ON GRAPHS WITH EQUAL DOMINATION AND EDGE INDEPENDENCE NUMBERS

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ABSTRACT. Let G be a simple graph. A set D of vertices of G is dominating if every vertex not in D is adjacent to some vertex in D. A set M of edges of G is called independent, or a matching, if no two edges of M are adjacent in G. The domination number $\gamma(G)$ is the minimum order of a dominating set in G. The edge independence number $\alpha_0(G)$ is the maximum size of a matching in G. If G has no isolated vertices, then the inequality $\gamma(G) \leq \alpha_0(G)$ holds. In this paper we characterize regular graphs, unicyclic graphs, block graphs, and locally connected graphs for which $\gamma(G) = \alpha_0(G)$.

1. TERMINOLOGY

We consider finite, undirected, and simple graphs G with the vertex set $V = V(G)$ and the edge set $E = E(G)$. For $A \subseteq V(G)$ let $G[A]$ be the subgraph induced by A. A subgraph H of G with $V(H) = V(G)$ is called a factor of G. $N(x) = N(x, G)$ denotes the set of vertices adjacent to the vertex x and $\bar{N}(x) = \bar{N}(x, G) = N(x) \cup \{x\}$. More generally, we define $N(X) = N(X, G) = \bigcup_{x \in X} N(x)$ and $\bar{N}(X) = \bar{N}(X, G) = N(X) \cup X$ for a subset X of $V(G)$. The vertex v is an end vertex if $d(v, G) = 1$, and an isolated vertex if $d(v, G) = 0$, where $d(x) = d(x, G) = |N(x)|$ is the degree of $x \in V(G)$. Let $\Omega = \Omega(G)$ be the set of end vertices, and $I = I(G)$ be the set of isolated vertices, respectively. We denote by $\delta = \delta(G)$ the minimum degree and by $n = n(G) = |V(G)|$ the order of G. An empty graph is one with no edges. We write C_n for a cycle of length n and K_n for the complete graph of order n. A star is a complete bipartite graph $K_{1,m}$ with $m \geq 2$, and the unique vertex v of this star of degree m is called the center.

A set $D \subseteq V(G)$ is a dominating set of G if $\bar{N}(D, G) = V(G)$, and is a covering set of G if every edge of G has at least one end in D. The domination number, $\gamma = \gamma(G)$, and the covering number, $\beta = \beta(G)$, of G is the order of the smallest dominating set, and the smallest covering of G, respectively. If G is a graph without isolated vertices, then it is easy

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to check that $\gamma(G) \leq \beta(G)$. A set $M \subseteq E(G)$ is an independent set, or a matching, if no two edges of M are adjacent in G. The order of a maximum matching is called the edge independence number $\alpha_0 = \alpha_0(G)$. A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v; otherwise, v is M-unsaturated. An M-alternating path in G is a path whose edges are alternately in $E(G) - M$ and M. An M-augmenting path is an M-alternating path whose origin and terminus are M-unsaturated.

2. Preliminary results

The next two famous theorems of König [4] from 1931, and Berge [1] from 1957 are very important for our research.

König's Theorem. [4] *If G is a bipartite graph, then $\alpha_0(G) = \beta(G)$.*

Berge's Theorem. [1] *A matching M in G is a maximum matching if and only if G contains no M-augmenting path.*

Theorem 1. *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \alpha_0(G).$$

Proof. If T is a spanning forest of G without isolated vertices, then it follows from König's Theorem

$$\gamma(G) \leq \gamma(T) \leq \beta(T) = \alpha_0(T) \leq \alpha_0(G),$$

and the proof is complete. □

It is the purpose of this paper to characterize some classes of graphs for which the equality $\gamma(G) = \alpha_0(G)$ holds.

In the sequel, we will need some further notions, results and observations.

Proposition 1. *[6] If G is a graph without isolated vertices, then we have $2\gamma(G) \leq n(G)$.*

Corollary 1. *If G is a graph without isolated vertices, then*

$$\gamma(G) \leq \alpha_0(G) \leq \beta(G).$$

Proof. The well-known fact that $\alpha_0(G) \leq \beta(G)$ and Theorem 1 yield the desired inequalities. □

Corollary 2. *For a bipartite graph without isolated vertices we have* $\gamma(G) = \alpha_0(G)$ *if and only if* $\gamma(G) = \beta(G)$.

Proof. Since $\gamma(G) \leq \beta(G)$, we deduce from König's Theorem

$$\gamma(G) \leq \beta(G) = \alpha_0(G),$$

and this yields the desired result. □

In [9] we have characterized all trees T with $\gamma(T) = \beta(T)$. According to Corollary 2 we get the same characterization for trees T with $\gamma(T) = \alpha_0(T)$.

Theorem 2. *[9] Let* T *be a tree of order* $n \geq 2$. *Then* $\gamma(T) = \alpha_0(T)$ *if and only if* $T^* = T - \bar{N}(\Omega(T), T) = \emptyset$ *or each component of* T^* *is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of* $N(\Omega(T), T)$.

The next lemma plays a central role in our proofs.

Lemma 1. *Let* G *be a connected graph with* $\gamma(G) = \alpha_0(G)$. *If* H *is a factor of* G *without isolated vertices, then*

$$\gamma(H) = \alpha_0(H) = \gamma(G).$$

Proof. Since the factor H has no isolated vertices, Theorem 1 yields the inequality $\gamma(H) \leq \alpha_0(H)$. This implies

$$\alpha_0(G) = \gamma(G) \leq \gamma(H) \leq \alpha_0(H) \leq \alpha_0(G),$$

and therefore $\gamma(H) = \alpha_0(H) = \gamma(G)$. □

Finally we note the simple but useful

Proposition 2. *If* C_n *is a cycle of length* n, *then* $\gamma(C_n) = \alpha_0(C_n)$ *if and only if* $n = 3, 4, 5$ *or* 7.

Since for a graph G without isolated vertices we have $\gamma(G) = \alpha_0(G)$ if and only if $\gamma(H) = \alpha_0(H)$ for each component H of G, we will only deal with connected graphs; one can easily generalize the results to non-connected graphs without isolated vertices.

Theorem 3. *Let G be a connected and r-regular graph G with $r > 0$. Then we have $\gamma(G) = \alpha_0(G)$ if and only if $G = K_2$ or $G = C_3, C_4, C_5, C_7$.*

Proof. The sufficiency is obvious. The converse is immediate for $r = 1$, and follows from Proposition 2 for $r = 2$. Now we assume that $r \geq 3$. Since G is regular, we can deduce that $|N(X, G)| \geq |X|$ for all subsets X of $V(G)$ (see [8, p. 125]). Hence by a theorem of Tutte [7] (see also [8, p. 105]), there exists a factor of G whose components are either 1-regular or 2-regular. Thus, G has a factor F whose components are 1-regular or cycles of odd length. If F contains an odd cycle of length ≥ 9, then it follows from Proposition 2 and Theorem 1 that $\gamma(F) < \alpha_0(F)$, a contradiction to Lemma 1. This implies that the components of F are graphs of the form K_2, C_3, C_5, or C_7. It is easy to check that $F = C_5$ or $F = C_7$ is not possible, so that the factor F consists of at least two components. If there exists an edge in G which joins two cycles of F, it is not hard to observe that F, together with such an edge, form a new factor F' with $\gamma(F') < \alpha_0(F')$, a contradiction to Lemma 1. This means that in G the cycles of F are only joined with the 1-regular components of F. On the other hand, we conclude from $r \geq 3$ that in G all vertices of each cycle C_3 of F are adjacent to an 1-regular component of F.
Now we shall show that there exists a factor J such that all components have the form K_2, C_5, or C_7. Let C be a cycle of length three in F with the three vertices c_1, c_2, and c_3. Furthermore, let x_1y_1, x_2y_2, and x_3y_3 be the edges of three 1-regular components of F. If there exist without loss of generality the edges c_1x_1, c_2x_2, c_3x_3, or c_1x_1, c_2x_1, c_3x_3, or c_1x_1, c_2x_1, c_3x_1 in G, then in each case we add these three edges to F and get again a factor F^* such that $\gamma(F^*) < \alpha_0(F^*)$ which is not possible. On the other hand, if G contains without loss of generality the edges c_1x_1 and c_2y_1, then we transform C and the 1-regular component with the vertices x_1 and y_1 in a cycle C_5, and we have a new factor with one fewer cycle of length three. Therefore we get step by step a factor J of the desired form.
Let J consists of i 1-regular components, j cycles C_5, and k cycles C_7. Since G is connected, every cycle C_5 or C_7 of J is joined in G with a 1-regular component of J. This means that $i > 0$ and therefore $5i + 15k > 4i + 14k$. From the last inequality it follows

$$\gamma(G) = \gamma(J) = i + 2j + 3k > \frac{2}{5}(2i + 5j + 7k) = \frac{2}{5}n(G).$$

But this is a contradiction to a recent inequality of McCuaig and Shepherd [5] that $\gamma(G) \leq \frac{2}{5}n(G)$, if $\delta(G) \geq 3$. □

From Theorem 3 and Corollary 1 we deduce immediately the following result.

Corollary 3. *[9] For a connected, r-regular graph G with $r > 0$ we have $\gamma(G) = \beta(G)$ if and only if $G = K_2$ or $G = C_4$.*

4. Corona graphs

Before proceeding we introduce the following notation. Let G be a graph and $\mathcal{F} = \{H_x | x \in V(G) \text{ and } H_x \neq \emptyset\}$ a family of graphs disjoint from each other and from G indexed by the vertices of G. The corona $G \circ \mathcal{F}$ of the graph G and the family $\mathcal{F}$ is the disjoint union of G and the graphs H_x, $x \in V(G)$, with additional edges joining each vertex v of G to all vertices of H_v. If all graphs of the family $\mathcal{F}$ are isomorphic to one and the same graph H (written $H \cong H_x$ for all $x \in V(G)$), then we shall write $G \circ H$ instead of $G \circ \mathcal{F}$.

Theorem 4. *For a graph G and a family $\mathcal{F} = \{H_x | x \in V(G)\}$ indexed by the vertices of G we have $\gamma(G \circ \mathcal{F}) = \alpha_0(G \circ \mathcal{F})$ if and only if every graph H_x of $\mathcal{F}$ is empty or $H_x \cong K_2$, and if exactly $H_{x_1}, ..., H_{x_t}$ are isomorphic to K_2, then the induced subgraph $G[x_1, ..., x_t]$ contains no edges.*

Proof. Since $V(G)$ is a dominating set of the corona graph $G \circ \mathcal{F}$, it is easy to see that $\gamma(G \circ \mathcal{F}) = n(G) = n$. Let $H_{x_1}, ..., H_{x_t}$ be isomorphic to K_2 and let $H_{x_{t+1}}, ..., H_{x_n}$ be empty graphs. If we choose a vertex $y_i \in H_{x_i}$ for all $i = 1, ..., n$, then $M = \{x_1y_1, ..., x_ny_n\}$ form a matching in $G \circ \mathcal{F}$. Now it is not difficult to check that $G \circ \mathcal{F}$ contains no M-augmenting path, and therefore by Berge's Theorem M is a maximum matching. Hence we conclude $\gamma(G \circ \mathcal{F}) = \alpha_0(G \circ \mathcal{F}) = n$.
On the other hand, if any H_x is not empty with $|H_x| \geq 3$, then it is easy to find a matching M^* with $|M^*| \geq n + 1$. Thus,

$$\gamma(G \circ \mathcal{F}) = n < n + 1 \leq \alpha_0(G \circ \mathcal{F}),$$

a contradiction. Furthermore, if the induced subgraph $G[x_1, ..., x_t]$ contains an edge, it is immediate that $\gamma(G \circ \mathcal{F}) < \alpha_0(G \circ \mathcal{F})$. □

Proposition 3. *[3] Let G be a connected graph of even order. Then we have $\gamma(G) = \frac{1}{2}n(G)$ if and only if $G = C_4$ or $G = H \circ K_1$ for an arbitrary connected graph H.*

Corollary 4. *Let G be a connected graph with a perfect matching that means $\alpha_0(G) = \frac{1}{2}n(G)$. Then $\gamma(G) = \alpha_0(G)$ if and only if $G = C_4$ or $G = H \circ K_1$ for an arbitrary connected graph H.*

Proof. If $G = C_4$ or $G = H \circ K_1$ for an arbitrary connected graph H, then it follows from Theorem 4 that $\gamma(G) = \alpha_0(G)$. The converse is immediate by Proposition 3. □

5. UNICYCLIC GRAPHS

In [9] we have proved that if every cycle of a graph G is adjacent to an end vertex, then $\gamma(G) = \beta(G)$ if and only if $G^* = G - \bar{N}(\Omega(G), G) = \emptyset$ or each component of G^* is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of $N(\Omega(G), G)$. Here we shall show that the same class of graphs is characterized by the weaker condition $\gamma(G) = \alpha_0(G)$. The proof is based on the next lemma.

Lemma 2. *Let T be a tree of order $n \geq 3$. If T is not a star, then*

$$\gamma(T - \Omega(T)) < \alpha_0(T).$$

Proof. In the sequel we shall use the short notation $\Omega = \Omega(T)$. Since the result is immediate if $\gamma(T) < \alpha_0(T)$, it remains to prove Lemma 2 for $\gamma(T) = \alpha_0(T)$. Then it follows from Theorem 2 that $T^* = T - \bar{N}(\Omega, T) = \emptyset$ or each component of T^* is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of $N(\Omega, T)$. Now we consider three cases.

Case 1. The set $T^* = \emptyset$. Then $T = T_1 \circ \mathcal{F}$ at which each graph H_x of $\mathcal{F}$ is empty and T_1 is a tree. From the assumption that T is not a star we conclude $n(T_1) \geq 2$. Hence by Proposition 1 and Theorem 4 we obtain

$$\gamma(T - \Omega) = \gamma(T_1) \leq \frac{1}{2}n(T_1) < n(T_1) = \alpha_0(T).$$

Case 2. The set T^* consists of isolated vertices I. In this case $|N(\Omega, T)| = \gamma(T) = \alpha_0(T)$ holds. For $a \in I$ let $U = N(\Omega, T) - N(a, T)$. Then the set $U \cup \{a\}$ is a dominating set of $T - \Omega$ with

$$|U \cup \{a\}| < |N(\Omega, T)| = \alpha_0(T).$$

Case 3. The set T^* contains $m \geq 1$ stars S_i with the centers a_i. It is easy to see that $D = N(\Omega) \cup \{a_1, ..., a_m\}$ is a minimum dominating set of T. If $N(a_1, T) = W$ and $N(W, T) - \{a_1\} = X \subseteq N(\Omega, T)$, then $|X| \geq |W|$. All isolated vertices of T^* are adjacent to at least two neigbours of Ω. But since T is a tree, each isolated vertex of T^* is adjacent to at most one vertex of X. Therefore $D_1 = D - (X \cup \{a_1\}) \cup W$ is a dominating set of $T - \Omega$ with $|D_1| < \gamma(T) = \alpha_0(T)$, and the lemma is proved. □

Theorem 5. *If every cycle of a connected graph G of order $n(G) \geq 2$ is adjacent to an end vertex, then $\gamma(G) = \alpha_0(G)$ if and only if $G^* = G - \bar{N}(\Omega(G), G) = \emptyset$ or each component of G^* is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of $N(\Omega(G), G)$.*

Proof. If G^* has the above form, then we have proved in [9] that $\gamma(G) = \beta(G)$, and hence Corollary 1 implies $\gamma(G) = \alpha_0(G)$.
For the converse we consider two cases.
Case 1. Assume that there exists a component H of G^* of order $n(H) \geq 3$ which is not a star. Let I be the set of isolated vertices of G^* and $R = G^* - H - I$. With $H' = H - \Omega(H)$ and $\Omega(G) = \Omega$, Theorem 1 and Lemma 2 yield

$$\begin{aligned}\gamma(G) &\leq |N(\Omega, G)| + \gamma(R) + \gamma(H') \\ &\leq |N(\Omega, G)| + \alpha_0(R) + \gamma(H') \\ &< |N(\Omega, G)| + \alpha_0(R) + \alpha_0(H) \leq \alpha_0(G),\end{aligned}$$

a contradiction to $\gamma(G) = \alpha_0(G)$.
Case 2. Assume that there exists a component H of G^* of order 2 or a component H which is a star such that the center of H is adjacent to an element of of $N(\Omega, G)$. If we define R as in the case 1, then we see

$$\gamma(G) \leq |N(\Omega, G)| + \gamma(R) < |N(\Omega, G)| + \alpha_0(R) + 1 \leq \alpha_0(G),$$

which is impossible. This completes the proof of Theorem 5. □

A graph is cactus, if all cycles are edge disjoint.

Theorem 6. *Let G be a connected cactus of order $n \geq 2$ without cycles of length three, four, five, and seven. Then $\gamma(G) = \alpha_0(G)$ if and only if $G^* = G - \bar{N}(\Omega(G), G) = \emptyset$ or each component of G^* is an isolated vertex or a star, where the centers of these stars are not adjacent to a vertex of $N(\Omega(G), G)$.*

Proof. The sufficiency follows at once from Theorem 5.
For the converse, we show that every cycle is adjacent to an end vertex. Let C be a cycle of G, and suppose to the contrary that C is not adjacent to a vertex of $\Omega(G)$. Then, since G is a cactus, and $C \neq C_3, C_4, C_5, C_7$, the subgraph $G - V(C)$ has no isolated vertices, and $\gamma(C) < \alpha_0(C)$ by Proposition 2. Consequently, $F = (G - V(C)) \cup C$ is a factor of G without isolated vertices, and hence Theorem 1 yields $\gamma(F) < \alpha_0(F)$. This is a contradiction to $\gamma(G) = \alpha_0(G)$ and Lemma 1. Now the result follows from Theorem 5. □

Let us recall that a unicyclic graph is a connected graph with exactly one cycle. If G is a unicyclic graph with cycle C and x is a vertex of G, then we denote by $c(x)$ the distance from x to C. In the next theorem we determine the unicyclic graphs in which the edge independence number is equal to the domination number.

Theorem 7. *Let G be a unicyclic graph, $\Omega = \Omega(G)$, $G^* = G - \bar{N}(\Omega, G)$, and C the only cycle of G. Then $\gamma(G) = \alpha_0(G)$ if and only if one of the following conditions holds:*

(1) $G = C_3, C_4, C_5, C_7$.

(2) *C is adjacent to an end vertex, and G^* fulfills the conditions of Theorem 5.*

(3) *$C = C_4$, $c(x) \geq 3$ for all $x \in \Omega$, $\min\{d(a,G), d(b,G)\} = 2$ for all pairs of adjacent vertices $a, b \in V(C)$, and all components $T_1, ..., T_k$ of the subgraph $G_0 = G - V(C)$ are trees with $\gamma(T_i) = \alpha_0(T_i)$ for $i = 1, ..., k$ such that no minimum dominating set of G_0 contains a vertex from $N(V(C), G) \cap V(G_0)$.*

(4) *$C = C_3$ or $C = C_5$, $c(x) \geq 2$ for all $x \in \Omega$, all components $T_1, ..., T_k$ of $G_0 = G - V(C)$ are trees with $\gamma(T_i) = \alpha_0(T_i)$ for $i = 1, ..., k$, and for each T_i there exists one vertex $w_i \in \Omega \cap V(T_i)$ such that $c(w_i) = 2$. Furthermore, in the case $C = C_3$ there exists at least one vertex $a \in V(C)$ with $d(a, G) = 2$, and in the case $C = C_5$ from any two neighbours on C there is one of degree 2.*

Proof. The sufficiency follows from Proposition 2 and Theorem 5 in the cases (1) and (2). For (3) we refer the reader to [9, Theorem 6]. Now let G be of the form (4). Since for all trees T_i there are vertices $w_i \in \Omega \cap V(T_i)$ with $c(w_i) = 2$, there exist minimum dominating sets D_i and maximum matchings M_i such that $N(V(C), G) \subseteq \bigcup_{i=1}^{k} D_i = D$ and $M = \bigcup_{i=1}^{k} M_i$ saturates $N(V(C), G)$, and all edges of M which are incident with a vertex of $N(V(C), G)$ are end edges of G and T_i. If we choose in the case $C = C_3$ an arbitrary vertex $v \in V(C)$ and an arbitrary edge $e \in E(C)$, then $D' = D \cup \{v\}$ is a dominating set of G, and $M' = M \cup \{e\}$ is a matching of G. It is easy to see that D' is a minimum dominating set of G, and from $\gamma(T_i) = \alpha_0(T_i)$ for all $i = 1, ..., k$ we deduce $|D'| = |M'|$. Furthermore, there is no M'-augmenting path in G, so that Berge's Theorem shows $\alpha_0(G) = |M'| = \gamma(G)$. With the same arguments we can prove the case that $C = C_5$.

Conversely, we assume that G is not of the form (1) or (2). Then $c(x) \geq 2$ for all $x \in \Omega$, and because of Lemma 1, it is immediate that $\gamma(T_i) = \alpha_0(T_i)$ for all $i = 1, ..., k$. If $C \neq C_3, C_4, C_5, C_7$, then $G - V(C)$ has no isolated vertex and $\gamma(C) < \alpha_0(C)$. Consequently, $F = (G - V(C)) \cup C$ is a factor of G without isolated vertices such that $\gamma(F) < \alpha_0(F)$, a contradiction to Lemma 1. This implies that $C = C_3, C_4, C_5, C_7$.

In the case $C = C_4$ we refer the reader to [9, Theorem 6].

Now let $C = C_3$. Assume that there exists a tree $T_j = T$ such that $c(x) \geq 3$ for all $x \in \Omega \cap V(T)$. Let $u \in V(T)$ be the unique vertex adjacent to C. Then the induced subgraph $H = G[V(C) \cup \{u\}]$ has the property $\gamma(H) = 1 < 2 = \alpha_0(H)$ and $G - V(H)$ contains no isolated vertices. This is

again a contradiction to Lemma 1. Now suppose, to the contrary, that there is no vertex $x \in V(C)$ with $d(x,G) = 2$. Let $\Omega_2 = \{v | v \in \Omega,\ c(v) = 2\}$, and $H = G[V(C) \cup \Omega_2 \cup N(\Omega_2, G)]$. We see that $\gamma(H) = k < k+1 = \alpha_0(H)$, and $G - V(H)$ is a subgraph without isolated vertices. According to Lemma 1, this is impossible.
By similar observations we get the desired result for $C = C_5$.
Finally, let $C = C_7$. Analogously to the case $C = C_3$ it is possible to show that there is no tree $T_j = T$ with $c(x) \geq 3$ for all $x \in \Omega \cap V(T)$. Now let u be a vertex with $c(u) = 1$ and $U = N(u, G) \cap \Omega$. Then the subgraph $H = G[V(C) \cup U \cup \{u\}]$ fulfills the inequality $\gamma(H) = 3 < 4 = \alpha_0(H)$. This contradicts Lemma 1, since the graph $G - V(H)$ has no isolated vertices. With this contradiction we have exhausted all the possibilities and the proof is complete. □

6. BLOCK GRAPHS

We now turn our attention to block graphs. We begin with some definitions and notations. A vertex v of a graph G is called a cut vertex of G if $G - v$ has more components than G. A connected graph with no cut vertex is called a block. A block of a graph G is a subgraph of G which is itself a block and which is maximal with respect to that property. A graph G is a block graph if every block of G is a complete graph.

Theorem 8. *Let G be a connected block graph of order $n(G) \geq 2$, $\Omega = \Omega(G)$, and $G^* = G - \bar{N}(\Omega, G)$. Then we have $\gamma(G) = \alpha_0(G)$ if and only if $G^* = \emptyset$ or each component of G^* is an isolated vertex, a star, or a triangle, where the centers of the stars are not adjacent to a vertex of $N(\Omega, G)$ and not all vertices of each triangle are adjacent to a vertex of $N(\Omega, G)$, but at least one vertex of each triangle is adjacent to a vertex of $N(\Omega, G)$.*

Proof. The sufficiency is not hard to check. For the converse we first observe that all components of G^* are block graphs. Let H be a component of G^*. If H is a tree, then the proof of Theorem 5 yields the desired result.
Now we assume that H is not a tree. Let I be the set of isolated vertices of G^* and $R = G^* - H - I$.
If H is a block with $n(H) \geq 4$, then $\gamma(H) = 1 < \alpha_0(H)$, and therefore we deduce from Theorem 1

$$\begin{aligned}\gamma(G) &\leq |N(\Omega, G)| + \gamma(R) + 1 \\ &< |N(\Omega, G)| + \alpha_0(R) + \alpha_0(H) \leq \alpha_0(G),\end{aligned}$$

a contradiction to $\gamma(G) = \alpha_0(G)$. If $H = K_3$ and all vertices of H are adjacent to a vertex of $N(\Omega, G)$, then analogously we obtain a contradiction. Furthermore, the connectivity of G implies that at least one vertex of H is adjacent to a vertex of $N(\Omega, G)$.

Finally, we discuss the case that H is not a tree with at least one cut vertex. Let B_1 be a block of H of order $n(B_1) \geq 3$, B_2 a further block of H with $V(B_1) \cap V(B_2) \neq \emptyset$, and $H' = G[V(B_1) \cup V(B_2)]$. From the fact that G is a block graph we deduce that $G - H'$ contains no isolated vertex. Hence Theorem 1 yields

$$\begin{aligned}\gamma(G) &\leq \gamma(G-H') + \gamma(H') \leq \alpha_0(G-H') + 1 \\ &< \alpha_0(G-H') + \alpha_0(H') \leq \alpha_0(G).\end{aligned}$$

This contradiction to $\gamma(G) = \alpha_0(G)$ completes the proof of Theorem 8. □

7. Locally connected graphs

A graph G is locally connected if for each $v \in V(G)$, $N(v,G) \neq \emptyset$ and the induced subgraph $G[N(v,G)]$ is connected (for some basic results on this class of graphs see [2]).

Theorem 9. *Let G be a connected and locally connected graph. Then $\gamma(G) = \alpha_0(G)$ if and only if $G = K_2$ or $G = K_3$.*

Proof. The sufficiency is obvious, and the converse is immediate if the order $n(G) \leq 3$. For $n(G) \geq 4$ we choose a vertex $u \in V(G)$ with

$$|N(u,G)| = \min\{|N(x,G)| \,|x \in V(G) \text{ with } |N(x,G)| \geq 3\}.$$

Since G is locally connected, we conclude for the induced subgraph $H = G[\bar{N}(u,G)]$ the inequality $\gamma(H) = 1 < 2 \leq \alpha_0(H)$.
In the case that $G' = G - \bar{N}(u,G)$ contains no isolated vertices, this inequality yields a contradiction to Lemma 1.
If the subgraph G' has an isolated vertex w, then by the definition of u, it follows $d(w,G) = 2$ or $d(w,G) = |N(u,G)|$. Now let $a_1, ..., a_p$ and $b_1, ..., b_m$ be the isolated vertices of G' with $d(a_i, G) = 2$ and $d(b_j, G) = d(u,G)$ for $1 \leq i \leq p$ and $1 \leq j \leq m$, respectively. Furthermore, define $A = \{a_1, ..., a_p\}$, $B = \{b_1, ..., b_m\}$, and $F = G[\bar{N}(u,G) \cup A \cup B]$. By construction, the graph $G - F$ contains no isolated vertices and therefore, by Lemma 1, it suffices to prove $\gamma(F) < \alpha_0(F)$.
For $d(u,G) = 3$ it is easy to check that $\gamma(F) = 1 < 2 \leq \alpha_0(F)$ or $\gamma(F) = 2 < 3 \leq \alpha_0(F)$. In the remaining case $d(u,G) \geq 4$, we assume without loss of generality that $|N(a_i,G) \cap N(a_j,G)| \leq 1$ for $i \neq j$. Now we consider two cases.
Case 1. If $N(a_i,G) \cap N(a_j,G) = \emptyset$ for all $i \neq j$, then it is easy to see:
If $p = 0$ and $m \geq 1$ or $p = 1$, then $\gamma(F) \leq 2 < 3 \leq \alpha_0(F)$.
If $p = 2$ and $d(u,G) = 4$, then $\gamma(F) \leq 2 < 3 \leq \alpha_0(F)$.
If $p = 2$ and $d(u,G) \geq 5$, then $\gamma(F) \leq 3 < 4 \leq \alpha_0(F)$.
If $p \geq 3$, then $\gamma(F) \leq p + 1 < p + 2 \leq \alpha_0(F)$.

Case 2. There exist two vertices $a_i, a_j \in A$ with $|N(a_i, G) \cap N(a_j, G)| = 1$. Suppose without loss of generality that $N(a_1, G) = \{x_1, x_2\}$ and $N(a_2, G) = \{x_2, x_3\}$. If there is no vertex $a_j \in A$ with $j \geq 3$ and $N(a_j, G) = \{x_1, x_3\}$, then the subgraph $J = G[\{x_1, x_2, x_3, a_1, a_2\}]$ of F has the property $\gamma(J) = 1 < 2 = \alpha_0(J)$. Since $d(u, G) \geq 4$, the subgraph $F - J$ contains no isolated vertices and thus we get the desired inequality $\gamma(F) < \alpha_0(F)$. If there exists a vertex $a_3 \in A$ with $N(a_3, G) = \{x_1, x_3\}$, then for $L = G[\{x_1, x_2, x_3, a_1, a_2, a_3\}]$ the inequality $\gamma(L) = 2 < 3 \leq \alpha_0(L)$ holds, and again it follows $\gamma(F) < \alpha_0(F)$. This completes the proof of Theorem 9. □

Corollary 5. *If G is a connected and locally connected graph, then $\gamma(G) = \beta(G)$ if and only if $G = K_2$.*

Proof. The sufficiency is obvious. If $\gamma(G) = \beta(G)$, then it follows from Corollary 1 that $\gamma(G) = \alpha_0(G)$. Since $\gamma(K_3) \neq \beta(K_3)$, Theorem 9 yields the desired result. □

A graph is chordal if it contains no cycle of length greater than three as an induced subgraph.

Lemma 3. *If G is a chordal block, then G is locally connected.*

Proof. Assume to the contrary that there exists a vertex $u \in V(G)$ such that $G[N(u, G)]$ is not connected. Since G is a block, the subgraph $G - u$ is connected. If $(a, ..., b)$ is a shortest path in $G - u$ between two components of $G[N(u, G)]$, then $(u, a, ..., b, u)$ is an induced cycle of length greater than three. This is a contradiction to the hypothesis that G is chordal. □

From Theorem 9 and Lemma 3 we deduce immediately the following result.

Corollary 6. *If G is a chordal block, then $\gamma(G) = \alpha_0(G)$ if and only if $G = K_2$ or $G = K_3$.*

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