Distinct Degrees Determined by Subgraphs

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Abstract. We show that if a graph G has n non-isomorphic 2-vertex deleted subgraphs then G has at most n distinct degrees. In addition we prove that if G has 3 non-isomorphic 3-vertex deleted subgraphs then G has at most 3 different degrees.

By a graph we shall mean an undirected graph with no loops or multiple edges. Let G be a graph and let G_v be the subgraph of G obtained by deleting vertex v and all its incident edges. A graph H is said to be a *reconstruction* of G if $V(G) = V(H)$ and $G_v \simeq H_v$ for all $v \in V(G)$. If $G \simeq H$ for all reconstructions H, then G is said to be *reconstructible.* The reconstruction (Ulam's) conjecture states that any finite simple graph on three or more vertices is reconstructible. Though the conjecture is yet to be settled, surveys of progress can be found in [1] and [5].

An immediate but useful result relating vertex-deleted subgraphs to their supergraph is the following.

Fact. *If a graph G has n non-isomorphic vertex-deleted subgraphs, then G has at most n distinct degrees.*

Various generalizations of reconstructability have been put forward, including the possible reconstruction of graphs from their n-vertex-deleted subgraphs per Kelly's conjecture in [4]. Giles [2] showed that trees can be reconstructed from their 2-vertex-deleted subgraphs and McAvaney [6, 7] has offered evidence that cartesian product graphs may be so reconstructible.

First we prove the analogous fact for the 2-vertex-deleted subgraphs of a graph G.

Theorem 1. *Let G be a graph on $p+2$ vertices, $p \geq 3$, which has n non-isomorphic subgraphs on p vertices. Then G has at most n different degrees.*

Note that when $n = 1$ the statement clearly holds and Theorem 2 of [3] states that if a graph G with order $p+2$ has 2 subgraphs of order p then G or its complement must be one of $K_{p+2} - e$, $K_{\frac{p+1}{2}, \frac{p+3}{2}}$, $\frac{p+1}{2} K_2 \cup K_1$, $K_{1,n}$ or a strongly regular graph thus covering the case $n = 2$, so henceforward $n \geq 3$.

Proof. The proof will consist of three lemmas.

Lemma 1.1. *Let G be a graph with $p+2$ vertices. If G contains vertices with degrees $\alpha, \alpha+1, \alpha+2, \ldots, \alpha+n$, for any α, then G has more than n non-isomorphic subgraphs on p vertices.*

Proof. Suppose G contains vertices with the given degrees and has at most n subgraphs on p vertices. Then by deleting vertices with all possible pairs of distinct degrees we have deleted total degrees of $2\alpha+i$, for $i = 1, 2, 3, \ldots, 2n-1$. When an edge exists between u and v, the two vertices deleted, the actual number of edges deleted is clearly $\deg(u)+\deg(v)-1$. Then G must contain the necessary edges so that these $2n-1$ totals reduce to at most n different numbers of actual edges deleted. These n values must be of the form $2\alpha+i$, for $i = 1, 3, 5, \ldots, j-1, j, j+2, \ldots, 2n-2$, where $0 \le j \le 2n$ is even.

If $j > n+2$ then all even totals $t \le n+2$ must be reduced to $t-1$. (Note that there can be only one vertex of degree α, else we could delete two of these giving another total of $i = 0$, whence $j = 0$.) Thus the vertex of degree α is adjacent to all vertices of degree $\alpha+k$, for all even k, and no others. The vertices of degree $\alpha+2$ must also be adjacent to all vertices of degree $\alpha+k$ including others of degree $\alpha+2$ and no others. Let $|\alpha+k|$ denote the number of vertices of degree $\alpha+k$, thus we have

$$\begin{aligned} \alpha &= |\alpha+2| + |\alpha+4| + \ldots \\ \alpha+2 &= |\alpha| + |\alpha+2| - 1 + |\alpha+4| + \ldots \\ &= 1 + |\alpha+2| - 1 + |\alpha+4| + \ldots \\ &= \alpha, \text{ a contradiction.} \end{aligned}$$

Similarly, if $j \le n-2$, there is a single vertex of degree $\alpha+n$. That vertex, along with all vertices of degree $\alpha+n-2$ must be adjacent to all vertices of degree $\alpha+k$, where k has parity opposite that of n. Now we find the contradiction that $\alpha+n-2 = \alpha+n$.

If n is even we consider the cases $j = n, n+2$. If $j = n$, the vertex of degree α is adjacent to all vertices of degree $\alpha+k$, for all even k, except possibly the vertex of degree $\alpha+n$. The vertices of degree $\alpha+2$ are also adjacent to all vertices of degree $\alpha+k$, for all even k, except possibly those of degree $\alpha+n-2$ and $\alpha+n$. Thus

$$\alpha = |\alpha+2| + |\alpha+4| + \ldots + |\alpha+n-2| + \begin{cases} 0 \\ 1 \end{cases} \tag{1}$$

$$\alpha+2 = |\alpha+2| + |\alpha+4| + \ldots + \beta|\alpha+n-2| + \begin{cases} 0 \\ 1 \end{cases}, \tag{2}$$

where $\beta \le 1$. Then (2) $-$ (1) yields

$$2 + (1-\beta)|\alpha+n-2| = \begin{cases} 1 \\ 0 \\ -1 \end{cases}.$$

However, since $\beta \leq 1, 2+(1-\beta)|\alpha+n-2| \geq 2$, a contradiction.

Now suppose $j = n+2$. Then the vertex of degree α is adjacent to all vertices of degree $\alpha + k$, for all even k, and the vertices of degree $\alpha + 2$ are adjacent to all vertices of degree $\alpha + k$, for all even k, except possibly the vertex of degree $\alpha + n$. So $\alpha + 2 \leq \alpha$, a contradiction.

If n is odd we consider the cases $j = n-1, n+1$. If $j = n-1$ then the vertex of degree $\alpha + n$ is adjacent to all vertices of degree $\alpha + k$, for all even k, and the vertices of degree $\alpha + n - 2$ are adjacent to all vertices of degree $\alpha + k$, for all even k, except the one of degree α. Then

$$\alpha + n = 1 + |\alpha + 2| + |\alpha + 4| + \ldots \text{ and}$$
$$\alpha + n - 2 = |\alpha + 2| + |\alpha + 4| + \ldots, \text{ a contradiction.}$$

Finally, if $j = n+1$, the vertex of degree α is adjacent to all vertices of degree $\alpha + k$, k even, and all vertices of degree $\alpha + 2$ are adjacent to all vertices of degree $\alpha + k$ except $k = n - 1$. Then we have a similar contradiction where $\alpha = \alpha + 2$. Thus the lemma is proved. ∎

Lemma 1.2. *Let G be a graph on p + 2 vertices which has three non-isomorphic subgraphs on p vertices. Then G has at most three different degrees.*

Proof. Suppose G has four degrees: $\alpha, \alpha + m_1, \alpha + m_2, \alpha + m_3$, where $0 < m_1 < m_2 < m_3$. Then in deleting two vertices of distinct degrees there are total deleted degrees of $2\alpha + i$, for $i = m_1, m_2, m_3, m_1 + m_2, m_1 + m_3, m_2 + m_3$. These must be reduced to at most three totals of actual edges deleted.

Assume first that there are only five totals, i.e. $m_1 + m_2 = m_3$. If $m_2 \neq m_1 + 1$, then $m_2 + m_3 = m_1 + m_3 + 1$ which implies $m_2 = m_1 + 1$, a contradiction. Using $m_2 = m_1 + 1$ then the five values are $2\alpha + i$, for $i = m_1, m_1 + 1, 2m_1 + 1, 3m_1 + 1, 3m_1 + 2$. So the three numbers of edges must be $2\alpha + i$, where $i = m_1, 2m_1 + 1, 3m_1 + 1$. Then in G the vertices of degree $\alpha + m_1$ cannot be adjacent to vertices of any other degree and two vertices of degree $\alpha + m_1$ cannot be adjacent to each other since $2m_1 - 1 = m_1$ implies that $m_1 = 1$ and the degrees are $\alpha, \alpha+1, \alpha+2, \alpha+3$, and we apply the previous lemma. But then $\alpha + m_1 = 0$.

Assuming second that there are six totals, $m_1 + m_2 = m_3 \pm 1$. As before, $m_2 = m_1 + 1$. If $m_1 + m_2 = m_3 + 1$ then the six values are $m_1, m_1 + 1, 2m_1, 2m_1 + 1, 3m_1, 3m_1 + 1$. Any vertex of degree α or $\alpha + 2m_1$ is adjacent to no vertex of degree α or $\alpha + m_1$ or $\alpha + 2m_2$ and is adjacent to every vertex of degree $\alpha + m_1 + 1$ so the number of vertices of degree $\alpha + m_1 + 1$ equals both α and $\alpha + 2m_1$, a contradiction. If $m_1 + m_2 = m_3 - 1$, then the six values are $m_1, m_1 + 1, 2m_1 + 1, 2m_1 + 2, 3m_1 + 2, 3m_1 + 3$. Then in G the vertices of degree $\alpha + m_1$ cannot be adjacent to any other vertex, a contradiction. ∎

Lemma 1.3. *Let G be a graph on $p+2$ vertices which has $n \geq 4$ nonisomorphic subgraphs on p vertices. Then G has at most n different degrees.*

Proof. Let the degrees be $\alpha, \alpha+m_1, \alpha+m_2, \ldots, \alpha+m_n$, where $0 < m_1 < m_2 < \ldots < m_n$. If we delete in turn all possible pairs of vertices with different degrees, we have total degrees of deleted pairs $2\alpha + i$, where $i \in \{m_j : 1 \leq j \leq n\} \cup \{m_j + m_k : 1 \leq j < k \leq n\}$. This contains at least $2n-1$ distinct values, namely the increasing sequence $A = \{m_1, m_2, m_1+m_2, m_1+m_3, m_2+m_3, m_2+m_4, \ldots, m_{n-2}+m_{n-1}, m_{n-2}+m_n, m_{n-1}+m_n\}$, but no more than $2n$ distinct values. These reduce to at most n numbers of edges, so A must contain $n-1$ pairs $x, x+1$ and one unpaired number. The unpaired term may be the first the last, or an intermediate term, but in each case it follows that $m_1, m_2, \ldots, m_{n-1}$ are consecutive integers.

If $m_1 > 1$ and $m_n > m_{n-1}+1$ then consider the increasing sequence $\{m_1, m_2, m_3, \text{(gap)}\, m_1+m_3, m_1+m_4, \ldots, m_1+m_{n-1}, \text{(gap)}\, m_1+m_n, m_2+m_n, \ldots, m_{n-1}+m_n\}$ if n is even, or $\{m_1, m_2, m_3, \text{(gap)}\, m_1+m_3, m_2+m_3, m_2+m_4, \ldots, m_2+m_{n-1}, \text{(gap)}\, m_2+m_n, m_3+m_n, \ldots, m_{n-1}+m_n\}$ if n is odd. In either case, the sequence contains $2n-1$ terms, yet consists of three subsequences of consecutive terms, each subsequence of odd length. Therefore there cannot be $n-1$ pairs $x, x+1$ so either $m_1 = 1$ or $m_n = m_{n-1}+1$.

If $m_n > m_{n-1}+2$, consider the increasing sequence $\{m_1, m_2, \ldots, m_{n-1}, m_1+m_{n-1}, m_2+m_{n-1}, m_n, m_1+m_n, \ldots, m_{n-1}+m_n\}$ and if $m_1 > 2$, consider the increasing sequence $\{m_1, m_2, \ldots, m_n, m_1+m_{n-2}, m_1+m_{n-1}, m_1+m_n, m_2+m_n, \ldots, m_{n-1}+m_n\}$. In either case, the number of total degrees of deleted pairs is at least $2n+1$, a contradiction.

Consider the case $m_j = j$ for $j = 1, \ldots, n-1$ and $m_n = n+1$. The sums of pairs of degrees are $1, 2, \ldots, 2n$ and the number of deleted edges must always be odd. Vertices of degrees α and $\alpha+2$ are adjacent to precisely those vertices of degree $\alpha+k$ where k is even, hence $\alpha = \alpha+2$.

Finally, consider the case $m_j = j+1$ for $j = 1, 2, \ldots, n$. Then the set of degree totals is $2, 3, \ldots, 2n+1$ and the number of deleted edges must always be even. Vertices of degrees α and $\alpha+2$ are adjacent to precisely those vertices of degrees $\alpha+k$, k odd, hence again $\alpha = \alpha+2$, and this contradiction completes the proof. ▮

Theorem 2. *Let $|G| = p+3$ and suppose G has exactly three nonisomorphic subgraphs of order p. Then G has at most three different degrees.*

We will also prove this theorem with a series of lemmas.

Lemma 2.1. *Let $|G| = p+3$ and suppose G has at least four distinct degrees,*

a, b, c, d, where $b-a, c-b, d-c$ are all ≥ 2. Then G has more than three subgraphs of order p.

Proof. Choose one vertex A, B, C, D, of each of the four degrees a, b, c, d and one other vertex K of degree k, where k may or may not be one of a, b, c, d. We delete K along with the six pairs in turn in $\{A, B, C, D\}$. Then the numbers of edges deleted are

(1) $a + b + k - \epsilon_A - \epsilon_B - \epsilon_{AB}$,

(2) $a + c + k - \epsilon_A - \epsilon_C - \epsilon_{AC}$,

(3) $a + d + k - \epsilon_A - \epsilon_D - \epsilon_{AD}$,

(4) $b + c + k - \epsilon_B - \epsilon_C - \epsilon_{BC}$,

(5) $b + d + k - \epsilon_B - \epsilon_D - \epsilon_{BD}$,

(6) $c + d + k - \epsilon_C - \epsilon_D - \epsilon_{CD}$,

where $\epsilon_X = 1$ or 0 as there is or is not an edge between K and X and $\epsilon_{XY} = 1$ or 0 as there is or is not an edge between X and Y.

Suppose that there are exactly three subgraphs of order p. Then there must be at least three pairs of equal expressions among the above six. It is clear that (1) $\neq$ (3), (1) $\neq$ (4), (1) $\neq$ (5), (1) $\neq$ (6), (2) $\neq$ (5), (2) $\neq$ (6), (3) $\neq$ (6), (4) $\neq$(6).

Possible equalities have implications as follows.

(1) = (2) $\Rightarrow c = b + 2, \quad \epsilon_C = \epsilon_{AC} = 1, \quad \epsilon_B = \epsilon_{AB} = 0.$

(2) = (3) $\Rightarrow d = c + 2, \quad \epsilon_D = \epsilon_{AD} = 1, \quad \epsilon_C = \epsilon_{AC} = 0.$

(2) = (4) $\Rightarrow b = a + 2, \quad \epsilon_B = \epsilon_{BC} = 1, \quad \epsilon_C = \epsilon_{AC} = 0.$

(3) = (4) gives no additional information.

(3) = (5) $\Rightarrow b = a + 2, \quad \epsilon_B = \epsilon_{BD} = 1, \quad \epsilon_A = \epsilon_{AD} = 0.$

(4) = (5) $\Rightarrow d = c + 2, \quad \epsilon_D = \epsilon_{BD} = 1, \quad \epsilon_C = \epsilon_{BC} = 0.$

(5) = (6) $\Rightarrow c = b + 2, \quad \epsilon_C = \epsilon_{CD} = 1, \quad \epsilon_B = \epsilon_{BD} = 0.$

Suppose (2) = (3). Then (1) $\neq$ (2), (3) $\neq$ (4), (3) $\neq$ (5), and (5) $\neq$ (6). But we may have (2) = (4) and/or (4) = (5). In fact we must have both of these equalities in order to reduce to at most three numbers of edges. But then (3) = (5), a contradiction.

If (2) = (4) then (1) $\neq$ (2) and (5) $\neq$ (6) so we must have (2) = (3) = (4) = (5) but again this requires (2) = (3) and (3) = (5). We have the same situation if (4) = (5) or (3) = (5) so the only remaining possible set of equalities is (1) = (2), (3) = (4), and (5) = (6).

Thus for any vertex V that we choose to delete along with some pair in $\{A, B, C, D\}$, the same equalities must hold, so, C must be adjacent to every vertex except possibly B. Then $c \geq p + 1$ and thus $d \geq p + 3$, a contradiction. ∎

Lemma 2.2. *Let $|G| = p + 3$ and suppose G has at least four distinct degrees, a, b, c, d, where at least one of the differences $b - a$, $c - b$, $d - c$, is ≥ 2. Then G has more than three subgraphs of order p.*

Outline of Proof: The cases to be considered are

(1) $b - a \geq 2, \quad c - b \geq 2, \quad d - c = 1$

(2) $b - a \geq 2, \quad c - b = 1, \quad d - c \geq 2$

(3) $b - a = 1, \quad c - b \geq 2, \quad d - c = 1$

(4) $b - a = 1, \quad c - b = 1, \quad d - c \geq 2$.

Each of these is easily proved using the same technique as in the proof of Lemma 2.1. Case (4) requires more subcases, but the technique still works. The other possibilities,

(5) $b - a = 1, \quad c - b \geq 2, \quad d - c \geq 2$

(6) $b - a \geq 2, \quad c - b = 1, \quad d - c = 1$,

are the complements of cases (1) and (4). ∎

When all of the differences between degrees are exactly one, the proof is not nearly so compact. In fact there are enough cases that computer support was used for the proof of the next lemma.

Lemma 2.3. *Let $|G| = p + 3$ and suppose G has the four degrees, a, $a + 1$, $a + 2$, $a + 3$, for some a. Then G has more than three subgraphs of order p.*

Outline of Proof: Suppose G has at most three subgraphs of order p. If we delete any vertex v from G, the remaining vertices can have degrees $a - 1, a, a + 1, a + 2, a + 3$. $G - v$ has at most three subgraphs of order p, so by Lemma 1.2 $G - v$ can have at most three degrees. We assume for now that there are at least two vertices of each degree in G. The possible degrees in $G - v$ are (1) $a - 1, a, a + 2$; (2) $a - 1, a + 1, a + 2$; (3) $a - 1, a + 1, a + 3$; (4) $a, a + 1, a + 3$; (5)

$a, a+1, a+2$; (6) $a, a+2, a+3$; (7) $a, a+2$. So we will say that the vertex v is of type 1,2,...,7, depending on the degrees which exist in $G-v$. In particular, if we choose one vertex of each of the four degrees in G, there are seven choices of type for each one and thus 7^4 total cases.

To deal with these cases in a reasonable manner, we consider a number of consequences of the above. The deletion of v from G of course deletes $\deg(v)$ edges from G. The two other vertices u_1, u_2 deleted from $G-v$ result in $\deg_{G-v}(u_1)+\deg_{G-v}(u_2)$ or $\deg_{G-v}(u_1)+\deg_{G-v}(u_2)-1$ more edges being deleted depending on the existence of a non-edge or edge between u_1 and u_2. So we consider all the sums $\deg_{G-v}(u_1)+\deg_{G-v}(u_2)+\deg_G(v)$. When considering in turn v_0, v_1, v_2, v_3, where $\deg_G(v_i) = a+i$, we find a number of different sums. If this set of sums is sufficiently spread, then the effect of potentially subtracting one from some of them will not be enough to reduce the number of different numbers of edges deleted to three.

If a vertex v_i is of type j then v_i must be adjacent to all vertices of certain degrees and non-adjacent to certain others. This information is summarized as follows.

Type	1	2	3	4	5	6	7
Adjacent to all v of degree	$a+1$ $a+3$	a $a+3$	a $a+2$	$a+2$	$a+3$	$a+1$	$a+1$ $a+3$
Non-adjacent to all v of degree	$a+2$	$a+1$	$a+1$ $a+3$	a $a+3$	a	a $a+2$	a $a+2$

Table 1

From the data in Table 1 we see that there are many other cases which cannot occur. For example, if v_2 and v_3 are both of type 1, v_2 must be adjacent to all vertices of degree $a+3$ but v_3 can be adjacent to no vertices of degree $a+2$, a contradiction. These contradictory results we summarize in Table 2. If two vertices are of types which appear in a single column, then we have a contradiction. With the aid of a simple computer program, the 7^4 cases were reduced to 65 which were not eliminated by one of these contradictions.

v_0	1,6,7			2,3			3,4			1,6,7	3,4	1,2,5,7
v_1		1,6,7		2,3	3,4	1,2,5,7		3,4		4,5,6,7		
v_2	2,3	1,6,7	1,2,5,7		2,3				3,4		4,5,6,7	
v_3			1,6,7			2,3	2,3	1,6,7	3,4			4,5,6,7

Table 2

Each of these 65 cases was handled in a manner similar to the following examples.

i) Let v_0 be of type 4, v_1 of type 2, v_2 of type 1, and v_3 of type 4. The sums $\deg_{G-v}(u_1)+\deg_{G-v}(u_2)+\deg_G(v)$ over all v chosen from $\{v_0,\ldots,v_3\}$ and then u_1 and u_2 chosen from different degrees of $G-v$ depending on the type of the v, are $3a+i$ for $i=1,2,3,4,6,7$. The only way that these sums might be reduced to three numbers of edges is if there exists an edge between any pair u_1, u_2 which gives a sum of 2, 4, or 7. For shorthand we write $2\to 1,\ 4\to 3,\ 7\to 6$. If we delete v_1, resulting in vertices of degrees $a-1,\ a+1,\ a+2$, and then a vertex of degree $a-1$ and one of degree $a+2$, we have a sum of 2. Thus every vertex of degree $a-1$ must be adjacent to every vertex of degree $a+2$. However, all vertices of degree a $(a+3$, resp.) in G have degree $a-1(a+2$, resp.) in $G-v_1$, and v_0 and v_3 are type 4. So v_0 cannot be adjacent to any vertices of degree $a+3$ and v_3 cannot be adjacent to any vertices of degree a, and so this case is eliminated.

ii) Let v_0 be of type 4, v_1 of type 5, v_2 of type 1, and v_3 of type 4. The sums are the same as in example (i), so $2\to 1,\ 4\to 3$, and $7\to 6$. A sum of 1 occurs when we delete v_0, and vertices u_1 and u_2 of degree a and $a+1$ in $G-v_0$, so no edges can exist between u_1 and u_2 in $G-v_0$. Now all vertices in G of degree $a+2$ are of degree $a+1$ in $G-v_0$ so (A) v_0 must be the only vertex of degree a which is adjacent to any vertices of degree $a+2$ in G. (B) Also, none of the vertices of degree $a+1$ in G can be adjacent to v_0 else letting u_1 be such a vertex and deleting v_0, u_1, v_2 gives $3a$ edges deleted, a contradiction. Now if we delete v_3 and u_1 and u_2 of degree a and $a+1$ in $G-v_3$, we have a sum of 4. If $\deg_G(u_1)=a$ and $\deg_G(u_2)=a+2$, there must be an edge between them, i.e. (C) each vertex of degree a is adjacent to each vertex of degree $a+2$ in G. Facts (A) and (C) contradict each other unless there is only one vertex of degree a. This contradicts our assumption on the number of vertices of a given degree. So this case is also eliminated.

Now if we consider the cases where there is exactly one vertex of given degree(s) we can use the same sort of analysis as above, and similar computer programs to reduce the number of cases to a more or less reasonable number to work out by hand. Note that examples (i) and (ii) above do not depend on the number of vertices of a given degree. Example (ii) appears to depend on the assumption of more than one vertex of degree a, but actually this case cannot occur by fact (B) if there is only one vertex of degree a. Since v_0 is of type 4, its deletion requires vertices of degree a in $G-v_0$. But since v_0 is not adjacent to any vertices of degree $a+1$, $G-v_0$ has no vertices of degree a. Types of vertices other than types 1,. . . ,7 are introduced which cover the possibility of $G-v$ having only degrees $a+1$ and $a+3$ and other special cases where one vertex of some degree exists. ∎

So Theorem 2 is proved. ∎

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