j; 2i, & 1 : p, & j, & \\
2i, \; 2; 2i+1, l : p, & & 3-l; &
\end{array}$$

then missing colours are distributed according to the scheme

$$
\begin{array}{lll}
1, \quad j:2, \quad j; \quad p, \quad 3-j, & \\
i, \quad j:a(i,j), 3-j; b(i,j), j & \text{for } i \in [2, p-2], \\
p-1, j:p-3, \ 3-j; p-2, j, & \\
p, \quad 1:p-2, \ 1; \quad p-2, 2, & \\
p, \quad 2:p, \quad 1; \quad p, \quad 2, &
\end{array}
$$

where

$$a(i,j) = 2i - 2 + 2(j-1)((i)_2 - 1),$$

$$b(i,j) = 2i + 2(j-2)((i)_2 - 1).$$

3. If $p = 4$ and $q \in [4, \infty)$, "colouring" relations will be

$$
\begin{array}{llll}
i, j; i+1, l: 3-i, & j * l & \text{for } i \in [1,2], j \neq l, \\
i, j; i+2, l: 3, & j * l & i \in [1,2], \\
1, j; 4, \quad l: 1, & (j+1)_q * (l+1)_q & j \neq l, \\
3, j; 4, \quad l: 2, & (j+2)_q * (l+2)_q & j \neq l, \\
i, j; k, \quad j: 4, & ((i+k+1)/2)_2 & i+k \in \{3,5,7\}
\end{array}
$$

and "omitting" relations

$$
\begin{array}{l}
1, j: 1, (j+1)_q; 2, j, \\
2, j: 1, j; \qquad 2, j, \\
3, j: 1, j; \qquad 2, (j+2)_q, \\
4, j: 1, (j+1)_q; 2, (j+2)_q.
\end{array}
$$

4,5. For $p = 4$ and $q = 3$ (or $q = 2$) we define φ directly by a 12×12 (or 8×8) matrix accompanied by a 2-column matrix showing corresponding pairs of omitted colours.

$$
\begin{pmatrix}
0&0&0&4&6&5&7&9&8&1&3&2\\
0&0&0&\bar{0}&5&4&9&8&7&3&2&1\\
0&0&0&\bar{1}&4&6&8&7&9&2&1&3\\
4&\bar{0}&\bar{1}&0&0&0&1&3&2&7&9&8\\
6&5&4&0&0&0&3&\bar{0}&1&9&8&7\\
5&4&6&0&0&0&\bar{1}&1&3&8&7&9\\
7&9&8&1&3&\bar{1}&0&0&0&\bar{0}&6&5\\
9&8&7&3&\bar{0}&1&0&0&0&6&\bar{1}&4\\
8&7&9&2&1&3&0&0&0&5&\bar{0}&\bar{1}\\
1&3&2&7&9&8&\bar{0}&6&5&0&0&0\\
3&2&1&9&8&7&6&\bar{1}&\bar{0}&0&0&0\\
2&1&3&8&7&9&5&4&\bar{1}&0&0&0
\end{pmatrix}
\begin{pmatrix}
\bar{0}&\bar{1}\\
6&\bar{1}\\
5&\bar{0}\\
5&6\\
2&\bar{1}\\
2&\bar{0}\\
2&4\\
2&5\\
4&6\\
4&\bar{1}\\
4&5\\
6&\bar{0}
\end{pmatrix}
\qquad
\begin{pmatrix}
0&0&7&3&5&6&1&8\\
0&0&8&4&6&5&2&1\\
7&8&0&0&1&2&5&6\\
3&4&0&0&8&7&6&5\\
5&6&1&8&0&0&7&4\\
6&5&2&7&0&0&4&3\\
1&2&5&6&7&4&0&0\\
8&1&6&5&4&3&0&0
\end{pmatrix}
\begin{pmatrix}
2&4\\
3&7\\
3&4\\
1&2\\
2&3\\
1&8\\
3&8\\
2&7
\end{pmatrix}
$$

6. For $p = 2r + 1$, $r \in [2, \infty)$, and $q = 2s + 1$, $s \in [1, \infty)$ we were not able to find an explicit colouring φ. Nevertheless, $\mathrm{Obs}_{d+2}(G) \neq \emptyset$ can be proved in a more complicated way.

First of all note that according to de Werra [8] for any graph (and even multigraph) H with e edges and with chromatic index $\chi'(H) = k$ there exists an *equitable* colouring of edges of H, i.e. a proper colouring $\tau \in [1, k]^{E(H)}$ such that each of its colour classes $\tau^{-1}(i)$, $i \in [1, k]$, satisfies the inequality $\lfloor \frac{e}{k} \rfloor \leq |\tau^{-1}(i)| \leq \lceil \frac{e}{k} \rceil$.

Let τ be an equitable colouring of edges of our graph $G = K(p \times q)$. As G has an odd number pq of vertices, it does not have a 1-factor, hence according to the well-known Vizing's theorem [7] $\chi'(G) = d + 1$ and at any vertex of G exactly one colour is omitted. The graph G has $\frac{dpq}{2}$ edges and a straightforward calculation (employing $p \geq 2$ and $q \geq 2$) shows that colour classes of τ are of cardinality $\frac{pq-3}{2}$ or $\frac{pq-1}{2}$. In other words it means that any colour is missing at one vertex or at three vertices.

Let $\mu(v)$ be the colour missing at a vertex v and let X be a set of representatives of the decomposition of $V(G)$ into subsets with the same missing colour. Then $|X| = d + 1$ and vertices of the set $Y = V(G) - X$ can be decomposed into pairs corresponding to the same colour missing at three vertices. Denote the vertices of Y by $y_1, \ldots, y_{q-1}$. We shall show by induction on i that for every $i \in [1, q-1]$ there exists a matching $M_i = \{\{x_j, y_j\} : j \in [1, i]\}$ in G between vertices of X and vertices of Y fulfilling

$$\tau\{x_i, y_i\} \notin \{\mu(x_j) : j \in [1, i-1]\}, \tag{1.i}$$

$$\tau\{x_i, y_i\} \notin \{\mu(y_j) : j \in [1, i-1]\}, \tag{2.i}$$

$$\{\tau\{x_i, y_i\}, \mu(y_i)\} \notin \{\{\tau\{x_j, y_j\}, \mu(x_j)\} : j \in [1, i-1]\}, \tag{3.i}$$

$$\{\tau\{x_i, y_i\}, \mu(y_i)\} \notin \{\{\tau\{x_j, y_j\}, \mu(y_j)\} : j \in [1, i-1]\}, \tag{4.i}$$

$$\mu(x_i) \notin \{\mu(y_j) : j \in [1, i]\}. \tag{5.i}$$

Suppose therefore $i \in [1, q-1]$ and edges of M_j for $j \in [1, i-1]$ have been already found.

Since φ is proper, at most $i - 1$ vertices of X are joined to y_i by edges with colours $\mu(x_j)$ for $j \in [1, i-1]$ and consequently to satisfy $(1.i)$ at most $i - 1$ vertices of X are defended to be chosen as x_i. Further, due to $(2.i)$ at most $\min\{i-1, s\}$ vertices of X are inadmissible (recall that Y as well as $\mu(Y)$ has s elements). If $(1.i)$ is fulfilled, then $(3.i)$ could be disturbed only if $\tau\{x_i, y_i\} = \tau\{x_j, y_j\}$ and $\mu(y_i) = \mu(x_j)$, hence for at most one value of j; that is why $(3.i)$ decreases the number of possibilities for the choice of x_i at most by 1. The same is true for the requirement $(4.i)$ with respect to $(2.i)$. As $|\mu(Y)| = s$ and $\mu(y_j)$ is missing at exactly one vertex of X for each $j \in [1, i]$, the condition $(5.i)$ discriminates against at most $\min\{i, s\}$ vertices of X. The last restriction $x_i \notin \{x_j : j \in [1, i-1]\}$ follows from the fact that M has to be a matching.

The degree of y_i in G is d, the number of neighbours of y_i in X is at least $d-(q-2)$ and the number of vertices which can be chosen in the role of x_i is at least

$$
\begin{aligned}
& d-(q-2)-((i-1)+\min\{i-1,s\}+1+1+\min\{i,s\}+(i-1)) \\
\geq & d-(q-2+2i+2s) \geq d-(q-2+2(q-1)+2s)=(p-5)q+5 \geq 5;
\end{aligned}
$$

it means that a new edge $\{x_i, y_i\}$ of M_i can be correctly determined and the induction works.

Define a $(d+2)$-edge-colouring $\bar{\tau}$ of G by

$$
\begin{aligned}
\bar{\tau}\{x_i, y_i\} &= d+2 && \text{for } i \in [1, q-1], \\
\bar{\tau}(e) &= \tau(e) && \text{otherwise.}
\end{aligned}
$$

Evidently, $\bar{\tau}$ is proper as τ is, and the set $\mathrm{Om}_v(\bar{\tau})$ consists of $\mu(v)$ and

$$
\begin{aligned}
& d+2 && \text{if } v \in X-\{x_j : j \in [1, q-1]\}, \\
& \tau\{x_i, y_i\} && \text{if } v \in \{x_i, y_i\} \text{ for some } i \in [1, q-1].
\end{aligned}
$$

To check that $\bar{\tau}$ distinguishes vertices of G take $v_1, v_2 \in X-\{x_l : l \in [1, q-1]\}$, $v_1 \neq v_2$, and $i, j, k \in [1, q-1]$, $i > j$. Then $\mu(v_1) \neq \mu(v_2)$ (note that μ restricted to X is a bijection from X onto $[1, d+1]$) and consequently

$$\mathrm{Om}_{v_1}(\bar{\tau}) \neq \mathrm{Om}_{v_2}(\bar{\tau}).$$

Due to $\mu(x_i) \neq \mu(x_j)$ and $(1.i)$ we get

$$\mathrm{Om}_{x_i}(\bar{\tau}) \neq \mathrm{Om}_{x_j}(\bar{\tau}).$$

The requirements $(3.i)$ and $(4.i)$ represent immediately

$$\mathrm{Om}_{y_i}(\bar{\tau}) \notin \{\mathrm{Om}_{x_j}(\bar{\tau}), \mathrm{Om}_{y_j}(\bar{\tau})\}.$$

According to $(5.k)$ $\mu(x_k) \neq \mu(y_k)$ and

$$\mathrm{Om}_{x_k}(\bar{\tau}) \neq \mathrm{Om}_{y_k}(\bar{\tau}).$$

From $(2.i)$ and $(5.i)$ it follows

$$\mathrm{Om}_{x_i}(\bar{\tau}) \neq \mathrm{Om}_{y_j}(\bar{\tau}).$$

Finally, we have also

$$\mathrm{Om}_{v_1}(\bar{\tau}) \notin \{\mathrm{Om}_{x_k}(\bar{\tau}), \mathrm{Om}_{y_k}(\bar{\tau})\}$$

for

$$\mathrm{Om}_{v_1}(\bar{\tau}) \ni d+2 \notin \mathrm{Om}_{x_k}(\bar{\tau}) \cup \mathrm{Om}_{y_k}(\bar{\tau}).$$

Resuming all proved assertions we obtain $\bar{\tau} \in \mathrm{Obs}_{d+2}(G)$ and $\mathrm{obs}(G) = d+2$.

7. For $p = 3$ and $q = 2s+1$, $s \in [1,\infty)$, $\mathrm{Obs}_{2q+2}(G)$ contains the map φ given by

$1,j;2,l:1,$	$(j+l-1)_q$	for $j \in [1,s+1]-\{l\}$,
$1,j;2,l:2,$	$(j+l-1)_q$	$j \in [s+2,q]-\{l\}$,
$1,j;3,l:2,$	$(j+l-1)_q$	$j \in [1,s+1]$,
$1,j;3,l:1,$	$(j+l-1)_q$	$j \in [s+2,q]$,
$2,j;3,l:3-((j-l)_q)_2,(j+l)/2$		$j \equiv l \pmod 2, j \neq l$,
$2,j;3,l:3-((j-l)_q)_2,((j+l+q)/2)_q$		$j \not\equiv l \pmod 2$,
$1,j;2,j:3,$	$1,$	
$2,j;3,j:3,$	2	

with omitted colours

$1,j:1,(2j-1)_q;3,2$	for $j \in [1,s+1]$,
$1,j:2,(2j-1)_q;3,2$	$j \in [s+2,q]$,
$2,j:1,(2j-1)_q;2,j$	$j \in [1,s+1]$,
$2,j:2,(2j-1)_q;2,j$	$j \in [s+2,q]$,
$3,j:2,j;\quad 3,1.$	

8. For $p = 2r+1$, $r \in [1,\infty)$, and $q = 2s$, $s \in \{1\} \cup [3,\infty)$, we shall use again an idempotent quasigroup Q_s of order s (besides $s \in [3,\infty)$ it trivially exists for $s = 1$, too). We define a map $\varphi \in \mathrm{Obs}_{d+2}(G)$ by help of the notation

$$i,j,k;l,m,n:\alpha,\beta,\gamma$$

corresponding to

$$\varphi\{(i-1)q+(j-1)s+k,(l-1)q+(m-1)s+n\} = (\alpha-1)\cdot 4s+(\beta-1)s+\gamma$$

for some

$$i \in [1,p],\quad l \in [i+1,p],\quad j,m \in [1,2],\quad k,n \in [1,s],$$

$$\alpha \in [1,r+1],\quad \beta \in [1,4],\quad \gamma \in [1,\max\{s,2\}];$$

$\alpha \in [1,r]$ is always accompanied by $\gamma \in [1,s]$ and $\alpha = r+1$ by $\beta = 1$ and $\gamma \in [1,2]$. (In this case blocks of the matrix describing φ are block matrices 2×2 with blocks $s \times s$.) The codes for φ are

$$\begin{array}{llll}
i, & j,k;l, & m, & n: c(i,l,r), d(i,j,l,m), k*n, \\
i, & j,k;l, & m, & n: c(i,l,r), d(l,m,i,j), n*k, \\
i, & j,k;p, & m, & n: i, \quad e(j,m), \quad k*n, \\
i, & j,k;p, & m, & n: i-r, \quad f(j,m), \quad n*k, \\
p-1, & j,k;p, & m, & n: r, \quad f(j,m), \quad n*k, \\
i, & j,k;i+1, & m, & n: i, \quad g(j,m), \quad k*n, \\
i, & j,k;i+1, & m, & n: i-r, \quad g(m,j), \quad n*k, \\
i, & j,k;i+1, & j, & k: r+1, \quad 1, \quad (i)_2, \\
i, & j,k;i+1, & 3-j, & k: r+1, \quad 1, \quad (i)_2,
\end{array}$$

where

$$\begin{aligned}
c(i,l,r) &= ((2i+2l+(-1)^{i+l}-1)/4)_r, \\
d(i,j,l,m) &= ((-1)^{i+l}(2j+4m-9)+5)/2, \\
e(j,m) &= 7j+6m-4jm-8, \\
f(j,m) &= 2jm-j-3m+4, \\
g(j,m) &= 7-j-2m
\end{aligned}$$

and conditions for parameters are (i) $l \in [i+2, p-i]$ (for the first code), (ii) $l \in [\max\{p-i+1, i+2\}, p-1]$, (iii) $i \in [1,r]$, (iv) $i \in [r+1, p-2]$, (v) $(j,k) \neq (3-m, n)$, (vi) $i \in [1,r], (j,k) \neq (m,n)$, (vii) $i \in [r+1, p-2], (j,k) \neq (3-m,n)$, (viii) $i \in [1,r]$, (ix) $i \in [r+1, p-1]$.

For omitted colours (using an analogous coding scheme) we have

$$\begin{array}{lllll}
1, & j,k:1, & 7-3j,k;r+1,1, & 2, & \\
2, & j,k:1, & 7-3j,k;1, & 2j-1,k & \text{for } r=1, \\
i, & j,k:i-1, & 7-3j,k;i, & 7-3j,k & i \in [2,r], \\
r+1, & j,k:1, & 4-j, \ k;r, & 7-3j,k & r in [2,\infty), \\
i, & j,k:i-r-1, & j+1, \ k;i-r, & 4-j, \ k & i \in [r+2, p-2], \\
p-1, & j,k:r-1, & j+1, \ k;r, & 2j-1,k & r \in [2,\infty), \\
p, & j,k:r, & 5-2j,k;r+1,1, & 1. &
\end{array}$$

9. For $p = 2r+1$, $r \in [1,\infty)$, and $q = 4$ determine φ in the following way:

$$\begin{array}{llllll}
i, & j,k;l, & m,n: c(i,l,r), & d(i,j,l,m), & (k+n-1)_2, \\
i, & j,k;l, & m,n: c(i,l,r), & d(l,m,i,j), & (k+n-1)_2, \\
i, & j,k;p, & m,n: i, & e(j,m), & (k+n-1)_2, \\
i, & j,k;p, & m,n: i-r, & f(j,m), & (k+n-1)_2, \\
i, & j,k;i+1, & m,n: i, & g(j,m), & (k+n-1)_2, \\
i, & j,k;i+1, & m,n: i-r, & g(m,j), & (k+n)_2, \\
p-1, & j,k;p, & m,n: r, & f(j,m), & (k+n)_2, \\
i, & j,k;i+1, & k,j: r+1, & 1, & (i)_2;
\end{array}$$

parameters fulfil conditions (i) $l \in [i+2, p-i]$, (ii) $l \in [\max\{p-i+1, i+2\}, p-1]$, (iii) $i \in [1,r]$, (iv) $i \in [r+1, p-2]$, (v) $i \in [1,r]$, $(j,k) \neq (n,m)$, (vi) $i \in [r+1, p-2]$, $(j,k) \neq (n,m)$, (vii) $(j,k) \neq (n,m)$.

Omitted colours are given by

$$\begin{array}{llllll}
1, & j,k:1, & g(j,k),(j+k-1)_2; & r+1,1, & 2, \\
2, & j,k:1, & g(k,j),(j+k-1)_2; & 1, & f(j,k),(j+k)_2, \\
i, & j,k:i-1, & g(k,j),(j+k-1)_2; & i, & g(j,k),(j+k-1)_2, \\
r+1, & j,k:r, & g(k,j),(j+k-1)_2; & 1, & g(k,j),(j+k)_2, \\
i, & j,k:i-r-1, & g(j,k),(j+k)_2; & i-r, & g(k,j),(j+k)_2, \\
p-1, & j,k:r-1, & g(j,k),(j+k)_2; & r, & f(j,k),(j+k)_2, \\
p, & j,k:r, & f(k,j),(j+k)_2; & r+1,1, & 1
\end{array}$$

with parameters conditioned by (ii) $r = 1$, (iii) $i \in [2,r]$, (iv) $r \in [2,\infty)$, (v) $i \in [r+2, p-2]$, (vi) $r \in [2,\infty)$. ∎

The result of this paper could be considered as the first step in determining observability for any complete multipartite graph. Another open problem consists in getting the proof of our Theorem "purely" constructive (i.e. in finding an explicit map φ in the case 6).

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