# Some properties of graphs with local Ore condition

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ABSTRACT. The author and N.K. Khachatrian proved that a connected graph $G$ of order at least 3 is hamiltonian if for each vertex $x$ the subgraph $G_1(x)$ induced by $x$ and its neighbors in $G$ is an Ore graph.

We prove here that a graph $G$ satisfying the above conditions is fully cycle extendible. Moreover, $G$ is panconnected if and only if $G$ is 3-connected and $G \neq K_n \vee \overline{K}_n$ for some $n \geq 3$ where $\vee$ is the join operation. The paper is concluded with two conjectures.

## 1 Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider finite simple graphs only.

For each vertex $u$ of a graph $G$ we denote by $N(u)$ the set of all vertices of $G$ adjacent to $u$. The subgraph induced by the set $M(x) = N(x) \cup \{x\}$ we denote by $G_1(x)$. The degree in $G_1(x)$ of a vertex $u \in M(x)$ is denoted by $d_{G_1(x)}(u)$. A graph $G$ is *locally $n$-connected*, $n \geq 1$, if the subgraph induced by the set $N(x)$is $n$-connected for each $x \in V(G)$.

A path with $x$ and $y$ as end-vertices is called an $xy$-path. A graph $G$ is said to be *panconnected*, if for each pair of distinct vertices $x$ and $y$ of $G$ and for each $\ell$, $d(x,y) \leq \ell \leq |V(G)| - 1$, there is an $xy$-path of length $\ell$ in $G$.

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Let $P$ be a path of $G$. We denote by $\vec{P}$ the path $P$ with a given orientation, and by $\overleftarrow{P}$ the path $P$ with the reverse orientation. If $u, v \in V(P)$, then $u\vec{P}v$ denotes the consecutive vertices of $P$ from $u$ to $v$ in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $v\overleftarrow{P}u$. We use $u^+$ to denote the successor of $u$ on $\vec{P}$ and $u^-$ to denote its predecessor.

Analogous notation is used with respect to cycles instead of paths.

A graph $G$ is said to be *fully cycle extendible* [6] if every vertex $x$ of $G$ lies on a triangle and for each cycle $C$ of length $\ell \leq |V(G)|$ there is a cycle $C'$ of length $\ell + 1$ such that $V(C) \subset V(C')$.

A graph $G$ is said to be $K_{1,3}$*-free* if $G$ has no induced subgraph isomorphic to $K_{1,3}$.

Let $A_1$ and $A_2$ be two disjoint subsets of vertices of a graph $G$. We denote by $e(A_1, A_2)$ the number of edges in $G$ with one end in $A_1$ and the other in $A_2$.

The following results are known.

**Theorem 1** (Clark [4]). *Let $G$ be a connected, locally connected and $K_{1,3}$-free graph of order at least 3. Then $G$ is fully cycle extendible.*

**Theorem 2** (Kanetkar and Rao [7]). *Let $G$ be a connected, locally 2-connected and $K_{1,3}$-free graph of order at least 3. Then $G$ is panconnected.*

**Theorem 3** (Ore [8]). *Let $H$ be a graph of order at least 3 such that $d(u) + d(v) \geq |V(H)|$ for each pair of nonadjacent vertices $u, v$ of $H$. Then $H$ is hamiltonian.*

A graph $H$ satisfying the conditions of Theorem 3 is called an Ore graph.

**Theorem 4** (Asratian[1] and Khachatrian [5]). *Let $G$ be a connected graph of order at least 3 where the subgraph $G_1(x)$ is an Ore graph for each vertex $x$. Then $G$ is hamiltonian.*

Denote by $B_0$ the set of all graphs satisfying the conditions of Theorem 4.

The following results are obtained in this paper.

**Theorem 5.** *Every graph $G \in B_0$ is fully cycle extendible.*

**Theorem 6.** *Let $G \in B_0$ and let $x$ and $y$ are two distinct vertices of $G$ with $d(x, y) \geq 2$. Then for each $\ell$, $d(x, y) \leq \ell \leq |V(G)| - 1$, there is an $xy$-path of length $\ell$.*

**Theorem 7.** *A graph $G \in B_0$ is panconnected if and only if $G$ is 3-connected and $G \neq K_n \vee \overline{K}_n$ for $n \geq 3$ where $\vee$ is the join operation.*

We show that for each Ore graph $H$ and each integer $t \geq 2$ there is a panconnected graph $G \in B_0$ with diameter $t$ such that $H$ is an induced subgraph of $G$.

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[1] In [5] the last name of the present author was transcribed as *Hasratian*

Finally, we consider the set $B_1$ of all connected, locally connected graphs $G$ of order at least 3 which satisfy

$$d_{G_1(x)}(u) + d_{G_1(x)}(v) \geq |M(x)| - 1$$

for each triple $x, u, v$ with $d(u,v) = 2$ and $x \in N(u) \cap N(v)$. The set $B_1$ contains all graphs satisfying the conditions of Theorem 1 or Theorem 4. Taking into account our results and the results of Clark and Kanetkar - Rao we formulate two conjectures for the characterization of fully cycle extendible and panconnected graphs from the set $B_1$.

We use similar arguments as in [1] and [2].

## 2 Results

By definition, a connected graph $G$ of order at least 3 belongs to the set $B_0$ if and only if

$$d_{G_1(x)}(u) + d_{G_1(x)}(v) \geq |M(x)| \tag{1}$$

for each triple of vertices $u, v, x$ where $d(u,v) = 2$ and $x \in N(u) \cap N(v)$.

**Lemma 1.** *The inequality (1) is equivalent to*

$$|N(u) \cap N(v) \cap N(x)| \geq |N(x) \setminus (N(u) \cup N(v))| - 1. \tag{2}$$

**Proof:** Let $u, v \in N(x)$ and $uv \notin E(G)$. Then

$$\begin{aligned} |N(u) \cap N(v) \cap N(x)| &= |N_{G_1(x)}(u) \cap N_{G_1(x)}(v)| - 1 \\ &= d_{G_1(x)}(u) + d_{G_1(x)}(v) - |N_{G_1(x)}(u) \cup N_{G_1(x)}(v)| - 1. \end{aligned}$$

Hence (1) is hold if and only if

$$\begin{aligned} |N(u) \cap N(v) \cap N(x)| &\geq |M(x)| - |N_{G_1(x)}(u) \cup N_{G_1(x)}(v)| - 1 \\ &= |N(x) \setminus (N(u) \cup N(v))| - 1. \end{aligned}$$

□

**Corollary 1.** *Let $G \in B_0$. Then*

*a) $|N(u) \cap N(v) \cap N(x)| \geq 1$ for each triple of vertices $u, v, x$ with $d(u,v) = 2$ and $x \in N(u) \cap N(v)$.*

*b) $|N(u) \cap N(v)| \geq 2$ for each pair of vertices $u, v$ with $d(u,v) = 2$.*

*c) $G$ is 2-connected and, therefore, $d(x) \geq 2$ for each vertex $x$ of $G$.*

**Proof:** Let $d(u,v) = 2$ and $x \in N(u) \cap N(v)$. Then $u, v \in N(x) \setminus (N(u) \cup N(v))$. Therefore, by Lemma 1,

$$|N(u) \cap N(v) \cap N(x)| \geq |N(x) \setminus (N(u) \cup N(v))| - 1 \geq 1.$$

Clearly, $w \neq x$ for each vertex $w \in N(u) \cap N(v) \cap N(x)$. Therefore $|N(u) \cap N(v)| \geq 2$. Since $|V(G)| \geq 3$ then the 2-connectedness of $G$ follows from (b). □

**Lemma 2.** *Let $G \in B_0$ and $x, y$ be two distinct vertices of $G$. Furthermore, let $P$ be an $xy$-path of length $\ell$, $d(x,y) \leq \ell \leq |V(G)| - 2$ and $v \in V(G) \setminus V(P)$, $N(v) \cap V(P) \neq \emptyset$. If $vx \notin E(G)$ or $vy \notin E(G)$ then there exists an $xy$-path $P'$ of length $\ell + 1$ such that $V(P) \subset V(P')$.*

**Proof:** Without loss of generality we suppose $vy \notin E(G)$. Let $\vec{P}$ be the path $P$ with orientation from $x$ to $y$ and let $w_1, \ldots, w_n$ denote the vertices of $W = N(v) \cap V(P)$ occurring on $P$ in the order of their indices.

**Case 1.** $n = 1$. Then $d(v, w_1^+) = 2$ and, by Corollary 1, there is a vertex $z \in (N(v) \cap N(w_1^+) \cap N(w_1)) \setminus V(P)$. The path $P' = x\vec{P}w_1 z w_1^+ \vec{P} y$ has the length e$\ell + 1$ and $V(P) \subset V(P')$.

**Case 2.** $n \geq 2$. Clearly, if $v$ is adjacent to two consecutive vertices of $P$ or $w_i^+ w_j^+ \in E(G)$ for some pair $i, j$, $1 \leq i < j \leq n$, then there exists an $xy$-path of length $\ell + 1$. Now suppose:

a) $v$ is not adjacent to two consecutive vertices of $P$,

b) $w_i^+ w_j^+ \notin E(G)$ for $1 \leq i < j \leq n$, that is: the set $W^+ = \{w_1^+, \ldots, w_n^+\}$ is independent.

Since $d(v, w_i^+) = 2$ for each $i = 1, \ldots, n$ then, from Lemma 1, we obtain

$$\sum_{i=1}^{n} |N(v) \cap N(w_i^+) \cap N(w_i)| \geq \sum_{i=1}^{n} |N(w_i) \setminus (N(v) \cup N(w_i^+))| - n. \quad (3)$$

If $N(v) \cap N(w_i^+) \cap N(w_i) \subseteq V(P)$ for each $i = 1, \ldots, n$ then

$$\sum_{i=1}^{n} |N(v) \cap N(w_i^+) \cap N(w_i)| \leq e(W, W^+) - n \quad (4)$$

and

$$\sum_{i=1}^{n} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq e(W, W^+) + n \quad (5)$$

because $N(w_i) \cap N(w_i^+) \cap N(v) \subseteq (W \cap N(w_i^+)) \setminus \{w_i\}$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \ldots, n$.

But (5) and (4) contradict (3). Hence

$$(N(v) \cap N(w_i^+) \cap N(w_i)) \setminus V(P) \neq \emptyset$$

for some $i$. Let $z \in (N(v) \cap N(w_i^+) \cap N(w_i)) \setminus V(P)$. Then the $xy$-path $P'$: $x\vec{P}w_i z w_i^+ \vec{P}y$ has the length $\ell + 1$ and $V(P) \subset V(P')$. □

**Proof of Theorem 5:** First let us show that every vertex $u$ of $G$ lies on a triangle. Consider an edge $e = ux$. By Corollary 1, we have $d(x) \geq 2$. Let $v \in N(x) \setminus \{u\}$. If $vu \in E(G)$ then $u$ lies on the triangle $xuvx$. If $vu \notin E(G)$ then $d(u,v) = 2$. Hence, by Corollary 1, there is a vertex $z \in N(u) \cap N(v) \cap N(x)$. Then $u$ lies on the triangle $xuzx$.

Now consider a cycle $\vec{C}$ of length $\ell$, $3 \leq \ell \leq |V(G)|$. Let $v \in V(G) \setminus V(C)$ and $N(v) \cap V(C) \neq \emptyset$. If $v$ is adjacent to two consecutive vertices of $C$ then there is a cycle $C'$ of length $\ell + 1$ such that $V(C) \subset V(C')$. Suppose $v$ is not adjacent to two consecutive vertices of $C$. Let $x$ be a vertex from the set $N(v) \cap V(C)$. Consider the $xy$-path $x\vec{C}y$ where $y = x^-$. We have $vy \notin E(G)$. Hence, by Lemma 2, there exists an $xy$-path $P'$ of length $\ell + 1$ such that $V(P) \subset V(P')$. Since $xy \in E(G)$ then the path $P'$ define the cycle $C'$ of length $\ell + 1$ such that $V(C) \subset V(C')$. □

**Proof of Theorem 6:** If $d(x,y) \geq 3$ and $P$ is an $xy$-path of length $\ell$, $d(x,y) \leq \ell \leq |V(G)| - 2$, then there is a vertex $v$ outside of $P$ with $N(v) \cap V(P) \neq \emptyset$. Since $d(x,y) \geq 3$ then $xv \notin E(G)$ or $yv \notin E(G)$. Hence, by Lemma 2, there exists an $xy$-path of length $\ell + 1$.

Now suppose $d(x,y) = 2$. Since $G_1(x)$ is an Ore graph then, by Theorem 3, $G_1(x)$ is hamiltonian. Let $C = u_0u_1 \ldots u_ru_0$ be a Hamilton cycle of $G_1(x)$ where $x = u_0$ and $r = d(x)$. Let $k$ be the maximum integer for which $u_ky \in E(G)$. Clearly, $k \geq 2$ because, by Corollary 1, $|N(x) \cap N(y)| \geq 2$. Then for any $\ell$, $2 \leq \ell \leq k+1$, the $xy$-path $u_0u_{k-\ell+2}u_{k-\ell+3} \ldots u_ky$ has the length $\ell$. Consider now an $xy$-path $P$ of length $\ell$, $\ell \geq k+1$, containing the vertices $u_1, \ldots, u_k$. If $\ell < |V(G)| - 1$ there is a vertex $v$ outside $P$ with $N(v) \cap V(P) \neq \emptyset$. Since $N(x) \cap N(y) \subseteq \{u_1, \ldots, u_k\} \subset V(P)$ then $vx \notin E(G)$ or $vy \notin E(G)$. Then, by Lemma 2, there exists an $xy$-path $P'$ of length $\ell + 1$ such that $V(P) \subset V(P')$. □

**Proof of Theorem 7:** Clearly, if a graph $G \in B_0$ is panconnected then $G$ is 3-connected and $G \neq K_n \vee \overline{K_n}$ for $n \geq 3$.

Now suppose that $G$ is a 3-connected graph from the set $B_0$. Let $x$ and $y$ be two distinct vertices of $G$. If $d(x,y) \geq 2$ then, by Theorem 6, there is an $xy$-path of length $\ell$ for each $\ell$, $d(x,y) \leq \ell \leq |V(G)| - 1$.

Let $d(x,y) = 1$. First we show that there is an $xy$-path $P'$ of length 2. By Corollary 1, we have $d(x) \geq 2$. Let $v \in N(x) \setminus \{y\}$. If $vy \in E(G)$ then $P' = xvy$. If $vy \notin E(G)$ then $d(v,y) = 2$ and $x \in N(v) \cap N(y)$. Hence, by Corollary 1, there is a vertex $z \in N(x) \cap N(y) \cap N(v)$. Then $P' = xzy$.

Let $\vec{P}$ be an $xy$-path $P$ of length $\ell$, $2 \le \ell \le |V(G)| - 1$, with orientation from $x$ to $y$. Suppose there is not exists an $xy$-path of length $\ell + 1$. Since $G$ is 3-connected there exists a vertex $v$ outside of $P$ such that

$$(N(v) \cap V(P)) \setminus \{x, y\} \neq \emptyset.$$

Let $w_1, \ldots, w_n$ denote the vertices of $W = N(v) \cap V(P)$ occurring on $P$ in the order of their indices. Since there is not an $xy$-path of length $\ell+1$ then, by Lemma 2, $w_1 = x$ and $w_n = y$, that is $n \ge 3$. Moreover, $w_i^+ \neq w_{i+1}$ for each $i = 1, \ldots, n-1$. Set $W_1 = \{w_1, \ldots, w_{n-1}\}$ and $W_2 = \{w_2, \ldots, w_n\}$. Using similar arguments as in the proof of Lemma 2, we can show the following:

$$\text{the sets } W_1^+ = \{w_1^+, \ldots, w_{n-1}^+\} \text{ and } W_2^- = \{w_2^-, \ldots, w_n^-\} \text{ are independent,} \tag{6}$$

$$N(v) \cap N(w_i) \cap N(w_i^+) \subseteq V(P) \text{ for each } i = 1, \ldots, n-1. \tag{7}$$

$$N(v) \cap N(w_j) \cap N(w_j^-) \subseteq V(P) \text{ for each } j = 2, \ldots, n. \tag{8}$$

$$\sum_{i=1}^{n-1} |N(v) \cap N(w_i) \cap N(w_i^+)| \ge \sum_{i=1}^{n-1} |N(w_i) \setminus (N(v) \cup N(w_i^+))| - (n-1). \tag{9}$$

$$\sum_{j=2}^{m} |N(v) \cap N(w_j) \cap N(w_j^-)| \ge \sum_{j=2}^{n} |N(w_j) \setminus (N(v) \cup N(w_j^-))| - (n-1). \tag{10}$$

Furthermore, from (6) and (7) we have

$$\sum_{i=1}^{n-1} |N(v) \cap N(w_i) \cap N(w_i^+)| \le e(W_1^+, W_1) \tag{11}$$

because $N(v) \cap N(w_i) \cap N(w_i^+) \subseteq (N(w_i^+) \cap W) \setminus \{w_i\}$ for each $i = 1, \ldots, n-1$.

Now let us prove that $w_i^+ = w_{i+1}^-$ for each $i = 1, \ldots, n-1$. First, note that

$$\text{if } w_i^+ \neq w_{i+1}^- \text{ then } w_{i+1}^- w_{i+1}^+ \in E(G), 1 \le i \le n-2. \tag{12}$$

Assuming $w_{i_0}^+ \neq w_{1+i_0}^-$ and $w_{1+i_0}^- w_{1+i_0}^+ \notin E(G)$ for some $i_0$, $1 \leq i_0 \leq n-2$, we obtain

$$\sum_{i=1}^{n-1} |N(w_i) \setminus (N(v) \cup N(w_i^+))| \geq e(W_1^+, W_1) + n \tag{13}$$

because $w_{1+i_0}^- \in N(w_{1+i_0}) \setminus (N(v) \cup N(w_{1+i_0}^+))$, $w_{1+i_0}^- \neq w_{i_0}^+$ and $v \in N(w_i) \setminus (N(v) \cup N(w_i^+))$ for each $i = 1, \ldots, n-1$. But (11) and (13) contradict (9). So, (12) is proved.

**Case 1.** $w_1^+ \neq w_2^-$. Then, by (12), $w_2^- w_2^+ \in E(G)$. Since, by (6), $w_2^- w_3^- \notin E(G)$ then $w_2^+ \neq w_3^-$. Hence, by (12), $w_3^- w_3^+ \in E(G)$.

Repetition of this argument shows that $w_i^+ \neq w_{1+i}^-$ and $w_i^- w_i^+ \in E(G)$ for each $i = 2, \ldots, n-1$.

Consider the set $D_1 = N(v) \cap N(w_1) \cap N(w_1^+)$. Since $d(v, w_1^+) = 2$ then, by Corollary 1, $|D_1| \geq 1$. If $w_i \in D_1$ for some $i$, $2 \leq i \leq n-1$, then the $xy$-path $w_1 v w_i w_1^+ \vec{P} w_i^- w_i^+ \vec{P} w_n$ has the length $\ell + 1$. Hence $w_i \notin D$ for each $i = 2, \ldots, n-1$. Since, by (7), $D_1 \subseteq V(P)$ then $D_1 = \{w_n\}$ because $w_1 \notin D_1$.

By a similar reasoning we have for the set $D_2 = N(v) \cap N(w_2) \cap N(w_2^+)$: $D_2 \subseteq V(P)$, $|D_2| \geq 1$ and if $n \geq 4$, then $w_i \notin D_2$ for each $i = 3, \ldots, n-1$. Clearly, $w_2 \notin D_2$.

**Subcase 1.1.** $w_1 \in D_2$. It means $v, w_1^+, w_2^+ \in N(w_1) \setminus (N(v) \cup N(w_1^+))$. Since $d(v, w_1^+) = 2$ then, using Lemma 1, we have

$$1 = |N(v) \cap N(w_1) \cap N(w_1^+)| \geq |N(w_1) \setminus (N(v) \cup N(w_1^+))| - 1 \geq 2$$

a contradiction.

**Subcase 1.2.** $w_1 \notin D_2$. Then $D_2 = \{w_n\}$ and $v, w_1^+, w_2^+ \in N(w_n) \setminus (N(v) \cup N(w_2^+))$. Using Lemma 1 we have

$$1 = |N(v) \cap N(w_2) \cap N(w_2^+)| \geq |N(w_n) \setminus (N(v) \cup N(w_2^+))| - 1 \geq 2,$$

a contradiction.

**Case 2.** $w_i^+ = w_{i+1}^-$ for each $i$, $1 \leq i \leq t-1 < n-1$, but $w_t^+ \neq w_{t+1}^-$. Then, by (6) and (8), we have

$$\sum_{j=2}^{n} |N(v) \cap N(w_j) \cap N(w_j^-)| \leq e(W_2^-, W_2). \tag{14}$$

If $w_t^- w_t^+ \notin E(G)$ then

$$\sum_{j=2}^{n} |N(w_j) \setminus (N(v) \cup N(w_j^-))| \geq e(W_2^-, W_2) + n \tag{15}$$

because $w_t^+ \in N(w_t)\setminus(N(v)\cup N(w_t^-))$, $w_t^+ \neq w_{t+1}^-$ and $v \in N(w_j)\setminus(N(v)\cup N(w_j^-))$ for each $j = 2,\ldots,n$. But (14) and (15) contradict (10).

Hence $w_t^- w_t^+ \in E(G)$. But then the $xy$-path $w_1\vec{P}w_{t-1}vw_tw_t^-w_t^+\vec{P}w_n$ has the length $\ell+1$, a contradiction.

So $w_i^+ = w_{i+1}^-$ for each $i = 1,\ldots,n-1$. Clearly, the path

$$P_i = w_1\vec{P}w_ivw_{i+1}\vec{P}w_n$$

has the length $\ell$ for each $i = 1,\ldots,n-1$. Repeating the arguments above with $P_i$ and $w_i^+$ instead of $P$ and $v$ we obtain $w_i^+w_j \in E(G)$ for each pair $i,j$, $1 \le i \le n-1$, $1 \le j \le n$. Hence, by (7), $|N(v)\cap N(w_i)\cap N(w_i^+)| = n-1$ for each $i = 1,\ldots,n-1$. Since

$$v, w_1^+,\ldots,w_{n-1}^+ \in N(x)\setminus(N(v)\cup N(w_1^+))$$

then, using Lemma 1, we obtain

$$n-1 = |N(v)\cap N(w_1)\cap N(w_1^+)| \ge |N(x)\setminus(N(v)\cup N(w_1^+))| - 1 \ge n-1$$

It means that

$$N(x)\setminus(N(v)\cup N(w_1^+)) = \{v, w_1^+,\ldots,w_{n-1}^+\}. \tag{16}$$

Let us prove that the set $V_0 = V(G)\setminus(V(P)\cup\{v\})$ is empty. Suppose $V_0 \neq \emptyset$. Since $G$ is connected then there exists a vertex $z \in V_0$ with $N(z)\cap(V(P)\cup\{v\}) \neq \emptyset$.

a) $N(z)\cap V(P) \neq \emptyset$. Since there is not an $xy$-path of length $\ell + l$ then, by Lemma 2, $z$ is adjacent to $x$. By (16), $z \notin N(x)\setminus(N(v)\cup N(w_1^+))$. Furthermore, $zw_1^+ \notin E(G)$ because there is not an $xy$-path of length $\ell+1$. Hence, $zv \in E(G)$. But then the $xy$-path $xzvw_2\vec{P}w_n$ has the length $\ell+1$, a contradiction.

b) $N(z)\cap V(P) = \emptyset$ and $zv \in E(G)$. Since $d(x,z) = 2$ then $|N(x)\cap N(z)\cap N(v)| \ge 1$ and there exists a vertex $z_1$ such that $z_1 \neq v$ and $z_1 \in (N(x)\cap N(v)\cap N(z))\setminus V(P)$. But then the $xy$-path $xz_1vw_2\vec{P}w_n$ has the length $\ell+1$, a contradiction.

Therefore $V_0 = \emptyset$, $V(G) = V(P)\cup\{v\}$ and $G = K_n \vee \overline{K}_n$. □

**Corollary 2.** *A connected graph $G$ of order at least 3 is panconnected if*

$$d_{G_1(x)}(u) + d_{G_1(x)}(v) \ge |M(x)| + 1$$

*for each triple of vertices $u, v, x$ with $d(u,v) = 2$ and $x \in N(u)\cap N(v)$.*

**Proof:** Clearly, $G \neq K_n \vee \overline{K}_n$ for $n \geq 3$. Using the same arguments as in the proof of Corollary 1 it is possible to prove that $|N(u) \cap N(v)| \geq 3$ for each pair of vertices $u, v$ of $G$ with $d(u,v) = 2$. It means that $G$ is 3-connected. Therefore, by Theorem 7, $G$ is panconnected. □

**Theorem 8.** *For each Ore graph $H_0$ and each integer $t \geq 2$ there exists a panconnected graph $G \in B_0$ with diameter $t$ such that $H_0$ is an induced subgraph of $G$.*

**Proof:** Let $H_0$ be an Ore graph of order $p$ and let $H_1, \ldots, H_{2t-1}$ be $p$-vertex complete graphs such that the sets $V(H_0), V(H_1), \ldots, V(H_{2t-1})$ are mutually disjoint. Consider the graph $G$ with $V(G) = \cup_{i=0}^{2t-1} V(H_i)$ and

$$E(G) = \cup_{i=0}^{2t-2}(E(H_i) \cup \{xy \setminus x \in V(H_i), y \in V(H_{i+1}))$$
$$\cup E(H_{2t-1}) \cup \{xy / x \in V(H_{2t-1}), y \in V(H_0)\}.$$

Clearly, $H_0$ is an induced subgraph of $G$ and $G \neq K_n \vee \overline{K}_n$ for $n \geq 3$. It is not difficult to check that $G \in B_0$ and $G$ is 3-connected. Therefore, by Theorem 7, $G$ is panconnected. □

Now consider the set $B_1$.

**Proposition 1.** $B_0 \subset B_1$.

**Proof:** If $G \in B_0$ then $G_1(x)$ is an Ore graph for each $x \in V(G)$. Hence $G_1(x)$ is 2-connected. It means that the subgraph induced by the set $N(x)$ is connected. Therefore $G$ is locally connected. □

The following observation was made by the author and N.K. Khachatrian.

**Proposition 2.** *If $G$ is a connected, locally connected and $K_{1,3}$-free graph then $G \in B_1$.*

**Proof:** Let $u, v \in N(x)$ and $uv \notin E(G)$. Then each vertex from the set $N(x) \setminus \{u, v\}$ is adjacent to $u$ or $v$ because $G$ is $K_{1,3}$-free graph. It means that

$$|N(x) \setminus (N(u) \cup N(v))| - 2 = 0 \leq |N(u) \cap N(v) \cap N(x)|.$$

Hence, by Lemma 1, $d_{G_1(x)}(u) + d_{G_1(x)}(v) \geq |M(x)| - 1$. So, $G \in B_1$. □

**Conjecture 1:** Every graph $G \in B_1$ is fully cycle extendible unless $G = K_{n-1} \vee \overline{K}_n$ for $n \geq 3$.

Conjecture 1 is motivated by Propositions 1 and 2 and Theorems 1 and 5.

Denote by $F_n$ the graph obtained from the complete bipartite graph $K_{n,n}$, $n \geq 3$, by deleting a perfect matching. Let

$$M_1 = \{G / F_n \subseteq G \subseteq K_n \vee \overline{K}_n \text{ for some } n \geq 3\}$$

and

$$M_2 = \{K_{n-1} \vee \overline{K}_n, n \geq 3\}.$$

**Conjecture 2:** A graph $G \in B_1$ is panconnected if and only if $G$ is 3-connected and $G \notin M_1 \cup M_2$.

Conjecture 2 is motivated by Propositions 1 and 2 and Theorems 2 and 7.

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