# Total Domination of the $m \times n$ Chessboard by Kings, Crosses, and Knights

*D.K. Garnick and N.A. Nieuwejaar*

*ABSTRACT*

For graph $G$, a *total dominating set* $S$ is a subset of the vertices in $G$ such that every vertex in $G$ is adjacent to at least one vertex in $S$. The *total domination number* of $G$ is the cardinality of a smallest total dominating set of $G$. We consider the total domination number of graphs formed from an $m \times n$ chessboard by letting vertices represent the squares, and letting two vertices be adjacent if a given chess piece can move between the associated squares. In particular, we bound from above and below the total domination numbers of the graphs induced by the movement of kings, knights, and crosses (a hypothetical piece that moves as does a king, except that it cannot move diagonally). We also provide some results of computer searches for the total domination numbers of small square boards.

## 1 Introduction.

Combinatorial problems on the chessboard have been proposed and studied for a long time. A classic problem is that of covering all of the squares of the chessboard with a minimum number of pieces of a given type. Typically the problem statement specifies that a piece covers all of the squares to which it could move, (according to the standard movement of chess pieces) as well the square on which it sits. It is this last provision that motivates this paper; in the game of chess, as well as in some real-world applications, vertices are not self-covering. Thus, we obtain some results on covering the chessboard under the assumption that a piece does not cover its own square.

We base our discussion on the standard graph theoretic description of the chessboard. We represent each square of an $m \times n$ board with a vertex; an edge connects two vertices, $x$ and $y$, if a piece placed on the square corresponding to $x$ covers the square corresponding to $y$. We denote such a graph representing the movement of kings on an $m \times n$ board by $K_{m,n}$. Similarly we denote the graphs for the placement of queens, bishops, knights, or rooks by $Q_{m,n}$, $B_{m,n}$, $N_{m,n}$, or $R_{m,n}$ (respectively). We use

$C_{m,n}$ to denote the *crosses'* graph. (The *cross* is a piece that is capable of moving only one square per turn, either horizontally or vertically; it is of interest because the graph representing its movement on an $m \times n$ board is the $m \times n$ grid.) For square boards, one subscript suffices; thus, $K_n$ is the kings' graph for the $n \times n$ board.

We now provide some definitions. Let $G = (V, E)$ be a simple graph. Given $x \in V$, $N(x)$ and $N[x] = N(x) \cup \{x\}$ denote the open and closed neighborhoods, respectively, of $x$. For $S \subset V$, define $N(S) = \bigcup_{x \in S} N(x)$; we define $N[S]$ similarly. We use $\lfloor\ \rfloor$ and $\lceil\ \rceil$ to denote the floor and ceiling functions respectively.

For graph $G$, $S \subset V(G)$ is a *dominating set* if every vertex in $V(G) - S$ is adjacent to at least one vertex in $S$. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. Thus, finding the minimum number of kings necessary to cover the standard chessboard reduces to the problem of determining the value of $\gamma(K_8)$. Yaglom and Yaglom [8] obtained the following values for $\gamma$:

$$\gamma(K_n) = \lfloor (n+2)/3 \rfloor^2 \qquad \gamma(B_n) = \gamma(R_n) = n$$

[1] surveys some results on the domination numbers of chessboard graphs.

A *total dominating set* of graph $G$ is a set $S \subset V(G)$ such that every vertex in $V(G)$ is adjacent to at least one vertex in $S$. Thus, total domination of the chessboard requires the pieces to be mutually protecting. The *total domination number*, $\tau(G)$, is the minimum cardinality of a total dominating set of $G$. One hundred years ago, W. W. Rouse Ball [7] obtained the following values of $\tau$:

| $G$ | $Q_8$ | $B_8$ | $N_8$ | $R_8$ |
|---|---|---|---|---|
| $\tau(G)$ | 5 | 10 | 14 | 8 |

Since every total dominating set of a graph is also a dominating set of that graph, then for all graphs $G$, $\gamma(G) \le \tau(G)$. Since $\gamma(R_n) = n$, then it follows that $\tau(R_n) = n$; a total dominating set in $R_n$ consists of all $n$ squares in any one row. $\tau(B_n)$ is determined in [2]. An upper bound on $\tau(Q_n)$ is established by diagonal dominating sets of queens as described in [3].

In this paper we bound from above and below the values of $\tau(K_{m,n})$, $\tau(C_{m,n})$, and $\tau(N_{m,n})$. We also provide exact values of these functions for small square chessboards, and use heuristic search algorithms to improve the upper bounds on boards of modest size.

## 2 Total domination of the kings' graph.

It is simple to determine the value of $\tau(K_{m,n})$ when the board is sufficiently narrow. We offer the following results without proof.

**Proposition 2.1**

$$\tau(K_{m,n}) = \begin{cases} n/2 & m \leq 3, n > 1, \text{ and } n \equiv 0 \pmod 4 \\ \lfloor n/2 \rfloor + 1 & m \leq 3, n > 1, \text{ and } n \not\equiv 0 \pmod 4 \\ 2\lceil n/3 \rceil & m = 4 \end{cases}$$

We proceed to a lower bound on $\tau(K_{m,n})$.

**Theorem 2.2** *For all $m, n \geq 5$, $\tau(K_{m,n}) > mn/7$.*

**Proof.** We note that for $x \in V(K_{m,n})$, $|N(x)|$ is at most 8. Further, we note that for $S \subset V(K_{m,m})$ and $x \in N(S) - S$, $|N(S \cup \{x\})| \leq |N(S)| + 6$. Since every king must be adjacent to at least one other king, $mn/\tau(K_{m,n})$ is at most 7. This ratio occurs only when the board is tiled with diagonally adjacent pairs of kings as shown in Figure 1; we call this graph of 14 vertices the *kings' tile*. Thus, any total dominating set, of $K_{m,n}$, which has cardinality $mn/7$ consists of a perfect tiling (no overlap or waste) of the $m \times n$ chessboard with kings' tiles. Since such a perfect tiling is not possible, we obtain the lower bound in the theorem statement. ■

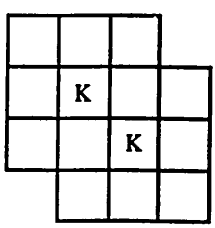


Figure 1: The kings' tile

We now provide an upper bound on $\tau(K_{m,n})$ based on a tiling of the plane with kings' tiles.

**Theorem 2.3** *For all $m, n > 4$, $\tau(K_{m,n}) \leq (mn + 2n + 89)/7$*

**Proof.** We begin with the tiling of the board depicted in Figure 2. The figure shows the tiling for the $24 \times 24$ chessboard; we will always assume

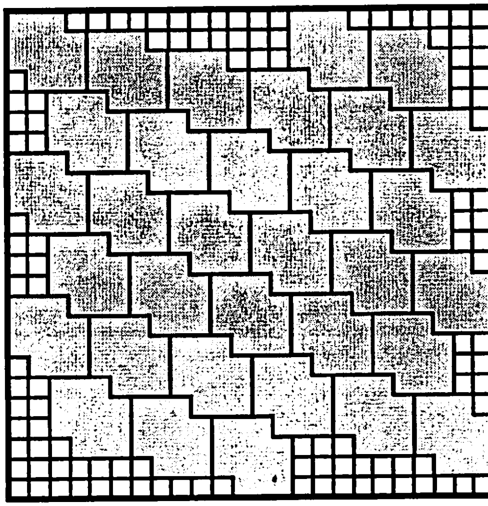

Figure 2: Tiling the chessboard with kings' tiles

that the tiling begins by placing a tile squarely into the top left corner of the board, and only considering non-overlapping tiles that lie entirely within the borders of the board.

We now determine the value of $a$, a lower bound on the number of squares on the board not part of tiles in the tiling scheme above. We look in turn at the uncovered squares near each of the four sides of the board.

1. Along the top of the board, there are at least $21\lfloor n/14 \rfloor$ uncovered squares;
2. Along the left side, there are at least $7\lfloor m/7 \rfloor$ uncovered squares;
3. Along the bottom, there are at least $21\lfloor n/14 \rfloor$ uncovered squares.
4. Along the right side, there are at least $7\lfloor m/7 \rfloor$ uncovered squares when $n$ is even, and $14\lfloor m/7 \rfloor$ when $n$ is odd.

Summing these, we have:

$$a = \begin{cases} 42\lfloor n/14 \rfloor + 14\lfloor m/7 \rfloor & n \text{ is even} \\ 42\lfloor n/14 \rfloor + 21\lfloor m/7 \rfloor & n \text{ is odd} \end{cases}$$

Thus, under the tiling, at most $mn - a$ squares are covered by tiles, and at most $(mn - a)/7$ kings are in the covering tiles.

We now determine an upper bound, $b$, on how many kings are needed to cover the squares on the board that are not in tiles under the tiling.

1. Five kings are sufficient to cover each set of 21 contiguous squares along the top that are not in tiles; the leftmost 3 uncovered squares can be covered by adding a king to the tile immediately below, and 4 more kings can cover the remaining 18 uncovered squares. Thus, along the top, $5\lceil n/14\rceil$ kings are sufficient to cover all of the uncovered squares.
2. Along the left side, two kings are sufficient to cover each set of 7 contiguous squares not in tiles. $2\lceil m/7\rceil$ kings can cover all such squares.
3. The uncovered squares along the bottom can be covered similarly to to those along the top, requiring $5\lceil n/14\rceil$ kings.
4. When $n$ is even, each set of 7 contiguous uncovered squares along the right side can be covered with two kings; $2\lceil m/7\rceil$ kings are sufficient for all such sets on the right side. When $n$ is odd, uncovered sets of 11 squares are joined by $1 \times 3$ strips of uncovered squares. Each strip can be covered by adding a single king to the tile immediately to the left of the strip; each set of 11 uncovered squares can be covered with two additional kings. Thus, when $n$ is odd, $3\lceil m/7\rceil$ kings are sufficient to cover the right side.

Summing these, we obtain:

$$b = \begin{cases} 10\lceil n/14\rceil + 2\lceil m/7\rceil & n \text{ is even} \\ 10\lceil n/14\rceil + 3\lceil m/7\rceil & n \text{ is odd} \end{cases}$$

Combining the number of kings used in the tiling, with the number of kings used to cover the untiled squares, we obtain:

$$\tau(K_{m,n}) \leq (mn - a)/7 + b$$

This simplifies (at worst) to the bound in the theorem statement. ∎

## 3 Total domination of the crosses' graph.

The value of $\tau(C_{m,n})$ is as follows for $1 \leq m \leq 2$:

**Proposition 3.1**

$$\tau(C_{m,n}) = \begin{cases} n/2 & m = 1 \text{ and } n \equiv 0 \pmod 4 \\ \lfloor n/2\rfloor + 1 & m = 1, n > 1, \text{ and } n \not\equiv 0 \pmod 4 \\ 2\lceil n/3\rceil & m = 2 \end{cases}$$

Using an argument similar to that for the kings' tile, there is a unique crosses' tile which provides coverage of four vertices per cross; the tile is shown in Figure 3. Based on a tiling of the $m \times n$ chessboard with crosses' tiles (as shown in Figure 4), we obtain the following bounds on $\tau(C_{m,n})$:

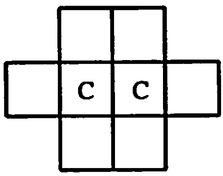


Figure 3: The crosses' tile

**Theorem 3.2** *For all* $m, n > 2$, $mn/4 < \tau(C_{m,n}) \leq (mn + 2m + 2n + 5)/4$.

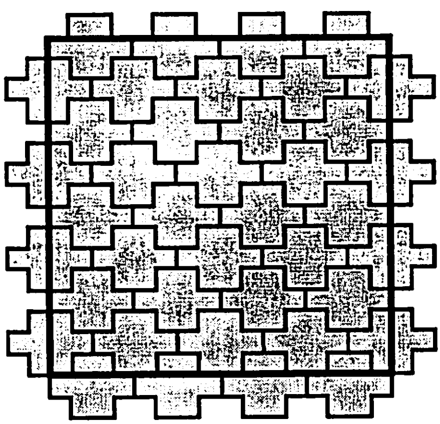

Figure 4: Tiling the chessboard with crosses' tiles

**Proof.** The lower bound derives from the maximum number of squares a cross can cover. To obtain the upper bound, we count the number of crosses needed to cover the region tiled by the odd rows of tiles (the tiles covering rows $i$ of the chessboard where $i \equiv 1 \pmod 4$) and the number of crosses needed to cover the region tiled by the even rows of tiles (the tiles covering rows $i$ of the board where $i \equiv 3 \pmod 4$). We count the partial tiles within the boundaries of the board as well as the full tiles.

1. In each odd numbered row of tiles, $2\lceil n/4 \rceil$ crosses are needed to cover the tiles, except when $n \equiv 1 \pmod 4$. In the special case a cross can be saved in each odd row of tiles since the last tile in the row covers

a single square; that square can be covered by an extra cross in the tile immediately to the left of the square. Since there are $\lfloor m/4 \rfloor + 1$ odd rows of tiles, then the number of crosses needed to cover the odd rows of tiles, is:

$$a = \begin{cases} 2\lfloor m/4 \rfloor \lceil n/4 \rceil + 2\lceil n/4 \rceil - \lfloor m/4 \rfloor - 1 & n \equiv 1 \pmod 4 \\ 2\lfloor m/4 \rfloor \lceil n/4 \rceil + 2\lceil n/4 \rceil & \text{otherwise} \end{cases}$$

2. Using a similar argument, the number of crosses needed in the even rows of tiles is:

$$b = \begin{cases} 2\lfloor (m+2)/4 \rfloor (\lceil (n+2)/4 \rceil - 1) & n \equiv 3 \pmod 4 \\ 2\lfloor (m+2)/4 \rfloor \lceil (n+2)/4 \rceil & \text{otherwise} \end{cases}$$

Summing $a$ and $b$, we obtain (at worst) the upper bound in the theorem statement. ■

## 4 Total domination of the knights' graph.

**Theorem 4.1** *For all $m, n > 4$, $mn/8 < \tau(N_{m,n})$ and*

$$\tau(N_{m,n}) \leq \begin{cases} (mn + 5m + 6n + 56)/8 & m \equiv n \pmod 2 \\ (mn + 5m + 5n + 43)/8 & m \text{ odd and } n \text{ even} \end{cases}$$

**Proof.** The lower bound derives from the fact that a knight can cover at most eight squares. The upper bound is based on the pattern of knights shown in Figure 5; the gray squares on the board are those not covered by knights in the pattern. Around the board we show how to cover the gray squares.

First we note that the number of knights in the pattern on an $m \times n$ board is as follows:

$$a = \begin{cases} \lceil mn/8 \rceil & m \text{ and } n \text{ are odd} \\ \lceil mn/8 \rceil - \lfloor n/8 \rfloor & m \text{ is even and } n \text{ is odd} \\ \lceil mn/8 \rceil - \lfloor m/8 \rfloor & m \text{ is odd and } n \text{ is even} \\ \lceil mn/8 \rceil - \lfloor m/8 \rfloor - \lfloor n/8 \rfloor & m \text{ and } n \text{ are even} \end{cases}$$

We now count the number of additional knights needed to cover the gray squares as shown in Figure 5.

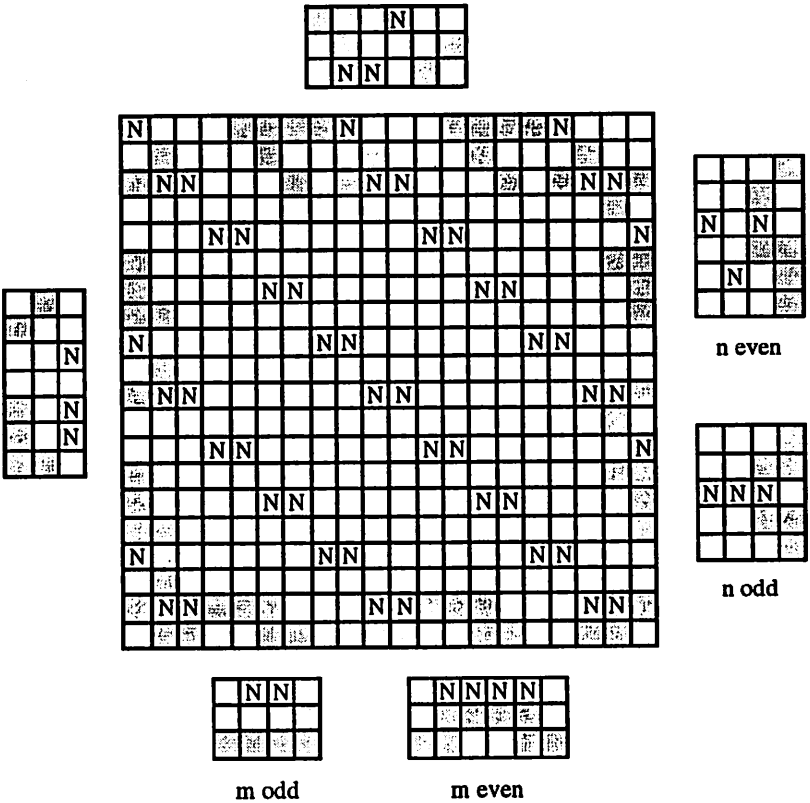

Figure 5: A pattern of knights on the $m \times n$ board

1. Every eight rows there is a pattern of six uncovered squares along the left edge of the board. Each of the $\lceil (m-1)/8 \rceil$ full or partial occurrences of the pattern can be covered by three knights.

2. Each of the $\lceil (n-4)/8 \rceil$ occurrences of eight uncovered squares along the top of the board can be covered by three knights.

3. Each of the $\lceil m/8 \rceil$ occurrences of six uncovered squares along the right edge of the board can be covered by three knights. This is true whether $n$ is even or odd.

4. When $m$ is even, each of the $\lceil n/8 \rceil$ occurrences of eight uncovered squares along the bottom can be covered by four knights. When $m$ is odd, each of the $\lceil n/8 \rceil$ occurrences of four uncovered squares can be covered by two knights.

Thus, the number of knights used to cover the gray squares is:

$$b \leq \begin{cases} 3\lceil m/8\rceil + 3\lceil (m-1)/8\rceil + 2\lceil n/8\rceil + 3\lceil (n-4)/8\rceil & m \text{ is odd} \\ 3\lceil m/8\rceil + 3\lceil (m-1)/8\rceil + 4\lceil n/8\rceil + 3\lceil (n-4)/8\rceil & m \text{ is even} \end{cases}$$

The sum of $a$ and $b$ simplifies (at worst) to the theorem statement if we assume that $m$ is the odd dimension whenever $m \not\equiv n \pmod 2$. ■

## 5 Algorithmic search and final remarks.

Using a backtracking search, we determined the following values of $\tau(K_n)$, $\tau(C_n)$, and $\tau(N_n)$:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_n$ | 2 | 2 | 4 | 5 | 8 | 9 | 12 | 15 | 18 | 21 | 24 |
| $C_n$ | 2 | 3 | 6 | 9 | 12 | 15 | 20 | 25 | 30 | 35 | |
| $N_n$ | | | 6 | 7 | 8 | 10 | 14 | 18 | | | |

To obtain these results, we used the following branch and bound techniques:

1. Abandon the search when it leaves behind an uncovered square;
2. Abandon the search when (the number of uncovered squares)/(the number of remaining pieces) is greater than the maximum number of squares a piece can cover;
3. Limit the placement of the pieces that cover the corner squares (to avoid checking solutions that are symmetric with respect to rotation and reflection of the board).

We also used the heuristic algorithm described in Figure 6 to strengthen the upper bounds on graphs of modest size. This algorithm is typical of hill-climbing algorithms that permit sideways moves in the search space (see [4] and [5]).

```
place a piece on each square of the board;
loop
    pick piece p at random;
    if p can be removed without uncovering a square then
        remove piece p;
    else if p can move to a new square without uncovering a square then
        move p to the new square;
```

Figure 6: Heuristic algorithm for covering a chessboard

Using this technique we improved the upper bounds for the following cases:

| $n$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_n$ | | | | 29 | 33 | 38 | 43 | 48 | 54 | 60 | 68 |
| $C_n$ | | | 42 | 49 | 56 | 64 | 72 | 81 | 90 | 105 | 111 |
| $N_n$ | 22 | 25 | 28 | 32 | 39 | 44 | 48 | 57 | 61 | 66 | 75 |

| $n$ | 21 | 22 | 23 | 24 | 25 | 30 | 40 | 50 | 60 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|
| $K_n$ | 72 | 80 | 87 | 95 | 102 | 146 | | | | |
| $C_n$ | 121 | 132 | 145 | 160 | 177 | 241 | 429 | 658 | 944 | 1284 |
| $N_n$ | 80 | 86 | 94 | 101 | 109 | 152 | | | | |

The successfully covered boards (from both the backtracking and heuristic searches) appear in [6].

We note that tori (boards with wrap-around rows and columns) of the proper dimensions can be perfectly tiled with kings or crosses tiles; similarly, the opposing edges of the pattern of knights from Figure 5 can be brought together when the dimensions of the board are appropriate. In particular, if $K^T_{m,n}$ is the $m \times n$ kings' torus, $C^T_{m,n}$ is the $m \times n$ crosses' torus, and $N^T_{m,n}$ is the $m \times n$ knights' torus, then:

**Remark 5.1** *For $m, n \geq 1$, $\tau(K^T_{7m,14n}) = 14mn$, $\tau(C^T_{4m,4n}) = 4mn$, and $\tau(N^T_{8m,8n}) = 8mn$.*

## References


[1] Cockayne, E. J., Chessboard domination problems, *Discrete Math.* **86** (1990), 13–20.

[2] Cockayne, E. J., B. Gamble, and B. Shepherd, Domination parameters for the bishops graph, *Discrete Math.*, to appear.

[3] Cockayne, E. J. and S. T. Hedetniemi, On the diagonal queens domination problem, *J. Combin. Theory Ser. A* **42** (1986), 137–139.

[4] Dinitz, J. H. and D. Stinson, A hill-climbing algorithm for the construction of one-factorizations and Room squares, *SIAM J. Alg. Disc. Math.* **8** (1987), 430–438.

[5] Garnick, D. K. and J. H. Dinitz, Heuristic algorithms for finding the irregularity strengths of graphs, *J. Combinat. Math. Combinat. Comput.* **8** (1990), 195–208.

[6] Garnick, D. K. and N. A. Nieuwejaar, *Total domination of small order kings' graphs, crosses' graphs, and knights' graphs*, Computer Science Research Rep. 92-2, Dept. of Comp. Sci., Bowdoin College, Brunswick, Maine (oct. 1992).

[7] Rouse Ball, W. W., "Mathematical Recreations and Problems of Past and Present Times," Macmillan, London, 1892.

[8] Yaglom, A. M. and I. M. Yaglom, "Challenging Mathematical Problems with Elementary Solutions, Vol. 1: Combinatorial Analysis and Probablity Theory," Holden-Day, San Francisco, 1964.



D.K. Garnick
Department of Computer Science
Bowdoin College
Brunswick, ME 04011
U.S.A.

N.A. Nieuwejaar
Department of Mathematics and Computer Science
Dartmouth College
Hanover, NH 03755
U.S.A.