Minimizing the Number of Holes in 2-Distant Colorings

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ABSTRACT. A *2-distant coloring* of a graph is an assignment of positive integers to its vertices so that adjacent vertices cannot get either the same number or consecutive numbers. Given a 2-distant coloring of a graph G, a *hole* of f is a finite maximal set of consecutive integers not used by f, and $h(f)$ is the number of holes of f. In this paper we study the problem of minimizing the number of holes, i.e., we are interested in the number $h(G) = \min_f h(f)$ where the minimum runs over all 2-distant colorings f of G. Besides finding exact values for $h(G)$ for particular graphs, we also relate $h(G)$ to the path-covering number and the Hamiltonian completion number of G.

1 Introduction

Recently, motivated by the channel assignment problem and T-colorings of graphs, Roberts [11] introduced a variant of a graph coloring[1], called the *no-hole 2-distant coloring*, or *N-coloring* for short. In such a coloring, we want to assign positive integers to the vertices of a graph so that adjacent vertices get numbers at least 2 apart (the 2-distant requirement), and furthermore, the set of all the numbers used in the coloring forms a consecutive set of integers (the no-hole requirement). If we drop the no-hole requirement, we get a *2-distant coloring.* Lanfear [6] suggests a heuristic for obtaining a generalization of T-coloring, with minimum span which, at one point, seeks a 2-distant coloring of a particular graph so that the set of all colors used is a consecutive set of integers. He also suggests that such T-coloring

[1] Throughout this work, we are going to use the graph-theoretical terminology of Harary [5].

should have span which are close to the minimum span. It was primarily in response to this suggestion that Roberts began the investigation of N-colorings.

Roberts [11] studied existence and efficiency problems related to N-colorings. More specifically, he studied the question of what graphs have N-colorings, and the question of what graphs have near-optimal N-colorings, in the sense that the *span*, i.e, the difference between the largest and the smallest integers used, is at most one larger than the minimum span in a 2-distant coloring.

Later, Sakai and Wang [13] generalized the results in Roberts [11]. They introduced the *no-hole* $(r+1)$*-distant colorings*, or N_r*-colorings* for short, with obvious definition. They characterize all graphs that are N_r-colorable, by relating this concept to some Hamiltonian structures in the graph. For instance, they showed that a graph is N-colorable if and only if its complement contains a Hamiltonian path. Therefore not all graphs have N-colorings. One very natural question arises: If a graph has no N-coloring, "how far" is it from having one? One possible interpretation for the expression "how far" might be to estimate the minimum number of holes among all 2-distant colorings of the graph.

Formally, let $G = (V, E)$ be a graph. If f is a 2-distant coloring of G, a *hole* in f is a nonempty set H of consecutive positive integers, i.e., $H = \{a, a+1, ..., b\}$ for some integers a, b, $a < b$, so that there exist vertices u, v with $f(u) = a-1$, $f(v) = b+1$, and for any $c \in H$, there is no vertex w with $f(w) = c$. Denote by $h(f)$ the number of holes in the 2-distant coloring f of G, and by $h(G)$ the minimum of $h(f)$ over all 2-distant colorings f of G.

In this work we find $h(G)$ for some particular classes of graphs G studied in Roberts [11], and Sakai and Wang [13]. We also relate $h(G)$ to two well known invariants of G: the path-covering number and the Hamiltonian completion number.

2 $h(G)$ for Special Classes of Graphs

The first result provides an upper bound for $h(G)$ based on the chromatic number $\chi(G)$.

Lemma 1 *Let G be a graph. Then $h(G) \leq \chi(G) - 1$.*

Proof: We can partition the set of vertices of G into $V_1 \cup V_2 \cup ... \cup V_{\chi(G)}$, where, for each $i = 1, 2, ..., \chi(G)$, V_i is an independent subset of vertices of G. For each $i = 1, 2, ..., \chi(G)$, color the vertices in V_i with color $2i - 1$. It is easily verified that this gives us a 2-distant coloring of G with $\chi(G) - 1$ holes. Hence $h(G) \leq \chi(G) - 1$. □

It is easy to see that this upper bound is sharp; it is attained by K_n, the complete graph on n vertices.

Proposition 2 *If f is a 2-distant coloring of K_n, then $h(f) = n - 1$. In particular, $h(K_n) = n - 1$.*

Proof: Trivial. □

An obvious observation is that G is N-colorable if and only if $h(G) = 0$. Next we calculate $h(G)$ for the classes of graphs studied in Roberts [11] and in Sakai and Wang [13]. We shall be interested in the existence of *near-optimal* 2-distant colorings of G (i.e., 2-distant colorings with span at most $2\chi(G) - 1$) with exactly $h(G)$ holes.

Proposition 3 *Let G be a bipartite graph. Then*

$$h(G) = \begin{cases} 1, & \textit{if } G \textit{ is complete bipartite} \\ 0, & \textit{otherwise,} \end{cases}$$

and there is a near-optimal (even optimal) 2-distant coloring with exactly $h(G)$ holes.

Proof: Let G be a bipartite graph. By a result of Roberts [11],

$$h(G) = 0 \text{ if and only if } G \text{ is not complete bipartite,}$$

and in this case, G has a near-optimal N-coloring.

If G is complete bipartite, then $h(G) \geq 1$. But, by Lemma 1, $h(G) \leq \chi(G) - 1 = 1$. So $h(G) = 1$ and observe that the 2-distant coloring provided by Lemma 1 for this case, is near-optimal (even optimal), and has exactly one hole. □

The following Corollary is immediate.

Corollary 4 *If T is a tree then*

$$h(T) = \begin{cases} 1, & \textit{if } T \textit{ is a star} \\ 0, & \textit{otherwise,} \end{cases}$$

and there is a near-optimal 2-distant coloring of T with $h(T)$ holes. In particular,

$$h(P_n) = \begin{cases} 1, & \textit{if } n = 2 \textit{ or } n = 3 \\ 0, & \textit{otherwise.} \end{cases} \quad \square$$

Proposition 5

$$h(C_n) = \begin{cases} 2, & if\, n = 3 \\ 1, & if\, n = 4 \\ 0, & if\, n \geq 5, \end{cases}$$

and there is a near-optimal 2-distant coloring of C_n with $h(C_n)$ holes.

Proof: By a result of Roberts [11],

$$h(C_n) = 0 \Leftrightarrow n \geq 5,$$

and in this case, C_n has a near-optimal N-coloring.

If $n = 4$ then C_n is complete bipartite, and Proposition 3 says that $h(C_n) = 1$. If $n = 3$ then $C_n = K_n$, and Proposition 2 implies that $h(C_n) = 2$. In both cases, $h(C_n)$ is attained by a near-optimal 2-distant coloring. □

A graph G is a *unit interval graph* if we can assign an interval of unit length to each vertex so that edges correspond to pairs of intervals which overlap. The unit interval graphs have a variety of applications which are discussed for instance in Golumbic [2] and Roberts [10]. They are especially important in the T-coloring and the channel assignment problem literature. Roberts [9] showed that a graph $G = (V, E)$ is a unit interval graph if and only if it has a *compatible vertex ordering*, i.e., an ordering $v_1, v_2, ..., v_n$ of vertices of V so that if $i < j < k$ and $\{v_i, v_k\} \in E$, then $\{v_i, v_j\}$, $\{v_j, v_k\} \in E$. Moreover, all 1-unit sphere graphs are perfect (see Golumbic [2]). The next four results are devoted to calculating $h(G)$ for G a unit interval graph when we restrict the number of vertices. We first state the following lemma. We use the notation $n = n(G)$ for the number of vertices of G.

Lemma 6 *Let G be a perfect graph. For each $j = 1, 2, ..., \chi(G)$, if $n(G) = 2\chi(G) - j$, then $h(G) \geq j - 1$.*

Proof: Let f be a 2-distant coloring of G and let K be a clique of size $\chi(G)$ in G. When we restrict f to K, we have $\chi(G) - 1$ holes by Proposition 2. But $n(G) - n(K) = \chi(G) - j$, so we can fill at most $\chi(G) - j$ of these holes. This implies that

$$h(f) \geq (\chi(G) - 1) - (\chi(G) - j) = j - 1.$$

So, $h(G) \geq j - 1$. □

The next theorem characterizes unit interval graphs G with $2\chi(G) - j$ vertices and $h(G)$ exactly equal to the lower bound $j - 1$, for $j = 1, 2, ..., \chi(G)$.

Theorem 7 *Let G be a unit interval graph. For each $j = 1, 2, ..., \chi(G)$, if $n(G) = 2\chi(G) - j$, then $h(G) = j - 1$ if and only if G has a unique clique of size $\chi(G)$. In this case, there is a near-optimal 2-distant coloring with $h(G)$ holes.*

Proof: Let $v_1, v_2, ..., v_n$ be a compatible vertex ordering of vertices of G.

Suppose that G has a unique maximal clique, say, $v_p, v_{p+1}, ..., v_{p+\chi-1}$. Color these vertices with colors $1, 3, ..., 2\chi(G) - 1$ in this order. For vertices following $v_{p+\chi-1}$ in the compatible vertex ordering, start coloring them in order as $2, 4, 6, ...$ until $2\chi(G) - 2$ or until no more vertices following $v_{p+\chi-1}$ remain to be colored. For vertices preceding v_p, color backwards starting with v_{p-1} using the colors $2\chi(G) - 2, 2\chi(G) - 4, ...$ until 2 or until no more vertices preceding v_p remain to be colored. This colors all vertices of G since there are at most $\chi(G) - j$ vertices following $v_{p+\chi-1}$ and at most $\chi(G) - j$ vertices preceding v_p. Since G has a unique maximal clique, this is a 2-distant coloring, and since $n(G) = 2\chi(G) - j$, it is not difficult to see that this coloring has exactly $j - 1$ holes. Also, this coloring has span $(2\chi(G) - 1) - 1 = 2\chi(G) - 2$, so it is near-optimal. From Lemma 6, we conclude that $h(G) = j - 1$.

Now suppose that G does not have a unique clique of size $\chi(G)$. We may assume that there is an index p and two positive integers a, b satisfying $a + b = \chi(G)$ so that

$$K_1 = \{v_p, v_{p+1}, ..., v_{p+a+b-1}\}$$

$$K_2 = \{v_{p+a}, v_{p+a+1}, ..., v_{p+2a+b-1}\}$$

are two maximal cliques of G. Let us suppose by contradiction that $h(G) = j - 1$. Let f be a 2-distant coloring of G with $h(f) = j - 1$. Observe that

$$X = \{v_{p+a-1}, v_{p+a}, ..., v_{p+a+b-1}\}$$

is a clique of size $b+1$ and therefore f, when restricted to these vertices, has b holes. Let $x_0, x_1, ..., x_b$ be the ordering of vertices of X so that $f(x_0) < f(x_1) < ... < f(x_b)$. Let Y be the set given by,

$$Y = \{f(x_0) + 1, ..., f(x_{i-1}) + 1, f(x_{i+1}) - 1, ..., f(x_b) - 1\},$$

where $v_{p+a-1} = x_i$, $0 \leq i \leq b$.

First note that $f(x_i) \notin Y$ for any $i = 0, 1, ..., b$. Otherwise $f(x_i) = f(x_l) + 1$ or $f(x_i) = f(x_l) - 1$ for some $l = 0, 1, ..., b$, $i \neq l$. But this is impossible because x_i, x_l are adjacent.

We claim that no vertex in $K_1 \cup K_2$ can have a color in Y. To see this, let v be a vertex in $K_1 \cup K_2$. If $v \in K_1 \cap K_2$, then $v = x_i$ for some $i = 0, 1, ..., b$, and so $f(v) = f(x_i) \notin Y$. If $v \notin K_1 \cap K_2$, then v is adjacent to every vertex in $K_1 \cap K_2 = X - \{v_{p+a-1}\}$, and this implies that v cannot get any color in $\{f(x) \pm 1 : x \in K_1 \cap K_2\}$. So $f(v) \notin Y$.

It is not difficult to see that the relation "belongs to" induces a one-to-one correspondence between the elements of Y and the holes of f restricted to X. So, by the claim and since $|Y| = b$, the b holes of f restricted to X can only be "filled up" by colors given to the vertices outside $K_1 \cup K_2$. But

$$\begin{aligned} |V - K_1 \cup K_2| &= (2\chi(G) - j) - (2a + b) \\ &= (2\chi(G) - j) - (a + \chi(G)) \\ &= \chi(G) - a - j \\ &= b - j, \end{aligned}$$

so f has at least $b - (b - j) = j$ holes, a contradiction. □

Theorem 8 *Let G be a unit interval graph. If $n(G) > 2\chi(G) - 1$, then $h(G) = 0$. If $n(G) = 2\chi(G) - j$ for some $j = 1, 2, ..., \chi(G)$, then*

$$h(G) = \begin{cases} j - 1, & \text{if there is a unique clique of size } \chi(G) \\ j, & \text{otherwise.} \end{cases}$$

Furthermore, there is a near-optimal 2-distant coloring with exactly $h(G)$ holes.

Proof: If $n(G) > 2\chi(G) - 1$ then by a result of Sakai and Wang [13], $h(G) = 0$ and G has a near-optimal N-coloring.

Suppose that $n(G) = 2\chi(G) - j$ for some $j = 1, 2, ..., \chi(G)$. If G has a unique clique of size $\chi(G)$, Theorem 7 implies that $h(G) = j - 1$ and there is a near-optimal 2-distant coloring with $h(G)$ holes. If G has more than one clique of size $\chi(G)$, then by Lemma 6 and Theorem 7, $h(G) \geq j$. So, in order to prove that $h(G) = j$ it is sufficient to present a 2-distant coloring of G with exactly j holes. Let $v_1, v_2, ..., v_n$ be a compatible vertex ordering. Start coloring the vertices from the beginning of the compatible ordering in order with the colors in the sequence below (from left-to-right and top-down),

$$\begin{array}{llllll} 2\chi(G) - 1, & 2\chi(G) - 3, & 2\chi(G) - 5, & \ldots & 5, & 3, \quad 1, \\ 2\chi(G), & 2\chi(G) - 2, & 2\chi(G) - 4, & \ldots & 6, & 4, \end{array}$$

until no more vertices remain to be colored. We have enough colors since $n(G) = 2\chi(G) - j$ for some $j = 1, 2, ..., \chi(G)$ and the sequence above has $2\chi(G) - 1$ colors (notice that color 2 is the only color in $\{1, ..., 2\chi(G)\}$ not in this sequence).

Let us show that the coloring above is a 2-distant coloring. The only possible problem could occur if a vertex colored $2q$ is adjacent to a vertex colored $2q \pm 1$ for some $1 < q \leq \chi(G)$. The only vertex colored $2q$, if there is any, is $v_{2\chi-q+1}$. If there is a vertex adjacent to $v_{2\chi-q+1}$ colored $2q \pm 1$, it must be either $v_{\chi-q}$ colored $2q+1$, or $v_{\chi-q+1}$ colored $2q-1$. But then $v_{\chi-q+1}, v_{\chi-q+2}, \ldots, v_{2\chi-q+1}$ would be a clique of size $\chi(G)+1$, a contradiction. Hence we have a 2-distant coloring.

Notice that for $1 \leq q \leq \chi(G)-1$, the hole between colors $2q+1$ of $v_{\chi-q}$, and $2q-1$ of $v_{\chi-q+1}$ can be "filled up" by color $2q$ assigned to $v_{2\chi-q+1}$, if this vertex exists. Since $n(G) = 2\chi(G) - j$, only $v_{\chi+2}, v_{\chi+3}, \ldots, v_{2\chi-j}$ can have color $2q$ for $j+1 \leq q \leq \chi(G)-1$. So, only $\chi(G)-j-1$ of the $\chi(G)-1$ distinct holes above will be "filled up". Therefore the 2-distant coloring has exactly $(\chi(G)-1) - (\chi(G)-j-1) = j$ holes.

Finally, this coloring has span $2\chi(G)-1$ if $j \neq \chi(G)$, and span $2\chi(G)-2$ if $j = \chi(G)$. Hence, it is near-optimal. □

3 $h(G)$ and Hamiltonian Structures

In 1960, Ore [7] introduced the notion of *path-covering number* of a graph G, denoted by $\mu(G)$, defined as the minimum number of vertex-disjoint paths containing all the vertices of G. This number turns out to be closely related to the *Hamiltonian completion number* of a graph G, denoted by hc(G), defined as the minimum number of edges that need to be added to G to make it Hamiltonian.

Proposition 9 (Goodman, Hedetniemi and Slater [4]) *For any graph G, either G is Hamiltonian, in which case $\mu(G) = 1$ and $hc(G) = 0$, or $\mu(G) = hc(G)$.*

The problem of finding the Hamiltonian completion number was first studied, simultaneously, by Goodman and Hedetniemi [3] and Boesch et al. [1]. They were motivated by considerations of traversing data structures and point diagnostic schemes for network integrity.

We close by establishing a relation between $\mu(G)$ and $h(G)$.

Proposition 10 *There exists a 2-distant coloring f of a graph G such that $h(f) = k$ if and only if the vertices of G^c can be covered by $k+1$ vertex disjoint paths in G^c.*

Proof: Let f be a 2-distant coloring of a graph G such that $h(f) = k$. Let $v_1, v_2, \ldots, v_n$ be an ordering of vertices of G so that $f(v_1) \leq f(v_2) \leq \ldots \leq f(v_n)$ and there exist integers $a_1 < a_2 < \ldots < a_k$ such that $\{f(v_{a_i}) +$

$1, f(v_{a_i})+2, ..., f(v_{a_i+1})-1\}$, $i = 1, 2, ..., k$, are the k holes of f. Therefore

$$\begin{array}{rllll} P_1: & v_1, & v_2, & \cdots & v_{a_1} \\ P_2: & v_{a_1+1}, & v_{a_1+2}, & \cdots & v_{a_2} \\ P_3: & v_{a_2+1}, & v_{a_2+2}, & \cdots & v_{a_3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ P_k: & v_{a_{k-1}+1}, & v_{a_{k-1}+2}, & \cdots & v_{a_k} \\ P_{k+1}: & v_{a_k+1}, & v_{a_k+2}, & \cdots & v_n, \end{array}$$

are $k+1$ vertex disjoint paths in G^c covering all the vertices (note that for each $i = 0, 1, 2, ..., k$, $j = 1, 2, ..., a_{i+1} - a_i - 1$, $f(v_{a_i+j+1}) - f(v_{a_i+j}) \le 1$, i.e., v_{a_i+j+1} and v_{a_i+j} are adjacent in G^c, where $a_0 = 0$ and $a_{k+1} = n$, so P_{i+1} is a path in G^c).

Conversely suppose that $P^i : v_1^i, v_2^i, ..., v_{n_i}^i$, for $i = 1, 2, ...k+1$, are vertex-disjoint paths in G^c covering all the vertices. Consider f defined by

$$f(v_j^1) = j, \quad 1 \le j \le n_1$$

and for $2 \le i \le k+1$,

$$f(v_j^i) = \sum_{l=1}^{i-1} n_l + i + j - 1, \quad 1 \le j \le n_i,$$

It is easily verified that f is a 2-distant coloring and $h(f) = k$. □

From Proposition 9, we can state the following corollary of Proposition 10.

Corollary 11 *For any graph G,*

$$h(G) = \mu(G^c) - 1 = \begin{cases} 0, & \textit{if } G^c \textit{ Hamiltonian} \\ hc(G^c) - 1, & \textit{otherwise.} \end{cases} \quad \square$$

Therefore, the problem of determining $h(G)$ is equivalent to the problem of determining $\mu(G^c)$ or $\mathrm{hc}(G^c)$. These are known to be difficult problems in the sense that they contain the classical Hamiltonian cycle problem as a special case. However, Boesch et al. [1] and Goodman and Hedetniemi [3] have shown that for any graph G,

$$\mathrm{hc}(G) = \min_{T \in S} \mathrm{hc}(T), \tag{1}$$

where S is the set of all spanning trees of G, and furthermore, they have presented an efficient algorithm for finding $\mathrm{hc}(T)$ for trees T (linear time on the number of vertices). Raychaudhuri [8] also provided an algorithm for finding the Hamiltonian completion number for trees by observing that $\mu(T) = n(T) - p(T)$, where $p(T)$ is the maximum number of edges in a vertex-disjoint union of paths of T, and by formulating the problem of finding $p(T)$, for trees T, as a maximum flow problem in a network with upper capacities on arcs. A heuristic based on the equation (1) is presented in Goodman et al. [4], and also a linear time algorithm (on the number of vertices) for finding $\mathrm{hc}(G)$ for *unicyclic graphs*, i.e., graphs having exactly one cycle. In a subsequent paper, Slater et al. [14] provided an algorithm for finding $\mathrm{hc}(G)$ when G is a *cactus*, i.e., a connected graph in which any two cycles intersect in at most one point.

Consequently, if G is a *cotree* (complement of a tree), a *counicyclic graph* (complement of a unicyclic graph) or a *cocactus* (complement of a cactus), we have algorithms (linear time for cotrees and counicyclic graphs) to calculate $h(G)$, and for arbitrary graphs we have the heuristic based on equation (1).

As we have already pointed out, the problem of determining $h(G)$ is not an easy one and, consequently, a reasonable way to attack a problem of this difficulty would be to try to understand it as deeply as possible by studying several different classes of graphs. In Section 2, we followed this strategy by concentrating on a few particular families of graphs G and obtaining exact values for $h(G)$. It would be interesting to keep investigating other classes of graphs, for instance the class of r-unit sphere graphs which has important applications in communications. A graph G is said to be *r-unit sphere* if there is a function $g : V \to R^r$ so that $\{x, y\}$ is an edge if and only if the Euclidean distance in R^r between $g(x)$ and $g(y)$ is at most 1.

Another interesting direction for further research would be to extend the results in this paper to $(r+1)$-distant colorings, i.e., study the minimum number of holes over all $(r+1)$-distant colorings of a graph (some results in this direction can be found in Sakai [12]). A more general context would be to study the minimum number of holes over all T-colorings of a graph for an arbitrary set T. For more about T-colorings we refer the reader to Tesman [15].

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