On the Spectrum of Maximal k-Multiple Free Sets of Integers

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ABSTRACT. A set of integers is k-multiple-free if it never contains two integers x and kx, where k is a given integer greater than 1. Such a set S is maximal in $[1,n] = \{1,2,\ldots,n\}$ if $S \cup \{t\}$ is not k-multiple free for any t in $[1,n]\backslash S$. In this paper we investigate the size of maximal k-multiple-free subsets of $[1,n]$, prove that the smallest such set has $\frac{(k^5-k^3+1)n}{k(k+1)(k^3-1)}+0(\log n)$ members, and show that given k and n, if s is any integer between the minimum and maximum possible orders, there is a maximal k-multiple-free subset of $[1,n]$ with s elements.

1 Introduction

Throughout this paper n and k are fixed integers; $k > 1$. The base k expansion of n is

$$n = \sum_{i=0}^{m} a_i k^i,$$

where $0 \leq a_i \leq k-1$ and $a_m \neq 0$. The set of all integers from p to q inclusive is denoted $[p,q]$. We write $\{i\}_j$ for a function which equals 1 when j divides i and 0 otherwise.

A subset S of a set T is called *k-multiple-free* if $S \cap kS = \emptyset$, where $kS = \{ks \colon s \in S\}$; a k-multiple-free subset S of T is called *maximal* if

$S \cup \{t\}$ is not k-multiple-free for any $t \in T\backslash S$. We write $M = M(k,n)$ for the set of all maximal k-multiple-free subsets of $[1,n]$, and define

$$f_k(n) = \max\{|S|: S \in M\}$$
$$g_k(n) = \min\{|S|: S \in M\}.$$

It was shown by Wang [4] (case $k=2$) and Leung and Wei [3] (case $k>2$) that

$$f_k(n) = n - \left\lfloor \frac{n}{k} \right\rfloor + f_k\left(\left\lfloor \frac{n}{k^2} \right\rfloor\right), \quad f_k(n) \geq 1 \tag{1}$$

whence

$$f_k(n) = \sum_{i \geq 0} (-1)^i \left\lfloor \frac{n}{k^i} \right\rfloor, \tag{2}$$

and

$$f_k(n) = \frac{k}{k+1}n + \frac{1}{k+1}\sum_{i\geq 0}(-1)^i a_i. \tag{3}$$

This was used to prove that for large n

$$f_k(n) = \frac{k}{k+1}n + O(\log n) \tag{4}$$

which was conjectured by Janous [1].

The formula (2) was proven independently by Lai [2], who also proved some properties of $g_k(n)$:

$$g_k(n+i) = g_k(n) + k - 1 \text{ if } i = k, n+k = bk^{\alpha}, \tag{5a}$$

where $k \nmid b$ and $\alpha \equiv O \pmod 3), = g_k(n) + i$ otherwise when $i \leq k$; (5b)

and in the case $n = k^m$,

$$g_k(k^m) = \left\lceil \frac{m+1}{3} \right\rceil + (k-2)\left\lceil \frac{m}{3} \right\rceil + \sum_{i=1}^{m-1} k^{i-1}(k-1)^2 \left\lceil \frac{m-i}{3} \right\rceil \tag{6}$$

In this paper we focus on two problems concerning $f_k(n)$ and $g_k(n)$:

(i) investigating the asymptotic behavior of $g_k(n)$;

(ii) determining the spectrum $\{|S|: S \in M\}$.

In Section 4 below we prove some formulae relating to problem (i), and other formulae for $f_k(n)$ and $g_k(n)$. We use these formulae to prove that the spectrum is $[g_k(n), f_k(n)]$, i.e. any integer between $g_k(n)$ and $f_k(n)$ can be realized as the order of a maximal k-multiple-free subset of $[1,n]$. Sections 2 and 3 contain preliminary lemmas.

2 Adjacency

We first consider a simpler idea than multiple-freedom. We say a subset S of $[1,n]$ is *adjacency-free* if S never contains both i and $i+1$ for any i, and such an S is *maximal adjacency-free* if $S \cup \{x\}$ is never adjacency-free for $x \in [1,n] \backslash S$. We write $A(n)$ for the set of maximal adjacency-free subsets of $[1,n]$.

Lemma 1. *There is a maximal adjacency-free subset S of $[1,n]$ if and only if*

$$\left\lceil \frac{n}{3} \right\rceil \leq |S| \leq \left\lceil \frac{n}{2} \right\rceil .$$

Proof: Since an adjacency-free subset can contain at most one of each of the sets $\{1,2\},\{3,4\},\ldots$, it can have at most $\lceil \frac{n}{2} \rceil$ elements. On the other hand, if S does not contain any of $\{x-1, x, x+1\}$, then $S \cup \{x\}$ is also adjacency-free, denying the maximality of S, so any maximal adjacency-free subset must contain at least one member of each of $\{1,2,3\},\{4,5,6\},\ldots$, which implies the lower bound.

To prove the density of the spectrum, we consider

$$S_x = \{2, 5, \ldots, 3x-1, 3x+1, 3x+3, \ldots\}.$$

In other words, S_x contains every third integer from 2 to $3x-1$, and every second integer thereafter. Then

$$|S_x| = x + \left\lceil \frac{n-3x+1}{2} \right\rceil .$$

So S_0 and $S_{\lceil \frac{n}{3} \rceil}$ attain the upper and lower bounds respectively. On the other hand,

$$\begin{aligned} |S_{x-2}| &= x - 2 + \left\lceil \frac{n-3x+7}{2} \right\rceil \\ &= x - 2 + \left\lceil \frac{n-3x+1}{2} \right\rceil + 3 \\ &= |S_x| + 1 \end{aligned}$$

so $S_0, S_2, S_4, \ldots$ between them every possible order from $\lceil \frac{n}{3} \rceil$ to $\lceil \frac{n}{2} \rceil$. □

3 Some coefficients related to divisibility

As usual we write $k^e \| x$ to mean $k^e | x$ but $k^{e+1} \nmid x$.

Lemma 2. *Suppose $s \geq 3$, $0 < a < k$, and*

$$\begin{aligned} B = B_s(a) &= \{x \in [1, ak^s - 1] : k^{3e+2} \| x \text{ for some } e \geq 0\} \\ b_s(a) &= |B_s(a)|. \end{aligned}$$

Then

$$
\begin{aligned}
b_s(a) + \left\lceil \frac{s-1}{6} \right\rceil = c_s(a) &= \frac{ak^2(k^s-1)}{(k+1)(k^3-1)}, \quad s \equiv 0 \pmod 6) \\
&= \frac{ak^3(k^{s-1}-1)}{(k+1)(k^3-1)}, \quad s \equiv 1 \pmod 6 \\
&= \frac{a}{k+1}\left(\frac{k(k^{s+1}-1)}{k^3-1} - 1\right), \quad s \equiv 2 \pmod 6 \\
&= \frac{a}{k+1}\left(\frac{k^2(k^s-1)}{k^3-1} + 1\right), \quad s \equiv 3 \pmod 6 \\
&= \frac{a}{k+1}\left(\frac{k^3(k^{s-1}-1)}{k^3-1} - 1\right), \quad s \equiv 4 \pmod 6 \\
&= \frac{ak(k^{s+1}-1)}{(k+1)(k^3-1)}, \quad s \equiv 5 \pmod 6.
\end{aligned}
$$

Proof: It will be convenient to write $t = \frac{s-2}{3}$. If e is a non-negative integer, define

$$B_e = \{x \in [1, ak^s - 1] \colon k^e \| x\}.$$

Then using the inclusion-exclusion principle,

$$
\begin{aligned}
|B_e| &= \sum_{i \geq 0} (-1)^i \left\lfloor \frac{ak^s - 1}{k^{e+i}} \right\rfloor \\
&= a \sum_{i=0}^{s-x} (-1)^i k^{s-e-i} - \sum_{i=0}^{s-e} (-1)^i \\
&= (-1)^{s-e} a \sum_{i=0}^{s-e} (-k)^i - \{s-e\}_2 \\
&= a \frac{(k^{s-e+1} - (-1)^{s-e})}{k+1} - \{s-e\}_2.
\end{aligned}
$$

So

$$
\begin{aligned}
b_s(a) &= \sum_{0 \leq e \leq t} |B_{3e+2}| \\
&= \frac{a}{k+1} \sum_{0 \leq e \leq t} (k^{s-3e-1} - (-1)^{s-3e-2}) \\
&\quad - \sum_{0 \leq e \leq t} \{s - 3e - 2\}_2.
\end{aligned}
$$

There are three cases, according to the residue class of s modulo 3. We assume $s = 3p$. Then

$$\begin{aligned}
\sum_{0\le e\le t} k^{s-3e-1} &= \sum_{0\le e\le p-1} k^{3p-3e-1} \\
&= k^2 \sum_{0\le e\le p-1} k^{3(p-1-e)} \\
&= \frac{k^2(k^s-1)}{k^3-1} \\
\sum_{0\le e\le t} (-1)^{s-3e-2} &= \sum_{0\le e\le p-1} (-1)^{3(p-e)} \\
&= 0 \text{ if } p\equiv 0 \pmod 2, \quad s\equiv 0 \pmod 6 \\
&\ 1 \text{ if } p\equiv 1 \pmod 2, \quad s\equiv 3 \pmod 6 \\
\sum_{0\le e\le t} (s-3e-2)_2 &= \sum_{0\le e\le p-1} (p-e)_2
\end{aligned}$$

which counts the number of non-negative integers less than p and congruent to e mod 2, i.e. $\lceil \frac{p}{2} \rceil$. Putting these together we have the result for cases $s \equiv 0$ or $3 \pmod 6$.

The other two cases, $s = 3p+1$ and $s = 3p+2$, are handled similarly. □

4 Main Results

Theorem 1. *Let k be a fixed integer, $k \ge 2$. If n is an integer greater than 1, let $a_0, a_1, \ldots a_n$ be integers defined by $n = a_0 + a_1k + \cdots + a_mk^m$, $0 \le a_i < k$. Let $\sum_{(i)}$ denote summation over $j \equiv i \pmod 6$. Then*

$$\begin{aligned}
g_k(n) = \frac{k^5-k^3+1}{k(k+1)(k^3-1)}n - \sum_{3\le j\le n} sgn(a_j)\left(\left\lceil\frac{j-2}{6}\right\rceil - \{j\}_3\right) + \frac{a_0}{k} \\
- \frac{1}{(k+1)(k^3-1)}\left\{ k\sum_{(0)}(a_j+1) + k^2\sum_{(1)} a_j + k^3\sum_{(2)} a_j \right. \\
\left. +(k^3+k-1)\sum_{(3)} a_j - (k^3-k^2-1)\sum_{(4)} a_j + (2k^3-1)\sum_{(5)} a_j \right\}.
\end{aligned}$$

Proof: From (5b),

$$\begin{aligned}
g_k(n) &= g_k(a_mk^m + \cdots + a_1k + a_0) \\
&= g_k(a_mk^m + \cdots + a_1k) + a_0 \qquad (7)
\end{aligned}$$

If $a_1 > 0$, write $a_1 k$ as $(a_1 - 1)k + k$. Then from (5a)

$$\begin{aligned} &g_k(a_m k^m + \cdots + a_1 k) + a_0 \\ &= g_k(a_m k^m + \cdots + a_2 k^2 + (a_1 - 1)k) + (k-1) + a_0 \end{aligned}$$

and repeating the same technique we get

$$g_k(a_m k^m + \cdots + a_2 k^2 + (a_1 - 2)k) + 2(k-1) + a_0$$

and eventually

$$g_k(n) = g_k(a_m k^m + \cdots + a_2 k^2) + a_1(k-1) + a_0 \tag{8}$$

(Although we assumed $a_1 > 0$, (8) clearly holds when $a_1 = 0$ also.) But by the same technique

$$\begin{aligned} &g_k(a_m k^m + \cdots + a_2 k^2) \\ &\quad = g_k(a_m k^m + \cdots + a_3 k^3) + a_2 k(k-1). \end{aligned} \tag{9}$$

For any $j \geq 3$ such that $a_j \geq 1$ we write

$$\begin{aligned} a_j k^j &= (a_j k^{j-1} - 1)k + k, \\ a_j k^j - k &= (a_j k^{j-1} - 2)k + k \\ &\cdots \\ a_j k^j - qk &= (a_j k^{j-1} - (q+1))k + k, \quad 0 \leq q \leq a_j k^{j-1} - 1 \\ &\cdots \\ a_j k^j - (a_j k^{j-1} - 1)k &= k. \end{aligned}$$

It is easy to see that if $j \geq 3$, $1 \leq a_j \leq k-1$ and $1 \leq q \leq ak^{j-1} - 1$, then

$$k^{3(e+1)} \| a_j k^j - qk \text{ for some } e \geq 0$$

if and only if

$$k^{3e+2} \| q.$$

Therefore, from (5a) and (5b) and Lemma 2 we have

$$\begin{aligned} &g_k(a_m k^m + \cdots + a_{j+1} k^{j+1} + a_j k^j) \\ &\quad = g_k(a_m k^m + \cdots + a_{j+1} k^{j+1}) \\ &\qquad + sgn(a_j) \left\{ (b_{j-1}(a_j) + \{j\}_3)k + (a_j k^{j-1}(a_j) - \{j\}_3)(k-1) \right\} \\ &\qquad\qquad \text{for any } j \geq 3. \end{aligned} \tag{10}$$

(The term $\{j\}_3$ on the right-hand side of (10) reflects the fact that when $1 \leq a_j \leq k$, $k^{3e} \| a_j k^j$ holds for some $e \geq 1$ if and only if $3|j$.) Simplifying

(10), we get

$$\begin{aligned} g(a_m k^m &+ \cdots + a_{j+1}k^{j+1} + a_j k^j) \\ &= g_k(a_m k^m + \cdots + a_{j+1}k^{j+1}) + a_j k^{j-1}(k-1) \\ &\quad + sgn(a_j) b_{j-1}(a_j) + sgn(a_j)\{j\}_3 \\ &= g_k(a_m k^m + \cdots + a_{j+1}k^{j+1}) + a_j k^{j-1}(k-1) \\ &\quad + sgn(a_j) c_{j-1}(a_j) - sgn(a_j)\left(\left\lceil \frac{j-2}{6} \right\rceil - \{j\}_3\right) \\ &= g_k(a_m k^m + \cdots + a_{j+1}k^{j+1}) + a_j k^{j-1}(k-1) \\ &\quad + c_{j-1}(a_j) - sgn(a_j)\left(\left\lceil \frac{j-2}{6} \right\rceil - \{j\}_3\right), \quad j \geq 3. \end{aligned} \tag{11}$$

By repeated application of (11), we get from (8) and (9) that

$$\begin{aligned} g_k(n) = &\sum_{3\leq j\leq m} a_j k^{j-1}(k-1) + \sum_{3\leq j\leq m} c_{j-1}(a_j) \\ &- \sum_{3\leq j\leq m} sgn(a_j)\left(\left\lceil \frac{j-2}{6} \right\rceil - \{j\}_3\right) + a_2 k(k-1) + a_1(k-1) + a_0. \end{aligned} \tag{12}$$

Applying the values of $c_{j-1}(a_j)$ from Lemma 2,

$$\begin{aligned} g_k(n) = &\frac{k-1}{k}n + \frac{k}{(k+1)(k^3-1)}n - \sum_{3\leq j\leq m} sgn(a_j)\left(\left\lceil \frac{j-2}{6} \right\rceil - \{j\}_3\right) \\ &+ \frac{a_0}{k} - \frac{k}{(k+1)(k^3-1)}(a_2k^2 + a_1 k + a_0) \\ &- \frac{1}{(k+1)(k^3-1)}\left\{ k\sum_{\substack{(0)\\ j\geq 3}} a_j + k^2 \sum_{\substack{(1)\\ j\geq 3}} a_j + k^3 \sum_{\substack{(2)\\ j\geq 3}} a_j \right. \\ &\left. +(k^3+k-1)\sum_{(3)} a_j - (k^3-k^2-1)\sum_{(4)} a_j + (2k^3-1)\sum_{(5)} a_j \right\}, \end{aligned}$$

which, on simplification, gives the desired result. □

Corollary 1.1. *For large n,*

$$g_k(n) = \frac{k^5 - k^3 + 1}{k(k+1)(k^3-1)}n + O(\log n).$$

□

Using a different viewpoint, we obtain other formulae for $g_k(n)$ and $f_k(n)$ and find the spectrum of M.

To do this, write

$$P = \{p \in [1,n] : k \nmid p\}$$

$$\ell(p) = \left\lfloor \log_k \frac{n}{p} \right\rfloor, \quad p \in P$$

$$Q_p = \{p, pk, pk^2, \ldots, pk^{\ell(p)}\}, \quad p \in P.$$

clearly $Q_p \cap Q_r = \emptyset$ if p and r are distinct elements of P, and

$$[1,n] = \bigcup_{p \in P} Q_p. \tag{13}$$

If S is any maximal k-multiple-free subset of $[1,n]$, write $S_p = S \cap Q_p$. Then S_p is maximal k-multiple-free in Q_p. Conversely, given a set of maximal k-multiple-free subsets S_p of Q_p, for $p \in P$, their union is maximal k-multiple-free.

On the other hand, consider the one-to-one correspondence φ from Q_p to $[1, \ell(p)+1]$ defined by $\varphi(pk^i) = i+1$. It is clear that S_p is maximal k-multiple-free in Q_p if and only if $\varphi(S_p)$ is maximal adjacency-free in $[1, \ell(p)+1]$. So from Lemma 1 we have

$$\begin{aligned} &\{|S_p| : S_p \text{ is a maximal } k\text{-multiple free subset of } Q_p\} \\ &= \{|V_p| = V_p \text{ is a maximal adjacency-free subset of } [1, \ell(p)+1]\} \\ &= \left[\left\lceil \frac{\ell(p)+1}{3} \right\rceil, \left\lceil \frac{\ell(p)+1}{2} \right\rceil\right]. \end{aligned} \tag{14}$$

We can choose S_p to have any value in the range (14), for each p in P. So we have

Theorem 2.

$$g_k(n) = \sum_{p \in P} \left\lceil \frac{\ell(p)+1}{3} \right\rceil; \quad f_k(n) = \sum_{p \in P} \left\lceil \frac{\ell(p)+1}{2} \right\rceil.$$

For any value s in $[g_k(n), f_k(n)]$, there is a maximal k-multiple-free subset of $[1,n]$ with s elements. □

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