On Harmonious Tree Labelings

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ABSTRACT. In this paper we have investigated harmonious labelings of p-stars, where p-star of length k is a star tree in which each edge is a path of length k. We have also demonstrate application of the labelings to k disjoint p-cycles.

1 Introduction

Harmonious labeling of graphs have been introduced in 1980 by Graham and Sloane [1] which is related with a problem of modular versions of certain additive bases for the integers $1, 2, \ldots, n$. Results on harmonious labelings can be found in a recent survey by Gallian [6]. A connected graph with v vertices and $e \geq v$ edges is called harmonious if it is possible to label the vertices x with distinct elements $h(x)$ of Z_e in such a way that, when edge (x, y) is labeled with $h(x)+h(y)$, the resulting edge labels are all distinct. If the graph is a tree we require exactly one vertex label repeated. It has been conjectured that all trees are harmonious and the proof of this conjecture would be even more complicated than the counter conjecture of graceful trees. This has been first observed in [1] and it will be confirmed for a class of trees considered in this study.

In this paper we have given harmonious labelings for a class of trees called p-stars for which the graceful labelings are known [4]. A p-star is a star tree in which each edge is a path of length k. The labelings given are for all values of p and k except for even values of p when $k \neq 2$. Let us call the p-star tree as $S(k, p)$.

2 Harmonious labeling of $S(k,p)$

Case a1. $p \equiv 1 \pmod 2$ and $k \equiv 1 \pmod 2$.

Let p_x, $1 \le x \le p$ be the p paths of length $(k-1)$ excluding the common vertex; the root of $S(k,p)$. Label the vertices of p_x with the integers

$$j+2i, j+2i-2, \ldots j+2, j, j+2i-1, j+2i-3, \ldots, j+3, j+1$$
$$j = 0, k, 2k, \ldots, (p-1)k \tag{1}$$

where $k = 2i+1$. The above labels covers all distinct integers from 0 to $pk-1$ and thus we are free to assign any integer to the root as the repeated label. The induce edge labels of the paths p_x can be found to be

$$2j+4i-2, 2j+4i-6, \ldots, 2j+6, 2j+2, 2j+2i-1,$$
$$2j+4i-4, 2j+4i-8, \ldots, 2j+8, 2j+4 \pmod{pk}$$
$$j = 0, k, 2k, \ldots, (p-1)k. \tag{2}$$

The missing edge labels from the above edge labels are:

$$0, k, 2k, \ldots, (p-1)k \pmod{pk} \tag{3}$$

From the set of induce edge labels of (1) we derive the possible end-vertex levels of the path p_x to be $\{j+1\}$, $j = 0, k, 2k, \ldots, (p-1)k$. Hence the missing edge labels (3) can be generated by assigning any integer from the set $\{k(i+1)-1\} \pmod{pk}$, $i = 0, 1, \ldots, p-1$ to the root of $t_{k,k}$. The other set of the end-vertex labels of p_x is $\{j+2i\}$, $j = 0, k, 2k, \ldots, (p-1)k$. Hence the missing edge labels (3) can also be generated by assigning any integer from the set $\{ik+1\}$, $i = 0, 1 \ldots, p-1$.

Case a2. $p \equiv 1 \pmod 2$ and $k \equiv 2 \pmod 4$.

In this case label the vertices of the paths p_x as

$$j+(k-1), j+(k-2), \ldots, j+(k-2i+1), j+(k-2i),$$
$$j+(k-2i-2), \ldots, j+2, j, j+1, j+3, \ldots, j+(k-2i-1)$$
$$j = 0, k, 2k, , (p-1)k \tag{4}$$

where $k = 4i+2$.

The induced edge labels of the paths p_x are found to be

$$2j+2k-3, 2j+2k-5, 2j+2k-7, \ldots, 2j+2k-4i+3,$$
$$2j+2k-4i+1, 2j+2k-4i-2, 2j+2k-4i-6, 2j+2k-4i-10, \ldots,$$
$$2j+6, 2j+2, 2j+1, 2j+4, 2j+8, \ldots, 2j+2k-4i-8, 2j+2k-4i$$
$$\pmod{pk}, \quad j = 0, k, 2k, , (p-1)k \tag{5}$$

The above set of integers contain all distinct elements in Z_e except the (missing edge labels) integers:

$$\{(j+1)k-1\} \pmod{pk}, \quad j = 0, 1, \ldots, p-1 \tag{6}$$

The one set of end-vertex labels of the paths p_x is $\{j+(k-2i-1)\}$, $j = 0, k, 2k., \ldots, (p-1)k$. Hence the missing edge labels (6) can be generated by assigning any integer from the set $\{k(j+1)-2i-2\}$, $j = 0, 1, \ldots, p-1$ (mod pk) to the root of $S(k,p)$. The other set of the end-vertex labels (see (4)) of the paths p_x is $\{j+(k-1)\}$, $j = 0, k, \ldots, (p-1)k$. Hence the missing edge labels can also be generated by assigning either integer from the set $\{jk\}$, $j = 0, k, 2k, \ldots, (p-1)k \pmod{pk}$.

Case a3. $p \equiv 1 \pmod 2$ and $k \equiv 0 \pmod 4$.

In this subcase label the vertices of the paths p_x as follows:

$$j+(i+2)/2, j+((i+2)/2)-1, \ldots, j+2, j, j+1, j+3, \ldots,$$
$$j+((i+4)/2)-1, j+((i+4)/2)+1, j+((i+4)/2)+3,$$
$$j+((i+2)/2)+4, j+((i+4)/2)+5, \ldots, j+k-1$$
$$j = 0, k, 2k, \ldots, (p-1)k$$

where $i = k/2$. Similar to the cases a1 and a2, if the root is adjacent to the vertices with labels $\{j+k-1\}$, $j = 0, k, \ldots, (p-1)k$ then assign the either label from the set $\{mk\}$, $m \equiv 0, 1, \ldots, (p-1)$ to the root. Otherwise if the root vertex is adjacent to the end-vertices of p_x with the labels $\{j+(i+2)2\}$ then assign either label from the set $\{2+j\}$, $j = 0, k, 2k, \ldots, (p-1)k$ to realize the missing edge labels. Therefore the tree $S(k,p)$ is harmonious for $p =$ odd and for any k.

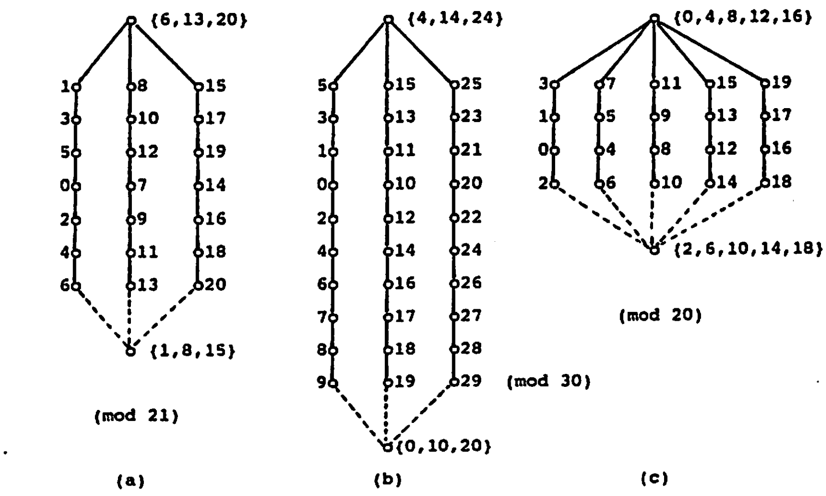

Figure 1

In Figure 1 we illustrate harmonious labelings of $S(k,p)$. The integer set assigned to the root of each tree gives the possible root-labels. The dashed line edges at the bottom of each tree gives another harmonious labelings if the root at the top of the figure is omitted.

3 Some properties of the labelings

In this section we give some interesting properties of the labelings of the previous section. Consider the vertices of $S(k,p)$ as rows of integers. Let $R_i = \{a_{11}, a_{12}, \ldots, a_{ip}\}$, $i = 1, 2, \ldots, k$ where $a_{i,j} \in Z$ is the label of ith vertex (row) of the jth path p_j, $1 \le i \le k$, $1 \le j \le p$ excluding the root.

Property I: Let h be the harmonious labelings given for $S(k,p)$, ($p =$ odd) (see Section 2). Then any cyclic shifts of the vertex labels in R_i, $1 \le i \le k$ also corresponds to an harmonious labeling of $S(k,p)$.

Proof: Take any three consecutive rows of labels R_{i-1}, R_i, R_{i+1}, $2 \le i \le k-1$:

$$\begin{aligned} R_{i-1} &= \{a_{i-1,1}, a_{i-1,2}, \ldots, a_{i-1,p}\} \\ R_i &= \{a_{i,1}, a_{1,2}, \ldots, a_{i,p}\} \\ R_{i+1} &= \{a_{i+1,1}, a_{i+1,2}, \ldots, a_{i+1,p}\} \end{aligned}$$

Consider the labels of the path p_x, $x - 1, 2, \ldots, p$ given in (1),(4) and (7) which may be rewritten as

$$\begin{aligned} R_{i-1} &= \{a'_{i-1,1}, a'_{i-1,2}, \ldots, a'_{i-1,1} + 2k, \ldots, a'_{i-1,1} + (p-1)k\} \\ R_i &= \{a'_{i,1}, a'_{i,1}, +k, a'_{i,1} + 2k, \ldots, a'_{i,1} + (p-1)k\} \\ R_{i+1} &= \{a'_{i+1,1}, a'_{i+1,1}, +k, a'_{i+1,1} + 2k, \ldots, a'_{i+1,1} + (p-1)k\} \end{aligned}$$

where $a'_{i-1,1}, a'_{i,1}, a'_{i+1,1}$ respectively are the $(i-1)$th, ith and $(i+1)$th vertex labels of the first path p_1 of $S(k,p)$.

The induced edge labels generated among the labels of the rows R_{i-1}, R_i, R_{i+1} are found to be

$$\begin{aligned} &\text{between } R_{i-1} \text{ and } R_i : (a'_{i-1,1} + a'_{i,1} + jk) \pmod{pk},\ j = 0, 1, \ldots, p-1 \\ &\text{between } R_i \text{ and } R_{i+1} : (a'_{i+1,1} + a'_{i,1} + jk) \pmod{pk},\ j = 0, 1, \ldots, p-1 \end{aligned} \tag{8}$$

Now let us shift s-steps the labels of R_i to the right cyclically. Let R'_i be the new row of labels. It can be verified that the edge labels between R_{i-1} and R'_i and between R'_i and R_{i+1} are

$$(a'_{i-1,1} + a'_{i,1} + (j + s + 1)\ {}_{(\mathrm{mod}\ s+2)} k) \quad j = 0, 1, \ldots, p-1.$$

which is same with (8). That is the effect of s-shift of the labels of R_i to the right cyclically is that the edge labels between R_{i-1} and R_i and R_i and R_{i+1} are shifted s steps to the left cyclically.

As an illustration to Property 1 the harmonious labeling of $S(5,4)$ has been obtained from the labeling of the same tree of Figure 1c by shifting, respectively the row vertex labels $R_1 = \{3,7,11,15,19\}$, $R_2 = \{1,5,9,13,17\}$, $R_3 = \{0,4,8,12,16\}$ and $R_4 = \{2,6,10,14,18\}$ $s = 3,1,2,3$ steps to the right cyclically.

Corollary. *Let $S(k,p)$ be a k-star with p odd. Let $t^{(i)}_{k,p}$ be the tree obtained from $S(k,p)$ by deleting edges adjacent to the root and joining the root to the ith row vertices (labels) by new edges. Then $t^{(i)}_{k,p}$ accepts the same labeling except only the repeated label of the root is changed.*

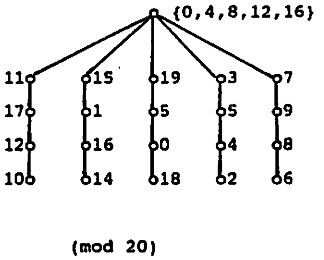

Figure 2

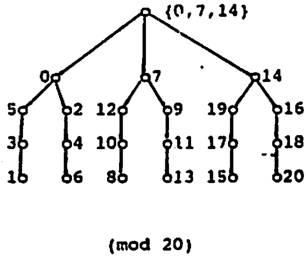

Figure 3

Harmonious labeling shown in Figure 3 for $t^{(4)}_{7,3}$ has been obtained from the labeling of $S(7,3)$ (see Figure 1a) where $\{0,7,14\}$ is the set of possible repeated labels for the root.

The next property deals with the harmonious labeling of n cycles. It is known that cycles C_n with n vertices, $n \geq 3$ is harmonious iff n is odd (cf Theorem 14 [1]).

Property 2: Consider again the labels of $S(k,p)$ ($p, k =$ odd). (1) may be rewritten in the form of rows:

$$R_1 = \{1, 1+k, 1+2k, \ldots, 1+(p-1)k\}$$
$$R_2 = \{3, 3+k, 3+2k, \ldots, 3+(p-1)k\}$$
$$R_i = \{2i-1, 2i-1+k, 2i-1+2k, \ldots, 2i-1+(p-1)k\}$$
$$R_{i+1} = \{0, k, 2k, \ldots, (p-1)k\}$$
$$R_{i+2} = \{2, 2+k, 2+2k, \ldots, 2+(p-1)k\}$$
$$\ldots$$
$$R_k = \{2i, 2i+k, 2i+2k, \ldots, 2i+(p-1)k\}$$

Now it is easy to see that if the labels in each row R_i, $1 \leq i \leq k$ are assigned (in that order) to the vertices of the k disjoint cycles C_p then the corresponding labeling is verified to be harmonious. In Fig.4 we illustrate the above property by giving harmonious labeling of $7C_3$ from the labeling of $S(7,3)$ (compare with the labeling of Figure 1a).

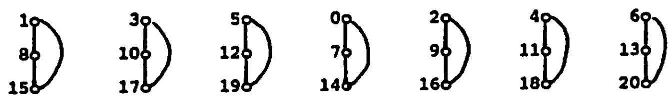

Figure 4

4 Harmonious labeling of $S(2,p)$

In this section we consider labelings of $S(2,p)$ when p is even. First the tree $S(2,2)$ is path on five vertices which is harmonious [1]. In Figure 5 we have given of all possible harmonious labelings of $S(2,2)$. For the other values of p and k we have only a partial answer.

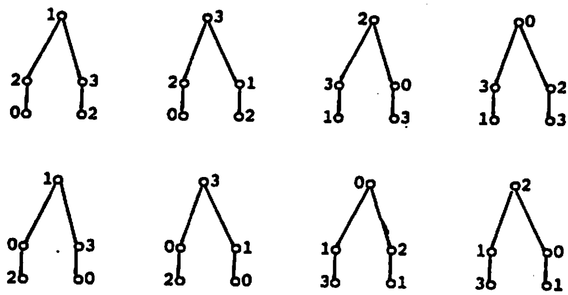

Figure 5

Let $k=2$ and $p \equiv 0 \pmod 2$. Divide the first row R_1 (respectively the second row R_2) of $S(2,p)$ into four sub-rows as $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, and $R_{1,4}$ (respectively $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$). Then label the vertices of row R_1 with

$$\begin{aligned}
R_{1,1} &= \{0\} \\
R_{1,2} &= \{4,6,8,\ldots,p-4,p-2,p-1\} \\
R_{1,3} &= \{p+3,p+4,p+6,p+8,\ldots,2p-4,2p-2\} \\
R_{1,4} &= \{2\}
\end{aligned}$$

and label the vertices of the second row R_2 with

$$R_{2,1} = \{1\}$$
$$R_{2,2} = \{3, 5, 7, \ldots, p-5, p-3, p+1\}$$
$$R_{2,3} = \{p+2, p+5, p+7, p+9, \ldots, 2p-3, 2p-1\}$$
$$R_{2,4} = \{0\}$$

where the label 0 is taken as an repeated label. The label of the root is p which is not used in R_1 and R_2. It is again straight matter to verify that the above labeling is harmonious.

Figure 6 illustrates harmonious labeling of $S(2,p)$ for $p = 10$. We have the following unproved claim when $p \equiv 0 \pmod 2$ and $k \geq 3$. There exists no harmonious labeling of $S(k,p)$ for $p =$ even in which the root has the repeated label.

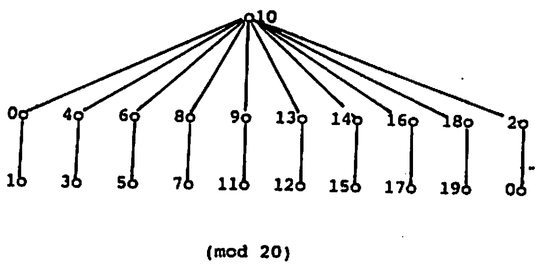

Figure 6

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