Distance Independent Domination in Graphs

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Abstract. Let $n \geq 1$ be an integer and let G be a graph of order p. A set I_n of vertices of G is n-independent if the distance between every two vertices of I_n is at least $n+1$. Furthermore, I_n is defined to be an n-independent dominating set of G if I_n is an n-independent set in G and every vertex in $V(G) - I_n$ is at distance at most n from some vertex in I_n. The n-independent domination number, $i_n(G)$, is the minimum cardinality among all n-independent dominating sets of G. Hence $i_1(G) = i(G)$ where $i(G)$ is the independent domination number of G. We establish the existence of a connected graph G every spanning tree T of which is such that $i_n(T) < i_n(G)$. For $n \in \{1, 2\}$ we show that, for any tree T and any tree T' obtained from T by joining a new vertex to some vertex of T, we have $i_n(T) \geq i_n(T')$. However we show that this is not true for $n \geq 3$. We show that the decision problem corresponding to the problem of computing $i_n(G)$ is NP-complete, even when restricted to bipartite graphs. Finally, we obtain a sharp lower bound on $i_n(G)$ for a graph G.

1. Introduction

For graph theory terminology not presented here we follow [15]. Specifically, $p(G)$ and $q(G)$ will denote, respectively, the number of vertices (order) and edges (size) of a graph G with vertex set $V(G)$ and edge set $E(G)$. If S is a set of vertices of G and v is a vertex of G, then the *distance from v to S*, denoted by $d_G(v, S)$ or simply $d(v, S)$, is the shortest distance from v to some vertex in S.

The theory of domination in graphs was formalised by Ore [53] and Berge [5] in 1958. A set D of vertices in a graph G is defined to be a *dominating set* of vertices of G if every vertex of $V(G) - D$ is adjacent to a vertex of D. The fact that every maximal independent set of vertices in a graph is also a minimal dominating set motivated Cockayne and Hedetniemi [18] in 1974 to initiate the study of "independent domination" in graphs. A dominating set of vertices in a graph that is also an independent set is called an *independent dominating set*. The minimum cardinality among all independent dominating sets of a graph G is called the *independent domination number* of G and is denoted by $i(G)$. The parameter $i(G)$ has received considerable attention in the literature (see, for instance, [1, 2, 6, 8, 9, 16, 17, 19–34, 38, 39, 46–49, 52, 56, 57, 59]).

In [41] and [44] a generalization of independent dominating sets and the independent domination number of a graph is considered. Let $n \geq 1$ be an integer and let G be a graph. A set D of vertices in G is defined to be an *n-dominating set* of G if every vertex of $V(G) - D$ is within distance n from some vertex of D. A set I of vertices of a graph G is defined to be *n-independent* in G if every vertex of I is at distance at least $n+1$ from every other vertex of I in G. It follows easily that every maximal n-independent set is also minimal n-dominating. A set

I is defined to be an *n-independent dominating set* of G if I is n-independent and n-dominating in G. The *n-independent domination number* $i_n(G)$ of G is the minimum cardinality among all n-indepndent dominating sets of G. Hence $i_1(G) = i(G)$ and 1-independent dominating sets of G are independent dominating sets of G. Results on the concept of n-domination in graphs have been presented by, among others, Bacsó and Tuza [3, 4], Bondy and Fan [7], Chang [10, 11], Chang and Nemhauser [12, 13, 14], Fink and Jacobson [35, 36], Fraisse [37], Henning, Oellermann and Swart [40–44], Jacobson and Peters [45], Meir and Moon [50], Mo and Williams [51], Slater [55], Topp and Volkmann [58] and He and Yesha [60].

There are potential applications of n-independent dominating sets to emergency aid centre location problems. Suppose a graph G is used to model a street system where vertices of G correspond to intersections and edges of G link vertices corresponding to adjacent intersections. A number of emergency aid centres are to be built at various points in the city so that each person living in the city is within n blocks of one of these centres. Furthermore, to avoid congestion in a crisis situation, these facilities are to be built in such a way that they are at least $n+1$ blocks apart. The problem of finding such a collection of potential sites for emergency aid centres amounts to finding a n-independent dominating set of vertices in G and an optimal solution has cardinality $i_n(G)$.

In Section 2, for each integer $n \geq 1$, we establish the existence of a connected graph G every spanning tree T of which is such that $i_n(T) < i_n(G)$. For $n \in \{1,2\}$ we show that, for any tree T and any tree T' obtained from T by adding a new vertex and joining this vertex with an edge to some vertex of T, we have $i_n(T) \leq i_n(T')$. However we show that this is not true for $n \geq 3$. In Section 3 we investigate the computational complexity of n-independent domination. We show that the decision problem corresponding to the problem of computing $i_n(G)$ is NP-complete, even when restricted to bipartite graphs. In Section 4 we investigate lower bounds on $i_n(G)$.

2. Spanning trees and subgraphs

We begin this section by establishing, for each integer $n \geq 1$, the existence of a connected graph G_n every spanning tree T of which satisfies $i_n(T) < i_n(G)$. For k a large integer, let H be the graph obtained from $K(1,k)$ by subdividing each edge $n-1$ times. Let $H_1, H_2, \ldots, H_{n+2}$ be $n+2$ disjoint copies of H and let v_i denote the vertex of $H_i (1 \leq i \leq n+2)$ of degree k. Further let G_n be the graph obtained from $\bigcup_{i=1}^{n+2}$ by adding the edge $v_1 v_{n+2}$ and the edges $v_i v_{i+1}$ for all i with $1 \leq i \leq n+1$. (The graph G_n is depicted in Figure 1.) Then every spanning tree T of G_n is isomorphic to $G_n - v_1 v_2$. Hence it is not difficult to verify that $i_n(T) = i_n(G_n - u_1 v_2) = nk + 2 < (n+1)k + 1 = i_n(G_n)$.

Proposition 1. *For $n \in \{1,2\}$, the tree T' obtained from a tree T by joining a new vertex to some vertex of T, satisfies $i_n(T') \geq i_n(T)$.*

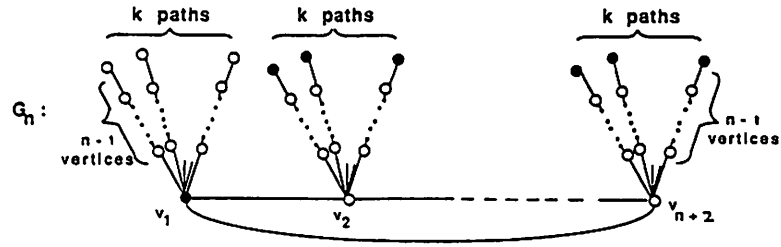

Figure 1. The graph G_n

Proof: Let v be the new vertex added to T to produce the tree $T' = T \cup \{v\} \cup \{uv\}$ where $u \in V(T)$. For $n \in \{1, 2\}$, we show that $i_n(T') \geq i_n(T)$. Let I_n be an n-independent dominating set of T' with $|I_n| = i_n(T')$. If $v \notin I_n$, then I_n is an n-independent dominating set of T and so $i_n(T) \leq |I_n| = i_n(T')$. Hence in what follows we may assume that $v \in I_n$ for otherwise there is nothing left to prove.

Since $v \in I_n$, $d(v, I_n - \{v\}) \geq n+1$ and so $d(u, I_n - \{v\}) \geq n$. If $d(u, I_n - \{v\}) > n$, then $(I_n - \{v\}) \cup \{u\}$ is an n-independent dominating set of T' and of T, and so $i_n(T) \leq |I_n| = i_n(T')$. If on the other hand $d(u, I_n - \{v\}) = n$, then for $n = 1$ this implies that $I_1 - \{v\}$ is an independent dominating set of T with $i_1(T) \leq |I_1 - \{v\}| < i_1(T')$. It remains for us to consider the case where $d(u, I_n - \{v\}) = n$ and $n = 2$.

If $I_2 - \{v\}$ is a 2-independent dominating set of T, then $i_2(T) < i_2(T')$. Suppose that $I_2 - \{v\}$ is not a 2-independent dominating set of T. Let S denote the set of all vertices of T that are at distance at least 3 from every vertex of $I_2 - \{v\}$. Since I_2 is a 2-dominating set of T', each vertex of S is at distance at most 2 from v in T'. Furthermore, since $d(u, I_2 - \{v\}) = 2$, it follows that $S \subseteq N(u)$. In particular, we observe, therefore, that each vertex of S is at distance at most 2 from every other vertex of S in T. This implies that, for any vertex $w \in S$, the set $(I_2 - \{v\}) \cup \{w\}$ is a 2-independent dominating set of T' and of T. Consequently, $i_2(T) \leq |I_2| = i_2(T')$. This completes the proof of the proposition. ∎

It is somewhat surprising that Proposition 1 is not true for $n \geq 3$. To see this, consider the tree $T_n (n \geq 3)$ constructed as follows. Let $n \geq 3$ be an integer, and let k be a large integer. Let F be the graph obtained from $K(1, k+1)$ by subdividing each edge $n - 1$ times. Further, let $F_1, F_2, \ldots, F_{2n-3}$ be $2n - 3$ disjoint copies of F, and let u_i and v_i, respectively, denote the vertex of degree $k + 1$ and an end-vertex of $F_i (1 \leq i \leq 2n - 3)$. The tree T_n is obtained from $\bigcup_{i=1}^{2n-3} F_i$ by adding two new vertices v_0 and v_{2n-2} and by adding the edges $v_i v_{i+1}$ for all i with $0 \leq i \leq 2n - 3$. (The tree T_n is shown in Figure 2.) Then it is not difficult to verify that $\{u_1, u_2, \ldots, u_{2n-3}\} \cup \{v_0, v_{2n-2}\}$ is an n-independent dominating set of T_n of cardinality $i_n(T_n) = 2n-1$. However the tree T_n obtained from T_n by adding a new vertex v and joining v with an edge to v_{n-2} is such that $i_n(T'_n) = 2n-2 < i_n(T_n)$. (The set $\{u_1, u_2, \ldots, u_{2n-3}\} \cup \{v\}$ is an n-independent dominating set of T_n of cardinality $2n-2$.)

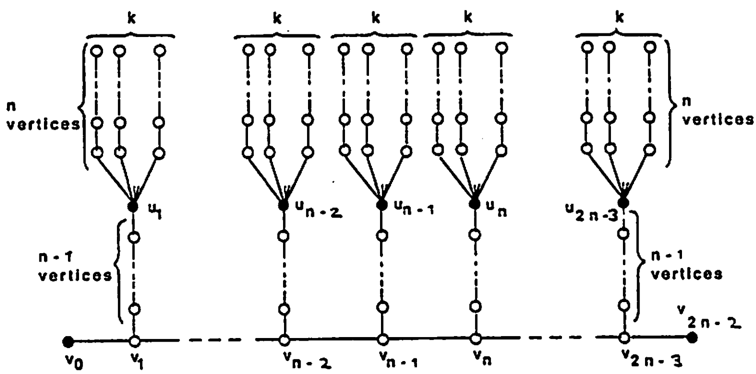

Figure 2. The tree T_n.
(The set of all darkened vertices form an
n-independent dominating set of cardinality $i_n(T) = 2n-1$.)

3. Complexity

From a computational point of view the problem of finding $i_n(G)$ appears to be very difficult. In fact, there is no known efficient algorithm for solving this problem. Let us consider the following decision problem corresponding to the problem of computing $i_n(G)$ for any fixed integer $n \geq 2$.

DISTANCE n-INDEPENDENT DOMINATING SET (DID)

Instance. Graph $G = (V, E)$, positive integer $k \leq |V|$.

Question. Is there an n-independent dominating set of cardinality k or less?

The purpose of this section is to establish the following result.

Theorem 1. *DID is NP-complete when restricted to bipartite graphs.*

Proof: It is obvious that DID is a member of NP since we can, in polynomial time, guess at a subset of vertices, verify that its cardinality is at most k, and then verify that it is an n-independent dominating set. To show that DID is an NP-complete problem, we will establish a polynomial transformation from the well-known NP-complete problem 3SAT. Let I be an instance of 3SAT consisting of the (finite) set $C = \{c_1, \ldots, c_m\}$ of three-literal clauses in the k-variables $x_1, \ldots, x_k$. We transform I to the instance (G_I, k) of DID in which G_I is the bipartite graph constructed as follows.

Let H be the graph obtained from a 4-cycle by attaching a path of length $n-1$ to each of two nonadjacent vertices of the 4-cycle. Let $H_1, \ldots, H_k$ be k disjoint copies of H. Corresponding to each variable x_i we associate the graph H_i. Let

x_i and $\overline{x}_i$ be the names of the two special vertices of H_i of degree 2 that are at distance n from the two end-vertices of H_i. Corresponding to each clause c_i we associate a special vertex named c_i. The construction of our instance of DID is completed by joining the vertex c_i to the three special vertices that name the three literals in the clause c_i and then subdividing each of these three edges $n-2$ times. The resulting graph G is depicted in Figure 3.

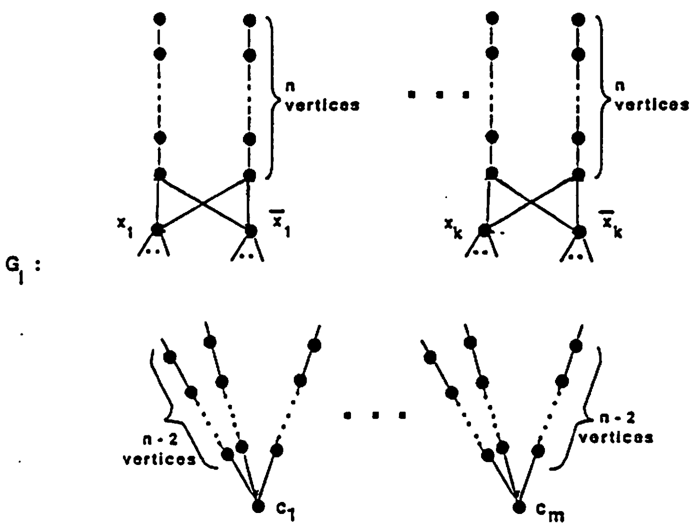

Figure 3. The graph G_I resulting from 3SAT instance I.

It is easy to see how the construction can be accomplished in polynomial time. All that remains to be shown is that I has a satisfying truth assignment if and only if $i_n(G_I) \leq k$.

First suppose I has a satisfying truth assignment. Let D be the set of k special vertices of G_I that correspond to literals which have the value T (in the instance I). We verify that D is an n-independent dominating set of G of cardinality k. Since $d(x_i, \overline{x}_i) = 2$, the only vertices whose n-domination by D gives any doubt are the vertices c_j. But these vertices are n-dominated by D because I has a satisfying truth assignment. This shows that $i_n(G_I) \leq |D| = k$.

Conversely assume that $i_n(G_I) \leq k$. Let D be an n-independent dominating set of G_I with $|D| = i_n(G_I)$. Since the end-vertices of H_i are at distance n from both x_i and $\overline{x}_i$, it follows from our construction of G_I that D contains a vertex of H_i for all $i (1 \leq i \leq k)$. This shows that $|D| \geq k$. Hence $i_n(G_I) = |D| = k$ and D consists of precisely one vertex from each H_i namely x_i or $\overline{x}_i$. However, since D is an n-dominating set of G_I, this implies that each vertex c_j is within distance n from some vertex of D in G_I. Thus we can use D to obtain a truth assignment

$t : \{x_1, \ldots, x_k\} \to \{T, F\}$. We merely set $t(x_i) = T$ if $x_i \in D$ and $t(x_i) = F$ if $x_i \notin D$. Since this truth assignment satisfies each of the clauses of C, I has a satisfying truth assignment. ∎

4. Bounds on $i_n(G)$ for a graph G

Since the problem of computing $i_n(G)$ appears to be a difficult one, it is desirable to find good upper bounds on this parameter. Before proceeding further we introduce some notation.

Let v be a vertex of G. The set of all vertices of G different from v and at distance at most n from v in G is defined in [37] as the *open n-neighborhood* of v in G and is denoted by $N_n(v)$. The *closed n-neighorhood* of v is the set $N_n[v] = N_n(v) \cup \{v\}$. The *n-degree*, $\deg_n v$, of v in G is given by $|N_n(v)|$. Hence $N_1(v) = N(v)$ and $\deg_1 v = \deg v$. The maximum n-degree among all the vertices of G is denoted by $\Delta_n(G)$ so $\Delta_1(G) = \Delta(G)$. Let v be a vertex with $\deg_n v = \Delta_n(G)$ and S be a maximal n-independent (and therefore n-independent dominating) set which contains v. Since $S \cap N_n(v) = \emptyset$, $|S| \le p - \Delta_n(G)$ and we have proved the following result.

Proposition 2. *For $n \geq 1$, if G is a graph of order p, then $i_n(G) \leq p - \Delta_n(G)$.*

To see that the above bound for $i_n(G)$ is best possible, consider the graph G obtained from a star $K_{1,k}, k \geq 2$, by subdividing $k-1$ of the edges n times and one edge $n-1$ times. Then $i_n(G) = k, p = p(G) = (n+1)k$ and $\Delta_n(G) = nk$; consequently, $i_n(G) = p - \Delta_n(G)$.

Next we present a lower bound on $i_n(G)$ in terms of the maximum n-degree $\Delta_n(G)$.

Theorem 2. *For $n \geq 1$, if G is a graph of order p and maximum n-degree $\Delta_n \geq 2n$, then*

$$i_n(G) \geq \frac{p}{\frac{n+1}{n}\Delta_n - 1}.$$

Furthermore, $i_n(G) = p/[(\frac{n+1}{n})\Delta_n - 1]$ if and only if all components of G are either paths or cycles on $l \equiv 0 \pmod{2n+1}$ vertices, or have order exactly $2n+1$.

Proof: Let X be a minimum n-independent dominating set of G. Let A be the set of vertices of $V(G) - X$ that are within distance n from exactly one vertex of X and B the set of vertices of $V(G) - X$ that are within distance n from at least two vertices of X. Note that $\{X, A, B\}$ is a partition of $V(G)$. For $x \in X$, set $A_x = \{u \mid u \in A \text{ and } d(u,x) \leq n\}$. By definition of A, $x_1 \neq x_2$ implies that $A_{x_1} \cap A_{x_2} = \emptyset$. We note that

$$|A_x| \leq \Delta_n \text{ for all } x \in X. \qquad (1)$$

For $x \in X$, set $B_x = \{u \mid u \in B \text{ and } d(u,x) \leq n\}$. We note that

$$|B_x| \leq \Delta_n - |A_x| \text{ for all } x \in X. \tag{2}$$

By definition of B, each vertex of B belongs to at least two sets B_x. We deduce that

$$2|B| \leq \sum_{x \in X} |B_x|$$

and hence, using (2), that

$$|B| \leq \sum_{x \in X} \frac{1}{2}(\Delta_n - |A_x|). \tag{3}$$

Using (1) and (3) we have

$$\begin{aligned} p &= |X| + |A| + |B| \\ &\leq |X| + \sum_{x \in X} |A_x| + \sum_{x \in X} \frac{1}{2}(\Delta_n - |A_x|) \qquad \text{(using (3))} \\ &= \sum_{x \in X} \frac{1}{2}(\Delta_n + |A_x| + 2) \\ &\leq \sum_{x \in X} (\Delta_n + 1) \qquad \text{(using (1))} \\ &\leq \sum_{x \in X} \left[\left(\frac{n+1}{n}\right)\Delta_n - 1\right] \qquad \text{(since } \Delta_n \geq 2n) \\ &= |X| \cdot \left[\left(\frac{n+1}{n}\right)\Delta_n - 1\right], \end{aligned} \tag{4}$$

so that

$$i_n(G) = |X| \geq \frac{p}{(\frac{n+1}{n})\Delta_n - 1}.$$

We now determine the connected extremal graphs G (the disconnected graphs are easily deduced). If G is extremal, then we have equality throughout in (4). In particular, this means that $\Delta_n = 2n$, so $i_n(G) = p/(2n+1)$. If $i_n(G) = 1$, then G has order $2n+1$. Assume, then, that $i_n(G) > 1$. We show that G is either a path or a cycle on $\ell \equiv 0 \pmod{2n+1}$ vertices. If $n = 1$, then $\Delta(G) = \Delta_1 = 2$ and $i(G) = i_1(G) = p/3$. This occurs if and only if G is either a path or a cycle on $\ell \equiv 0 \pmod 3$ vertices. Assume, then, that $n \geq 2$.

Let v be a vertex with $\deg_n v = \Delta_n = 2n$. For $i = 0, 1, \ldots, m = e(v)$, let $D_i = \{u \in V(G) \mid d(u,v) = i\}$. Since $i_n(G) > 1$, we know that $p > 2n+1$. Since $\deg_n v = 2n$, we have $e(v) \geq n+1$. Let $v_m \in D_m$ and consider a shortest

v-v_m path $P : v = v_0, v_1, \ldots, v_m$. Necessarily, $v_i \in D_i$ $(0 \le i \le m)$. Let P' be the v_0-v_n subpath of P and consider the vertex v_1. If $N_n[v] \subseteq N_n[v_1]$, then $\deg_n v_1 \ge |(N_n[v] - \{v_1\}) \cup \{v_{n+1}\}| = 2n+1 > \Delta_n$, which is impossible. It follows that there exists a vertex $w_n \in D_n$ at distance $n+1$ from v_1. Let $Q : v, w_1, \ldots, w_n$ be a shortest v-w_n path. Necessarily, $V(P) \cap V(Q) = \{v\}$, so $N_n[v] = V(P') \cup V(Q)$. Further, $w_i \in D_i$ and, since $d(v_1, w_n) = n+1$, there is no edge of the form $v_i w_i (1 \le i \le n)$ or $v_i w_{i+1} (1 \le i \le n-1)$. Moreover, there is no edge of the form $v_i w_{i-1} (2 \le i \le n)$, for otherwise $V(P') \cup V(Q) \cup \{v_{n+1}\} \subseteq N_n[w_{i-1}]$, so $\Delta_n < 2n+1 \le \deg_n w_{i-1}$, which is impossible. Thus there is no edge joining $V(P') - \{v\}$ and $V(Q) - \{v\}$. That is to say, $\langle N_n[v] \rangle \cong P_{2n+1}$. Necessarily, v_{n+1} is the only vertex of D_{n+1} that is adjacent with v_n, for otherwise, $\deg_n v_1 > \Delta_n$. We consider two possibilities.

Case 1. Suppose that $\deg w_n = 1$. Then $D_{n+1} = \{v_{n+1}\}$. Since p is a multiple of $2n+1$, and $|\bigcup_{i=0}^{n+1} D_i| = 2n+2$, we know that $m \ge n+2$. Let $n+1 \le k < m$ and assume that $D_k = \{v_k\}$ for all i with $n+1 \le i \le k$. We show that $D_{k+1} = \{v_{k+1}\}$. If $k \ge 2n-1$, then

$$\begin{aligned} 2n = \Delta_n &\ge \deg_n v_{k-n+1} \\ &= |\{v_{k-n}, v_{k-n-1}, \ldots, v_{k-2n+1}\}| + |\{v_{k-n+2}, \ldots, v_k\}| + |D_{k+1}| \\ &= 2n - 1 + |D_{k+1}|, \end{aligned}$$

so $|D_{k+1}| \le 1$. Hence $D_{k+1} = \{v_k + 1\}$. If, on the other hand, $k < 2n-1$, then

$$\begin{aligned} 2n = \Delta_n &\ge \deg_n v_{k-n+1} \\ &= |\{v_0, v_1, \ldots, v_{k-n}\}| + |\{w_1, \ldots, w_{2n-1-k}\}| + \\ &\qquad |\{v_{k-n+2}, \ldots, v_k\}| + |D_{k+1}| \\ &= 2n - 1 + |D_{k+1}|, \end{aligned}$$

so $|D_{k+1}| \le 1$. Once again, $D_{k+1} = \{v_{k+1}\}$. Hence, by induction, G is a path on $(m+n+1) \equiv 0 \pmod{2n+1}$ vertices.

Case 2. Suppose that $\deg w_n > 1$. Then w_n is adjacent to exactly one vertex in D_{n+1}, for otherwise, $\deg_n w_1 > \Delta_n$. If $|D_{n+1}| = 1$, then v_{n+1} is adjacent to v_n and to w_n, and therefore is within distance n from Δ_n vertices of $P' \cup Q$. It follows that $D_{n+2} = \emptyset$, for otherwise, $\deg_n v_{n+1} > \Delta_n$. Thus $G \cong C_{2n+2}$, which contradicts the fact that p is a multiple of $2n+1$. We deduce that $|D_{n+1}| = 2$. Let $D_{n+1} = \{v_{n+1}, w_{n+1}\}$ where $w_{n+1} w_n \in E(G)$. If $v_{n+1} w_{n+1} \in E(G)$, then $G \cong C_{2n+3}$, once again contradicting the fact that p is a multiple of $2n+1$. Hence $\langle \bigcup_{i=0}^{n+1} D_i \rangle \cong P_{2n+3}$.

Let $n+1 \le j < m$, and assume that $D_i = \{v_i, w_i\}$ for all i with $1 \le i \le k$ and that $\langle \bigcup_{i=0}^{k} D_i \rangle \cong P_{2k+1}$ (where $w_i w_{i-1} \in E(G)$ for $2 \le i \le k$). If $\deg w_k = 1$, then proceeding in a similar manner as in Case 1, we may conclude that G is a path on $(m+k+1) \equiv 0 \pmod{2n+1}$ vertices. Assume, then, that $\deg w_k > 1$. Then

w_k is adjacent to exactly one vertex in D_{k+1}, for otherwise, $\deg_n w_{k-n+1} > \Delta_n$. If $|D_{k+1}| = 1$, then v_{k+1} is adjacent to v_k and to w_k. It follows that $D_{k+2} = \emptyset$, $k+1 = m$ and $G \cong C_{2m}$. If $2m \not\equiv 0 \pmod{2n+1}$, then this produces a contradiction. Otherwise, G is a cycle on $2m \equiv 0 \pmod{2n+1}$ vertices.

If $|D_{k+1}| > 1$, then, necessarily, $|D_{k+1}| = 2$. Let $D_{k+1} = \{v_{k+1}w_{k+1}\}$ where $w_{k+1}w_k \in E(G)$. If $v_{k+1}w_{k+1} \in E(G)$, then $m = k+1$ and $G \cong C_{2m+1}$. If $(2m+1) \not\equiv 0 \pmod{2n+1}$, then this produces a contradiction; otherwise, G is a cycle on $(2m+1) \equiv 0 \pmod{2n+1}$ vertices. On the other hand, if $v_{k+1}w_{k+1} \notin E(G)$, then either $m = k+1$, in which case G is a path on $(2m+1) \equiv 0 \pmod{2n+1}$ vertices, or $m > k+1$, in which case $\langle \bigcup_{i=0}^{k+1} D_i \rangle \cong P_{2(k+1)+1}$.

Continuing in this way, we deduce that G is either a path or a cycle on $\ell \equiv 0 \pmod{2n+1}$ vertices. This completes the necessity. The sufficiency is clear. ∎

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