Cohen's Theorem and Z-Cyclic Whist Tournaments

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Abstract

Let p, q denote primes, $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, $q \geq 7$. In an earlier study we established that if $\gcd(q-1, p^{n-1}(p-1)) = 2$ and if a Z-cyclic $Wh(q+1)$ exists then a Z-cyclic $Wh(qp^n+1)$ exists for all $n \geq 0$. Here we consider $\gcd(q-1, p^{n-1}(p-1)) > 2$ and prove that if a Z-cyclic $Wh(q+1)$ exists then there exists a Z-cyclic $Wh(qp^n + 1)$ for all $n \geq 0$. The proof employed depends on the existence of an appropriate primitive root of p. Utilizing a theorem of S. D. Cohen we establish that such appropriate primitive roots always exist.

1. Introduction. A whist tournament for $v = 4m$ players, $Wh(v)$, is a schedule of games involving two players playing against two others such that

(i) the games can be arranged in $4m - 1$ rounds of m games each,

(ii) each player plays in exactly one game in each round,

(iii) each player partners every other player exactly once,

(iv) each player opposes every other player exactly twice.

Conditions (iii), (iv) will be referred to as the *whist conditions*. Each game in the whist tournament is denoted by a 4-tuple (a, b, c, d) in which the pairs $\{a, c\}$, $\{b, d\}$ designate partnerships and the four other pairs designate opponents. It is not uncommon to refer to the game (a, b, c, d) as a *whist table* since the problem originates from the card game of whist. As a mathematical structure the problem was introduced by E. H. Moore [12]. Existence of $Wh(v)$ for all $v \equiv 0, 1 \pmod 4$ was established in the late 1970's [7,10] but it has only been recently [2,3,4,5,6] that progress has been made on the existence of Z-cyclic whist tournaments. By a Z-cyclic

$Wh(4m)$ it is meant that the $4m$ players are elements in $Z_{4m-1} \cup \{\infty\}$ and the rounds are labeled so that round $j+1$ is obtained by adding $+1$ (mod $4m-1$) to each non-∞ element in round j.

In [3] we established that if p, q are primes, $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, $q \geq 7$ (the case $q = 3$ is dealt with in [2,4]) such that $\gcd(q-1, p^{n-1}(p-1)) = 2$ and if a Z-cyclic $Wh(q+1)$ exists then there exists a Z-cyclic $Wh(qp^n+1)$ for all $n \geq 0$. In the current study we consider $\gcd(q-1, p^{n-1}(p-1)) > 2$ and establish similar results, namely if there exists a Z-cyclic $Wh(q+1)$ then there exists a Z-cyclic $Wh(qp^n+1)$ for all $n \geq 0$. As for the existence of Z-cyclic $Wh(q+1)$ it is still the case that existence is known only for $q \in \{3, 7, 11, 19, 23, 31\}$ ($q = 3, 7, 11$ can be found in [3,12] and $q = 19, 23, 31$ in [9]). In Section 3 we introduce constructions that yield Z-cyclic $Wh(qp^n+1)$ if there exists a common primitive root of q and p^2 whose *power sequence* (defined in Section 2) possesses certain number theoretic properties that are compatible with the construction. Thus in contrast to our earlier study [3] the constructions, in general, are not valid for an arbitrary primitive root but rather depend on the existence of an appropriate primitive root. A theorem of Cohen [8] enables us to prove that such appropriate primitive roots always exist.

In Section 2 we introduce a structure for the ring Z_{qp^n} and list some lemmas that are useful for our constructions. Cohen's theorem is also listed. In Section 3 the methodology and the constructions are discussed and the main results are established. In Section 4 we provide a few specific examples and in Section 5 we provide a list of primitive roots that serves to substantiate our claim of the existence of primitive roots appropriate for our constructions.

2. Structure in the Ring Z_{qp^n}. In the sequel it will be understood that p, q denote primes such that $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, $q \geq 7$ with $\gcd(q-1, p^{n-1}(p-1)) = 2e$, $e > 1$. Consider the following subsets of Z_{qp^n}.

$$P = \{x : p \mid x\} \setminus \{0\},$$
$$Q^* = \{x : q \mid x, p \nmid x\} \setminus \{0\},$$
$$E = \{x : p \nmid x, q \nmid x\} \setminus \{0\}.$$

Thus $|P| = qp^{n-1} - 1$, $|Q^*| = p^{n-1}(p-1)$, $E = p^{n-1}(p-1)(q-1)$ and $Z_{qp^n} = P \cup Q^* \cup E \cup \{0\}$. Let W be any common primitive root of q and p^2.

Lemma 2.1. $\operatorname{ord}_{qp^n} W = p^{n-1}(p-1)(q-1)/2e$.

Proof.

$$\operatorname{ord}_{qp^n} W = \operatorname{lcm}(\operatorname{ord}_q W, \operatorname{ord}_{p^n} W) = p^{n-1}(p-1)(q-1)/(q-1, p^{n-1}(p-1)).$$

□

We note that $\text{ord}_{qp^n} W \equiv 0 \pmod 4$ and define t, s by the relations

$$4t = p^{n-1}(p-1)(q-1)/2e, \tag{2.1}$$

$$4s = p^{n-1}(p-1). \tag{2.2}$$

We list some useful results. The proofs of Lemmas 2.2–2.5 can be found in [3], that of Lemma 2.6 in [11] and the proof of Theorem 2.7 appears in [8].

Lemma 2.2. *$W^i \not\equiv -1 \pmod{qp^n}$ for all $0 \le i \le 4t-1$.*

Lemma 2.3. *Q^* is a cyclic set $\{q_1, q_2, \ldots, q_{4s}\}$ where*

(i) *$q_{i+1} = Wq_i$ for all $1 \le i \le 4s-1$ and $Wq_{4s} = q_1$, and*

(ii) *$q_{i+2s} + q_i \equiv 0 \pmod{qp^n}$ for all $1 \le i \le 4s$.*

Lemma 2.4. *If α is odd then* (i) *$W^\alpha - 1$ is coprime to both p and q, and* (ii) *$W^\alpha + 1$ is coprime to p, and is a multiple of q if and only if α is an odd multiple of $\frac{q-1}{2}$.*

Lemma 2.5. *If α is even then* (i) *$W^\alpha - 1$ is a multiple of p if and only if α is a multiple of $p-1$,* (ii) *$W^\alpha - 1$ is a multiple of q if and only if α is a multiple of $q-1$,* (iii) *$W^\alpha + 1$ is a multiple of p if and only if α is an odd multiple of $\frac{p-1}{2}$, and* (iv) *$W^\alpha + 1$ is coprime to q.*

Lemma 2.6. (Mann's Lemma) *Let $4u+1$ be a power of a prime and let x be a primitive element of $GF(4u+1)$. Then there exist odd integers c, d such that $x^c + 1 = x^d(x^c - 1)$.*

Theorem 2.7. (Cohen's Theorem) *Let γ be a prime such that $\gamma > 211$. If $g(x)$ is a quadratic polynomial over $GF(\gamma)$ not of the form $a(x+b)^2$ where a is a non-square in $GF(\gamma)$ then $g(W)$ is a non-zero square for some primitive root W of γ.*

From the general theory of cyclotomy [13] we know that the set E is a multiplicative group, the group of *reduced residues*, and has a coset decomposition

$$E = \bigcup_{i=0}^{e-1} C_i, \tag{2.3}$$

where $C_0 = \{\pm 1, \pm W, \pm W^2, \ldots, \pm W^{4t-1}\}$, and $C_i = x_i C_0$, $i = 1, \ldots, e-1$, for certain *representative* elements x_i. Analogous to the general theory contained in [13] it can be shown (although we shall not in fact need this result) that for Z_{qp^n} we can choose $x_i = x^i$ where x is uniquely determined by $x \equiv W \pmod{p^n}$ and $x \equiv 1 \pmod q$. For our present purposes, however, all we need to note is that, in view of Lemma 2.2, we can express each C_i in the form

$$C_i = A_i \cup B_i, \tag{2.4}$$

with

$$A_i = \{x_i, x_iW, x_iW^2, \ldots, x_iW^{4t-1}\}, \tag{2.5}$$

and

$$B_i = \{-x_i, -x_iW, \ldots, -x_iW^{4t-1}\}, \tag{2.6}$$

with $x_0 = 1$. Now each A_i, B_i is a cyclic set, and, as such, *any* element $y_i \in A_i$ can be used as a representative, in which case the A_i, B_i would be cyclically permuted and their presentation will be as in (2.5), (2.6) with x_i replaced by y_i. This flexibility in the choice of the representative of a coset shall prove to be useful for the constructions introduced in Section 3. We close this section with the following definition.

Definition. Let γ be a prime and let W be a primitive root of γ. The sequence $\{W^i\}_{i=0}^{\gamma-2}$ is called the *power sequence* of W in Z_γ.

3. The Methodology and Main Results. For Z-cyclic whist tournaments, or indeed cyclic $Wh(v)$ in general, the basic approach is to produce an initial round that exhibits satisfaction of the whist conditions via the method of symmetric differences [1]. In general our approach is to form the initial round as the union of three collections of whist tables, one collection from each of the sets P, Q^*, and E. The sets P and Q^* will be dealt with exactly as in [3]. That is to say P will be handled inductively and for Q^* we have the following lemma that is proved in [3].

Lemma 3.1. *The collection of whist tables*

$$(q_1, q_{1+s}, q_{1+2s}, q_{1+3s}) \;\; \textit{times } 1, W, W^2, \ldots, W^{s-1},$$

satisfy the whist conditions for the set Q^*. (Here the q_i are as described in Lemma 2.3.)

We note that in [3], Lemma 3.1 was established under the hypothesis $(q-1, p^{n-1}(p-1)) = 2$. Nevertheless the construction is still valid here for the whist differences arising from these tables are of the form $q_1w^\beta(w^s-1)$, $q_1w^\beta(w^{2s}-1)$, and $q_1w^\beta(w^{3s}-1)$. Regardless of the parity of s, it is not possible that p divides any of these forms (compare Lemmas 2.4, 2.5).

Consequently we need only be concerned with the set E. To this end we introduce the following construction.

Construction 1. Form the collection of *et* whist tables

$$(x_i, x_iW^\alpha, -x_i, -x_iW^\alpha) \;\; \text{times } 1, W^2, \ldots, W^{4t-2}; \;\; i = 0, 1, \ldots, e-1, \tag{3.1}$$

where α is odd.

In this construction x_i merely denotes a representative for the coset C_i and is not necessarily that associated with the cyclotomic theory. Since α is odd, the collection (3.1) exhausts the set E. The whist differences that arise from the collection (3.1) are as follows.

$$\text{partner differences:} \quad \pm 2x_i, \pm 2x_i W^\alpha \text{ times } 1, W^2, \ldots, W^{4t-2}; \quad i = 0, 1, \ldots, e-1, \tag{3.2}$$

$$\begin{aligned}\text{opponent differences:} \quad & \pm x_i(W^\alpha - 1) \text{ \{twice\}times } 1, W^2, \ldots, W^{4t-2}; \\ & \pm x_i(W^\alpha + 1) \text{ \{twice\} times } 1, W^2, \ldots, W^{4t-2}; \\ & i = 0, 1, \ldots, e-1.\end{aligned} \tag{3.3}$$

Clearly each element in E occurs exactly once in (3.2) so the whist tables (3.1) satisfy the partner whist condition for the set E. We proceed to demonstrate that for suitable restrictions on α and on the x_i the opponent whist condition for the set E will be satisfied by the differences (3.3). First of all we note that if $W^\alpha \pm 1 \in E$ then all of the differences (3.3) belong to E. That suitable restrictions on α guarantee that $W^\alpha \pm 1 \in E$ can be seen as follows. Since W is a primitive root of p^2 then W is a primitive root of p^n for all $n \geq 1$. Specifically then W is a primitive root of p and we write $W \equiv w_p \pmod{p}$. In the Galois field Z_p we invoke Mann's Lemma to obtain a pair of odd integers, (α, β), such that, in Z_p, $w_p^\alpha + 1 = w_p^\beta(w_p^\alpha - 1)$ (or equivalently $w_p^\beta + 1 = w_p^\alpha(w_p^\beta - 1)$). Thus $w_p^\alpha + 1$, $w_p^\alpha - 1$ occupy positions of opposite parity in the power sequence of w_p and hence precisely one of $w_p^\alpha + 1$, $w_p^\alpha - 1$ is a non-zero quadratic residue (alt. square) in Z_p. Mann's Lemma guarantees at least one pair (α, β) but oftentimes there is more than one pair. For the time being we make a basic assumption; eventually we demonstrate that this assumption can be satisfied provided that $(p, q) \neq (13, 7)$.

Hypothesis A. There exists a primitive root w_p of p for which at least one of the pairs (α, β) obtained via Mann's Lemma is such that not both of α, β are multiples of $\frac{q-1}{2}$.

In general Hypothesis A places restrictions on the choices of q, p, and W. For instance it is impossible to satisfy Hypothesis A for the pair $(p, q) = (13, 7)$. Assuming Hypothesis A, Lemma 2.4 enables us to conclude that $W^\alpha \pm 1 \in E$. We now invoke the flexibility in the choice of the x_i by assuming that $x_0 = 1$ and $x_i \equiv 1 \pmod{p}$, $1 \leq i \leq e-1$.

Lemma 3.2. *If Hypothesis A is satisfied and if* $x_0 = 1$, $x_i \equiv 1 \pmod{p}$, $1 \leq i \leq e-1$ *then the opponent differences given in* (3.3) *cover each element of* E *exactly twice.*

Proof. Since each of the differences $\pm x_i(W^\alpha \pm 1)$ times $1, W^2, \ldots, W^{4t-2}$ occurs twice in (3.3) it suffices to show that $\pm x_i(W^\alpha \pm 1)$ times $1, W^2, \ldots, W^{4t-2}$, $i = 0, 1, \ldots, e-1$ covers the set E exactly once. Since we have assumed that Hypothesis A is satisfied we know that for α thus given there exist integers ℓ, λ such that

$$W^\alpha + 1 \equiv \pm x_\ell W^\lambda (W^\alpha - 1) \pmod{qp^n}. \tag{3.4}$$

As i varies

$$\pm x_i(W^\alpha - 1) \text{ times } 1, W^2, \ldots, W^{4t-2}, \tag{3.5}$$

gives $W^\alpha - 1$ times all elements of E with *even* parities, and

$$\pm x_i(W^\alpha + 1) = \pm x_i x_\ell W^\lambda (W^\alpha - 1). \tag{3.6}$$

Now $x_i \to x_i x_\ell$ permutes the C_i and, as $x_i \equiv 1 \pmod p$, we have $x_i x_\ell = W^{\text{even}} x_{i'}$. Thus from (3.6), $\pm x_i(W^\alpha + 1)$ times $1, W^2, \ldots, W^{4t-2}$ gives $W^\alpha - 1$ times all $\pm x_{i'} W^\lambda$ times $1, W^2, \ldots, W^{4t-2}$, i.e. $W^\alpha - 1$ times all elements of E with *odd* parities provided that λ is odd. But, mod p, $W^\alpha + 1 = \pm x_\ell W^\lambda (W^\alpha - 1)$ gives $w_p^\alpha + 1 = \pm w_p^\lambda (w_p^\alpha - 1)$ and so $\lambda = \beta$ or $\lambda = \beta + \frac{(p-1)}{2}$. Hence λ has the same parity as β, i.e. λ is odd. □

Thus we can formulate the following theorem.

Theorem 3.3. *If q, p are primes, $q \equiv 3 \pmod 4$, $q \geq 7$, $p \equiv 1 \pmod 4$ such that (1) a Z-cyclic $Wh(q+1)$ exists and (2) Hypothesis A is satisfied, then there exists a Z-cyclic $Wh(qp^n + 1)$ for all $n \geq 0$.*

Proof. Let W be a common primitive root of q and p^2 that is associated with Hypothesis A. We proceed by induction. For $n = 0$ we have the Z-cyclic $Wh(q+1)$. Assume the theorem true for $n-1$ and consider the n case. The initial round for the Z-cyclic $Wh(qp^n + 1)$ is the union of the tables of Lemma 3.1, the tables (3.1) of Construction 1 and those of a Z-cyclic $Wh(qp^{n-1} + 1)$ constructed on the set $P \cup \{0, \infty\}$. □

Note that if $p = 5$ then $w_p = 2, 3$ and in either case the (α, β) of Mann's lemma is $(1, 3)$. Thus we can choose $\alpha = 1$ in Construction 1 and Hypothesis A is automatically satisfied independent of the value of q.

Corollary 3.4. *If there exists a Z-cyclic $Wh(q+1)$, then there exists a Z-cyclic $Wh(q \cdot 5^n + 1)$ for all $n \geq 0$.*

We observe that if p is a prime such that in Z_p Mann's Lemma gives $\alpha = 1$ for at least one pair (α, β) then Hypothesis A is satisfied independent of the choice of q and the rest of our methodology guarantees that the tables (3.1) with $\alpha = 1$ satisfy the whist conditions for the set E. We demonstrate now that with the exception of $p = 13$, there exists at least one primitive root of p for which Mann's Lemma yields a pair with $\alpha = 1$.

Lemma 3.5. *Let γ be any prime such that $\gamma > 211$, then there exists at least one primitive root of γ, call it w, for which an α of Mann's Lemma equals 1. That is to say there exists a primitive root w of γ for which precisely one of $w+1$, $w-1$ is a square in $GF(\gamma)$.*

Proof. Let z denote a fixed, but otherwise arbitrary, non-square in $GF(\gamma)$. Consider the quadratic polynomial $g(x) = z(x^2 - 1)$ over $GF(\gamma)$. Applying Cohen's Theorem, there exists a primitive root of γ, call it w, such that $g(w)$ is a non-zero square. Thus $z(w^2 - 1) = w^{2\mu}$ for some $\mu \geq 0$. Since z is a non-square, $z = w^{2\tau+1}$ for some $\tau \geq 0$. Consequently $(w+1)(w-1) = w^2 - 1 = w^{\text{odd}}$ (a non-square). In $GF(\gamma)$ the product of two squares or the product of two non-squares is a square, hence precisely one of $w+1$, $w-1$ is a square. □

If for a given p, the w given by Lemma 3.5 is not a primitive root of p^2 then we set $w' = w + p$ (which will be a primitive root of p^2) and $(w'+1)(w'-1) \equiv (w+1)(w-1) \pmod{p}$ is a non-square in Z_p.

Lemma 3.6. *Let p be a prime such that $p \equiv 1 \pmod 4$, $p < 211$, $p \neq 13$, then there exists a primitive root of p, call it w, such that precisely one of $w+1$, $w-1$ is a square in Z_p.*

Proof. See the list in Section 5. It is to be noted that all of the w listed in Section 5 are also primitive roots of p^2. □

Corollary 3.7. (to Theorem 3.3) *If $p \neq 13$ and if there exists a Z-cyclic $Wh(q+1)$ then there exists a Z-cyclic $Wh(qp^n+1)$ for all $n \geq 0$.*

There remains only the case $p = 13$ to consider. $p = 13$ has four (4) primitive roots $2, 6, 7, 11$. For each of these the (α, β) of Mann's Lemma is $(3, 9)$. Thus Hypothesis A could be violated only for $q \in \{7, 19\}$. However, $q = 19$ causes no problem for in that case we can choose $\alpha = 3$ in Construction 1. Indeed

Corollary 3.8 (to Theorem 3.3) *If there exists a Z-cyclic $Wh(q+1)$, $q \geq 11$, then there exists a Z-cyclic $Wh(q \cdot 13^n + 1)$ for all $n \geq 0$.*

Proof. Choose $\alpha = 3$ in Construction 1. □

Finally we deal with $q = 7$, $p = 13$ via a new construction.

Theorem 3.9. *There exists a Z-cyclic $Wh(7 \cdot 13^n + 1)$ for all $n \geq 0$.*

Proof. The proof is inductive as is the proof of Theorem 3.3. We streamline the argument by focusing exclusively on the set E. Let W be a common primitive root of 7 and 13^2 such that $W \equiv 2 \pmod{13}$. Each coset representative x_i is taken so that $x_i \equiv 1 \pmod{13}$. Consider the collection of *et* whist tables

$$(x_i, x_iW, -x_iW, -x_iW^2) \text{ times } 1, W^2, \ldots, W^{4t-2};\ i = 0, 1, \ldots, e-1. \quad (3.7)$$

The differences arising from these tables are as follows.

partner differences: $\pm x_i(W+1), \pm x_i W(W+1)$ times $1, W^2, \ldots, W^{4t-2};\ i = 0, 1, \ldots, e-1$ (3.8)

opponent differences: $\pm x_i(W-1), \pm x_i W(W-1)$ times $1, W^2, \ldots, W^{4t-2};\ i = 0, 1, \ldots, e-1$ (3.9)

opponent differences: $\pm 2x_i W, \pm x_i(W^2+1)$ times $1, W^2, \ldots, W^{4t-2};\ i = 0, 1, \ldots, e-1.$ (3.10)

Invoking Lemmas 2.3, 2.4 we conclude that all of the differences in (3.8)–(3.10) belong to E. It is clear that all elements of E occur exactly once in each of (3.8), (3.9). Thus the whist conditions for E are satisfied by the whist tables (3.7) if $2x_iW$, $x_i(W^2+1)$ occupy positions of opposite parity in their respective sets A_k (or B_k), A_ℓ (or B_ℓ). But this condition is equivalent to the requirement that precisely one of $2w_p$, w_p^2+1 is a square in Z_{13}. Now $2w_p = 4 = 2^2$ and $w_p^2 + 1 = 5 = 2^9$. □

4. Some Examples. For reference we give a $Wh(8)$, $Wh(20)$, and a $Wh(56)$.

Example 4.1.

(a) $Wh(8)$: $(\infty,4,0,5), (1,2,3,6)$;

(b) $Wh(20)$: $(\infty,10,0,17), (1,6,2,9), (3,7,5,16), (4,13,12,18)$, $(8,11,14,15)$;

(c) $Wh(56)$ [3]: $(\infty,40,0,10), (5,25,20,30), (35,50,45,15)$, $(11,22,44,33), (1,8,54,47), (4,32,51,23)$, $(16,18,39,37), (9,17,46,38), (36,13,19,42)$, $(34,52,21,3), (26,43,29,12), (49,7,6,48)$, $(31,28,24,27), (14,2,41,53)$.

Example 4.2. $v = 92 = 7 \cdot 13 + 1$. $W = 80$, $e = 3$, $x_0 = 1$, $x_1 = 66$, $x_2 = 79$. For the initial round of a Z-cyclic $Wh(92)$ form the union of the tables:

(1) $Wh(8)$ on $P \cup \{0, \infty\}$: $(\infty,52,0,65), (13,26,39,78)$;

(2) Lemma 3.1 applied to Q^*: $(7,56,84,35), (14,21,77,70), (28,42,63,49)$;

(3) Tables (3.7) applied to E:
$(1,80,11,61), (30,34,57,10), (81,19,72,27), (64,24,67,82)$,
$(9,83,8,3), (88,33,58,90)$,

(66, 2, 89, 22), (69, 60, 31, 23), (68, 71, 20, 53), (38, 37, 54, 43),
(48, 18, 73, 16), (75, 85, 6, 25),
(79, 41, 50, 87), (4, 47, 44, 62), (29, 45, 46, 40), (51, 76, 15, 17),
(74, 5, 86, 55), (36, 59, 32, 12).

Example 4.3. $v = 248 = 13 \cdot 19 + 1$. $W = 2$, $e = 3$, $x_0 = 1$, $x_1 = 40$, $x_2 = 105$. For the initial round of a Z-cyclic $Wh(248)$ take the union of the following whist tables:

(1) For $P \cup \{0, \infty\}$ take the tables of Example 4.1(b) and multiply each element by 13;

(2) Apply Lemma 3.1 to Q^*: (19, 152, 228, 95), (38, 57, 209, 190), (76, 114, 171, 133);

(3) For E take the tables (3.1) with $\alpha = 3$ (alternatively we could use (3.7) since $W \equiv 2 \pmod{13}$).
(1, 8, 246, 239) times $1, W^2, W^4, \ldots, W^{34}$;
(40, 73, 207, 174) times $1, W^2, W^4, \ldots, W^{34}$;
(105, 99, 142, 148) times $1, W^2, W^4, \ldots, W^{34}$.

Example 4.4. $v = 276 = 11 \cdot 5^2 + 1$. $W = 2$, $e = 5$, $x_0 = 1$, $x_1 = 6$, $x_2 = 56$, $x_3 = 21$, $x_4 = 46$. For the initial round of a Z-cyclic $Wh(276)$ take the union of the following tables:

(1) For $P \cup \{0, \infty\}$ take the tables of Example 4.1(c) and multiply each element by 5;

(2) Apply Lemma 3.1 to Q^*: (11, 77, 264, 198) times $1, W, \ldots, W^4$;

(3) For E take the tables (3.1) with $\alpha = 1$
(1, 2, 274, 273) times $1, W^2, \ldots, W^{18}$,
(6, 12, 269, 263) times $1, W^2, \ldots, W^{18}$,
(56, 112, 219, 163) times $1, W^2, \ldots, W^{18}$,
(21, 42, 254, 233) times $1, W^2, \ldots, W^{18}$,
(46, 92, 229, 183) times $1, W^2, \ldots, W^{18}$.

5. Appropriate Primitive Roots for Lemma 3.6. For convenience of space we list the results as ordered pairs (p, w_p). (17, 5), (29, 2), (37, 5), (41, 6), (53, 2), (61, 6), (73, 11), (89, 6), (97, 13), (101, 2), (109, 10), (113, 5), (137, 5), (149, 2), (157, 5), (173, 2), (181, 21), (193, 10), (197, 2).

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