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On The Existence of $(v, 4, 1)$-RPMD

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ABSTRACT. It is well known that a necessary condition for the existence of a $(v, 4, 1)$-RPMD is $v \equiv 0$ or $1 \pmod 4$ and the existence of $(v, 4, 1)$-RPMDs for $v \equiv 1 \pmod 4$ has been completely settled.

In this paper, we shall introduce the concept of $(v, k, 1)$-nearly-RPMDs and use it to obtain some new construction methods for $(v, k, 1)$-RPMDs with $v \equiv 0 \pmod k$. As an application, we shall show that a $(v, 4, 1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod 4$, except for $v = 4, 8$ and with at most 49 possible exceptions of which the largest is 336.

It is also well known that a $(v, k, 1)$-RPMD exists for all sufficiently large v with $k \geq 3$ and $v \equiv 1 \pmod k$, and a $(v, k, 1)$-PMD exists with $v(v-1) \equiv 0 \pmod k$ for the case when k is an odd prime and v is sufficiently large. In this paper, we shall show that there exists a $(v, k, 1)$-RPMD for all sufficiently large v with $v \equiv 0 \pmod k$, and there exists a (v, k, λ)-PMD for all sufficiently large v with $\lambda v(v-1) \equiv 0 \pmod k$.

1 Introduction

A cyclic k-tuple $(a_1, a_2, \ldots a_k)$, is defined to be $\{(a_i, a_{i+1}), (a_k, a_1) : 1 \leq i \leq k-1\}$. The elements a_i, a_{i+t} are said to be t-apart in a cylic k-tuple $(a_1, a_2, \ldots, a_k)$ where $i + t$. is taken modulo k.

Let $M = \{n_i : 1 \leq i \leq h\}$ be a set of positive integers. A holey perfect Mendelsohn design (briefly $(v, k, 1)$-HPMD, or $(v, k, 1)$-GPMD), is a triple $(X, \mathbf{G}, B)$ where

(1) X is a v-set (of points),

(2) $\mathbf{G}$ is a collection of non-empty subsets of X (called holes) with sizes in M and partition X,

(3) $\mathbf{B}$ is a collection of cyclic k-tuples of X (called blocks)

(4) no block meets a hole in more than one point, and

(5) every ordered pair of points not contained in a hole appears t-apart in exactly one block.

The vector $(n_1, n_2, \ldots, n_k)$ is called the type of HPMD. A $(v, k, 1)$-HPMD of type $(1, 1, \ldots, 1, n)$ is called an incomplete perfect Mendelsohn design, denoted by $(v, n, k, 1)$-IPMD. A $(v, k, 1)$-HPMD of type $(1, 1, \ldots, 1)$ is called a $(v, k, 1)$-PMD.

A subset of blocks in a design is called a partial parallel class if the subset consists of pairwise disjoint. blocks.

Let $(X, \mathbf{A})$ be a $(v, k, 1)$-PMD and $(Y, \mathbf{B})$ be a $(n, k, 1)$-PMD. If $X \supset Y$ and $\mathbf{A} \supset \mathbf{B}$ we say that the first design contains the second as a subdesign.

Let $A = (a_1, a_2, \ldots, a_k) \in \mathbf{A}$ be a cyclic k-tuple, $B = (h_1, b_2, \ldots, b_k) \in \mathbf{B}$ and $C = (c_1, c_2, \ldots, c_k) \in \mathbf{C}$ be two ordered k-tuples, we define

$$\langle A, B\rangle = ((a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)) \text{ is a cyclic } k - \text{tuple.}$$
$$\langle \mathbf{A}, \mathbf{B}\rangle = \{\langle A, B\rangle : A \in \mathbf{A}, B \in \mathbf{B}\}$$
$$\langle A, B, C\rangle = \langle\langle A, B\rangle, C\rangle$$
$$\langle \mathbf{A}, \mathbf{B}, \mathbf{C}\rangle = \{\langle A, B, C\rangle : A \in \mathbf{A}, B \in \mathbf{B}, C \in \mathbf{C}\}$$

Suppose (b_{ij}) is an $n^2 \times k$ orthogonal array based on an n-set, say Y, we say that $(A \times Y, \mathbf{G}, \langle A, \mathbf{B}\rangle)$ is a cyclic $TD[k, 1; n]$ where $\mathbf{B} = \{(b_{i1}, b_{i2}, \ldots, b_{ik}) : 1 \leq i \leq n\}$.

It is easy to show that the numher of blocks in a $(v, k, 1)$-PMD is $v(v-1)/k$, and hence an obvious necessary condition for its existence is $v(v-1) \equiv 0 \pmod{k}$. We next define the notion of resolvability of a $(v, k, 1)$-PMD where $v \equiv 0$ or $1 \pmod{k}$.

Definition 1.1: If the blocks of a $(v, k, 1)$-PMD for which $v \equiv 1 \pmod{k}$ can be partitioned into v sets each containing $(v-1)/k$ blocks which are pairwise disjoint (as sets), we say that the $(v, k, 1)$-PMD is resolvable (briefly $(v, k, 1)$-RPMD) and each set of $(v-1)/k$ pairwise disjoint blocks will be called a parallel class.

A resolvahle PMD and parallel classes by Definition 1.1 are usually called an almost resolvahle PMD and almost parallel classes. For convenience, we use Definition 1.1 in this paper.

Definition 1.2: If the blocks of a $(v, k, 1)$-PMD for which $v \equiv 0 \pmod{k}$ can be partitioned into $(v-1)$ sets each containing v/k blocks which are pairwise disjoint (as sets), we shall also say that the $(v, k, 1)$-PMD is resolvahle (briefly $(v, k, 1)$-RPMD) and each set of v/k pairwise disjoint blocks will be called a parallel class.

N.S. Mendelsohn introduced perfect cyclic designs which were called Mendelsohn designs by Hsu and Keedwell (see [6] and [11]). The following theorem was proved in [3,7].

Theorem 1.1. *A $(v,3,1)$-RPMD exists if and only if $v \equiv 0$ or $1 \pmod 3$, $v \neq 6$.*

From Bennett, and Lamken, Mills,and Wilson (see [1,4,9]), we have

Theorem 1.2. *A $(v,4,1)$-RPMD exists for $v \equiv 1 \pmod 4$.*

Following the original work of Mendelsohn, investigations into the existence of $(v,4,1)$-PMDs were carried out by Bennett, Zhu, and the author. It is proved that a $(v,4,1)$-PMD exists for precisely all positive integers $v \equiv 0$ or $1 \pmod 4$ except $v = 4$ and 8, and possibly excepting $v = 12$ (see [1,5,14]). The only possible exception $v = 12$ was recently removed according to Bennett.

Consequently, we have

Theorem 1.3. *The necessary condition for the existence of a $(v,4,1)$-PMD, namely, $v(v-1) \equiv 0 \pmod 4$, is also sufficient except for $v = 4$ and 8.*

The following results were proved by the author (see [15]).

Theorem 1.4. *The necessary condition for the existence of a $(v,4,\lambda)$-RPMD where $\lambda > 1$, namely, $v \equiv 0$ or $1 \pmod 4$, is also sufficient with the exception of pairs (v,λ) where $v = 4$ and λ odd.*

We assume that the reader is familiar with the concept of group divisible design (GDD), a transversal design (TD) and a resolvable transversal design (RTD).

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. The following results are well known.

Lemma 1.5. *The existence of an $n^2 \times k$ orthogonal array is equivalent to $N(n) \geq k-2$.*

Let $\mathbf{B} = \{(b_{i1}, b_{i2}, \ldots, b_{ik}) \colon 1 \leq i \leq n\}$ where (b_{ij}) is an orthogonal array based on the set $\{1,2,\ldots,n\}$. If $\mathbf{B}$ can pe partitioned into n parts: $\mathbf{B}_j$, $1 \leq j \leq n$ such that $\langle A, \mathbf{B}_j \rangle$ is a partition of $A \times Y$, we say that (b_{ij}) (or $\mathbf{B}$) is resolvable. In this case we can always let

$$\mathbf{B}_1 = \{(i,i,\ldots,i) \colon 1 \leq i \leq n\} \text{ or}$$
$$\mathbf{B}_1 = \{(i+1, i+2, \ldots, i+k) \colon 0 \leq i \leq n-1\}$$

Lemma 1.6. *The existence of a resolvable $n^2 \times k$ orthogonal array is equivalent to $N(n) \geq k-1$.*

Lemma 1.7. *If $n \neq 2,6$, then $N(n) \geq 2$; If $n \neq 2,3,6,10$, then $N(n) \geq 3$.*

In this paper, we shall introduce the concept of $(v,k,1)$-nearly-RPMDs and use it to obtain some new construction methods for $(v,k,1)$-RPMD for $v \equiv 0 \pmod{k}$. As an aplication, we shall show that a $(v,4,1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod{4}$, except for $v = 4, 8$, and with at most 49 possible exceptions of which the largest is 336. We also show that there exists a $(v,k,1)$-RPMD for all sufficiently large v with $v \equiv 0 \pmod{k}$, and there exists a (v,k,λ)-PMD for all sufficiently large v with $\lambda v(v-1) \equiv 0 \pmod{k}$.

2 $(v,k,1)$-nearly-RPMD and construction methods

We first introduce the concept of $(v,k,1)$-nearly-RPMD.

Definition 2.1: Let $Y(\subset X)$ be n-set and $(X,Y,\mathbf{A}\cup\mathbf{B})$ be a $(v,n,k,1)$-IPMD, where $v \equiv 0 \pmod{k}$ and $n \equiv 1 \pmod{k}$. If $\mathbf{A}$ can be partitioned into $(v-n)$ parallel classes of X, and $\mathbf{B}$ can be partitioned into n partial parallel classes of $X \setminus Y$: $\mathbf{B}_j$, $1 \leq j \leq n$. We say that $(X,Y,\mathbf{A}\cup\mathbf{B})$ is nearly resolvable (briefly, $(v,n,k,1)$-nearly-IRPMD). Let $S_j = \{a\colon a \in B \in \mathbf{B}_j\}$, $C_j = X \setminus (Y \cup S_j)$ and $c_j = |C_j|$ we say C_j is the complement of the partial parallel class $\mathbf{B}_j$ and the vector $(c_1, c_2, \ldots, c_n)$ is called the complement type of the nearly-IRPMD. If $c_i \leq c_j$ when $i < j$ and $(c_n - c_1) \leq k$ we say the type $(c_1, c_2, \ldots, c_n)$ is standard.

It is easy to see the complements C_j, $1 \leq j \leq n$ partition $X \setminus Y$.

Definition 2.2: Let $(X,\mathbf{B})$ be a $(v,k,1)$-PMD where $v \equiv 0 \pmod{k}$. If $\mathbf{B}$ can be partitioned into v partial parallel classes $\mathbf{B}'_j$, $1 \leq j \leq n$, we say that $(X,\mathbf{B})$ is a $(v,k,1)$-nearly-RPMD. Let $S'_j = \{a\colon a \in B \in \mathbf{B}'_j\}$, $C'_j = X \setminus S'_j$ and $c'_j = |C'_j|$ we say C'_j is the complement of the partial parallel class $\mathbf{B}'_j$ and the vector $(c'_1, c'_2, \ldots, c'_v)$ is called the complement type of the nearly-RPMD.

It is easy to see the complements C'_j, $1 \leq j \leq v$, partition X.

Remark 2.3: Let $(X,Y,\mathbf{A}\cup\mathbf{B})$ be a $(v,n,k,1)$-nearly-IRPMD with the complement type $(c_1, c_2, \ldots, c_k)$. If $(Y,\mathbf{N})$ is an $(n,k,1)$-RPMD where $n \equiv 1 \pmod{k}$. Then $(X, \mathbf{A}\cup\mathbf{B}\cup\mathbf{N})$ is a $(v,k,1)$-nearly-RPMD with the complement type $(c'_1, c'_2, \ldots, c'_n)$ where $c'_i = c_i + 1$.

Definition 2.4: If the blocks of $(v,k,1)$-HPMD based on X can be partitioned into some partitions of X, we say that it is resolvable (briefly, $(v,k,1)$-HRPMD).

We now establish several constructions for $(v,k,1)$-RPMD for $v \equiv 0 \pmod{k}$ by using nearly-IRPMDs or nearly-RPMDs.

Lemma 2.1. *Let $I_u = \{1,2,\ldots,u\}$ and $I_v = \{1,2,\ldots,v\}$. Let $\mathbf{E}_i$ be a partition of $I_u \times \{i\}$ for $1 \leq i \leq v$, and $\mathbf{F}_j$ is a partial parallel class of I_v with the complement C_j for $1 \leq j \leq s$. If the complements partition I_v*

then $\mathbf{E} \cup I_u \times \mathbf{F}$ *is the union of* s *partitions of* $I_u \times I_v$, *where* $\mathbf{E} = \cup_{1 \le i \le v} \mathbf{E}_i$ *and* $\mathbf{F} = \cup_{1 \le j \le s} \mathbf{F}_j$.

Proof: Let $\mathbf{V}_j = \{E \colon E \in \mathbf{E}_i, i \in C_j\}$ it is easy to see that $\mathbf{V}_j \cup I_u \times \mathbf{F}_j$ is a partition of $I_u \times I_v$, and $\mathbf{E} \cup I_u \times \mathbf{F} = \cup_{1 \le j \le s}(\mathbf{V}_j \cup I_u \times \mathbf{F}_j)$. This completes the proof.

Theorem 2.2. *Let* $v, h \equiv 0 \pmod{k}$. *Suppose* $(I_h \times I_v, \mathbf{G}, \mathbf{D})$ *is an* $(hv, k, 1)$*-HRPMD of type* v^h *with* $(h-1)v$ *parallel classes:* $\mathbf{D}_i$, $1 \le i \le (h-1)v$, *suppose that*

(1) $\mathbf{D}_1$ *can be partitioned into* v *parts:* $\mathbf{E}_j$, $1 \le j \le v$ *such that every* $\mathbf{E}_j$ *is a partition of* $I_h \times \{j\}$ *for* $1 \le j \le v$;

(2) there exists a $(v, k, 1)$*-nearly-RPMD of* $(I_v, \mathbf{A})$.

Then there exists an $(hv, k, 1)$*-RPMD.*

Proof: Since the complements of a $(v, k, 1)$-nearly-RPMD, partition I_v, so $\mathbf{D}_1 \cup I_h \times \mathbf{A}$ can be partitioned into v parallel classes of $I_h \times I_v$ by Lemma 2.1. Hence $(I_h \times I_v, \mathbf{D} \cup I_h \times \mathbf{A})$ is an $(hv, k, 1)$-RPMD.

Theorem 2.3. *Let* $v, h \equiv 0 \pmod{k}$ *and* $n \equiv 1 \pmod{k}$. *Suppose that*

(1) there exists an $(hv, k, 1)$*-HRPMD of type* v^h;

(2) there exists a $(v, n, k, 1)$*-nearly-IRPMD with the complement type* $(c_1, c_2, \dots, c_n)$;

(3) there exist an $(n, k, 1)$*-RPMD.*

Then there exists a $(hv, n, k, 1)$*-nearly-IRPMD with the complement type* $(f_1, f_2, \dots, f_n)$ *where* $f_i = h(c_i + 1) - 1$ *for* $1 \le i \le n$.

Proof: Let $\mathbf{D}$ be the blocks of the HRPMD hased on $I_h \times I_v$, $(I_v, I_n, \mathbf{A} \cup \mathbf{B})$ be a $(v, n, k, 1)$-nearly-IRPMD with the complement type $(c_1, c_2, \dots, c_n)$ where $\mathbf{A}$ is the union of some parallel classes, and $\mathbf{B}$ can be partitioned into n partial parallel classes: $\mathbf{B}_j$, $1 \le j \le n$. Let $(I_n, \mathbf{N})$ be a $(n, k, 1)$-RPMD having n parallel classes: $\mathbf{N}_j$, $1 \le j \le n$. It is easy to see that $(I_h \times I_v, \{h\} \times I_n, \mathbf{D} \cup I_h \times (\mathbf{A} \cup \mathbf{B}) \cup I_{h-1} \times \mathbf{N})$ is a $(hv, n, k, 1)$-IPMD. Since $\mathbf{D} \cup I_h \times \mathbf{A}$ can be partitioned into some parallel classes of $I_h \times I_v$ and $I_h \times \mathbf{B} \cup I_{h-1} \times \mathbf{N}$ can be partitioned into n partial parallel classes: $I_h \times \mathbf{B}_j \cup I_{h-1} \times \mathbf{N}_j$, $1 \le j \le n$.
This completes the proof.

Theorem 2.4. *Let* $u \equiv 0 \pmod{k}$ *and* $p, n \equiv 1 \pmod{k}$. *Suppose*

(1) There exist a $(p, k, 1)$*-RPMD and an* $(n, k, 1)$*-RPMD,*

(2) There exists a $(v, n, k, 1)$-nearly-IRPMD with the complement type $(c_1, c_2, \ldots, c_n)$,

(3) $v = um$ such that $N(u) \geq k-1$, $N(m) \geq k-1$ and $n \leq m$.

Then there exist a $(pv, k, 1)$-RPMD and a $(pv, n, k, 1)$-nearly-IRPMD with the complement type $(f_1, f_2, \ldots, f_n)$ where $f_i = p(ci+1)-1$.

Proof: Let $\mathbf{B}(u)$ and $\mathbf{B}'(u)$ be two resolvable orthogonal array based on I_u with $\mathbf{B}_1(u) = \{(i, i, \ldots, i)\colon 1 \leq i \leq u\}$ and $\mathbf{B}'_1(u) = \{(1+i, 2+i, \ldots, k+i)\colon 0 \leq i \leq u-1\}$. Let $\mathbf{B}(m)$ be a resolvable orthogonal array based on I_m with $\mathbf{B}_1(m) = \{(i, i, \ldots, i)\colon 1 \leq i \leq m\}$. Let $(I_p, \mathbf{A})$ be a $(p, k, 1)$-RPMD having p parallel classes: $\mathbf{A}_j$, $1 \leq j \leq p$ where each $\mathbf{A}_j$ is a partition of $I_p \setminus \{j\}$. Let $(I_u \times I_m, \mathbf{M} \cup \mathbf{N})$ be a $(v, k, 1)$-nearly-RPMD where $\mathbf{M}$ is the union of $(v-n)$ parallel classes: $\mathbf{M}_j$, $1 \leq j \leq (v-n)$ and $\mathbf{N}$ is the union of n partial parallel classes: $\mathbf{N}_j$, $1 \leq j \leq n$.

Let $\mathbf{D} = \langle \mathbf{A} \setminus \mathbf{A}_p, \mathbf{B}(u), \mathbf{B}(m) \rangle \cup \langle \mathbf{A}_p, \mathbf{B}'(u), \mathbf{B}(m) \rangle$, $X = I_p \times I_u \times I_m$. Since $\langle A, \mathbf{B}(u), \mathbf{B}(m) \rangle$ or $\langle A, \mathbf{B}'(u), \mathbf{B}(m) \rangle$ is a cyclic $TD[k, 1; um]$ so it is easy to see that $(X, I_p \times (\mathbf{M} \cup \mathbf{N}) \cup \mathbf{D})$ is a $(pv, k, 1)$-PMD.

It is clear that $\langle \mathbf{A}_j, \mathbf{B}_s(u), \mathbf{B}_t(m) \rangle \cup \{j\} \times \mathbf{M}_i$ is a partition of X, so $(\mathbf{D} \setminus \mathbf{D}') \cup I_p \times \mathbf{M}$ is the union of $p(v-n)$ partitions of X, where $\mathbf{D}' = \cup_{1 \leq t \leq n} \{\mathbf{A} \setminus \mathbf{A}_p, \mathbf{B}_1(u), \mathbf{B}_t(m) \rangle \cup \langle \mathbf{A}_p, \mathbf{B}'_1(u), \mathbf{B}_t(m) \rangle\}$, which, we are to show, can be partitioned into $(p-1)n$ partit.ions of X.

First we are to show that $\langle \mathbf{A} \setminus \mathbf{A}_p, \mathbf{B}_1(u) \rangle \cup \langle \mathbf{A}_p, \mathbf{B}'_1(u) \rangle = (\mathbf{A} \setminus \mathbf{A}_p) \times I_u \cup \langle \mathbf{A}_p, \mathbf{B}'_1(u) \rangle$ can pe partitioned into $(p-1)$ partitions of $I_p \times I_u$. Let $A \in \mathbf{A}_p$, $B = (i+1, i+2, \ldots, i+k) \in \mathbf{B}'_1(u)$ we define $\langle A, B \rangle \cup \{\mathbf{A}_j \times \{s\}\colon (j, s) \in \langle A, B \rangle\}$, a partition of $I_p \times B$, is a small part, denoted by $\mathbf{E}(A, B)$. Since $\langle \mathbf{A}_p, \mathbf{B}'_1(u) \rangle$ is a partition of $I_{p-1} \times I_u$, it is not difficult to see that $(\mathbf{A} \setminus \mathbf{A}_p) \times I_u \cup \langle \mathbf{A}_p, \mathbf{B}'_1(u) \rangle$ is the union of $u(p-1)/k$ small parts: $\mathbf{E}(A, B)$, $A \in \mathbf{A}_p$, $B \in \mathbf{B}'_1(u)$. Since $\mathbf{B}'_1(u)$ is the union of k partitions of $I_u\colon \mathbf{P}_s = \{(i+1, i+2, \ldots, i+k)+s\colon i = 0, k, 2k, \ldots, u-k\}$, $0 \leq s \leq k-1$, so these small parts can be partitioned into $(p-1)$ partitions of $I_p \times I_u$. Hence $\cup_{A \in \mathbf{A}_p, 0 \leq s \leq k-1, 1 \leq t \leq n} \langle \mathbf{E}(A, \mathbf{P}_s), \mathbf{B}_t(m) \rangle$ is the union of $(p-1)n$ partitions of X. We take one partition of X from them, say, $\langle \mathbf{E}(A, \mathbf{P}_0), \mathbf{B}_1(m) \rangle = \mathbf{E}(A, \mathbf{P}_0) \times I_m$. Since the complements C_j of $\mathbf{N}_j$, $1 \leq j \leq n$, partition $I_u \times I_m$, and $C_j \equiv 0 \pmod{k}$, so we can partition $\mathbf{E}(A, \mathbf{P}_0) \times I_m$ (may be viewed as um/k small parts) into n parts $\mathbf{F}_j$, $1 \leq j \leq n$, such that each $\mathbf{F}_j$ contains c_j/k small parts. There is no loss of generality by assuming that $I_p \times \mathbf{N}_j \cup \mathbf{F}_j$ is a partition of X for $1 \leq j \leq n$. Therefore we have proved that the $(pv, k, 1)$-PMD is also resolvable.

Let $(Y, \mathbf{H})$ be an $(n, k, 1)$-RPMD having n parallel classes $\mathbf{H}_j$, where $\mathbf{H}_j \subset \{p\} \times \mathbf{N}_j$, $1 \leq j \leq n$. Since $I_p \times \mathbf{M} \cup \{\langle \mathbf{A} \setminus \mathbf{A}_p, \mathbf{B}(u), \mathbf{B}(m) \rangle\} \cup \langle \mathbf{A}_p, \mathbf{B}'(u), \mathbf{B}(m) \rangle$ is the union of $(pv-n)$ partitions of X, and $(I_p \times \mathbf{N}) \setminus \mathbf{H} =$

$\cup_{1\le j\le n}\{I_p \times \mathbf{N}_j \setminus \mathbf{H}_j\}$ is the union of n partial parallel classes of X. Hence there is a $(pv, n, k, 1)$-nearly-IRPMD. Now we have completed the proof.

Theorem 2.5. *Let $v \equiv 0$, $n, p \equiv 1 \pmod k$. suppose that*

(1) There exists a $(v, k, 1)$-RPMD;

(2) There exists a $(v, n, k, 1)$-nearly-IRPMD;

(3) There exists a $(p, k, 1)$-RPMD;

(4) $N(v - n) \ge k - 1$.

Then there exists a $((v - n)p + n, k, 1)$-RPMD.

Proof: Let $(I_p, \mathbf{A})$ be a $(p, k, 1)$-RPMD having p parallel classes: $\mathbf{A}_j$, $1 \le j \le p$ where $\mathbf{A}_j$ is a partition of $I_p \setminus \{j\}$ for $1 \le j \le p$. Let $Y = \{\infty_1, \infty_2, \ldots, \infty_n\}$, $\{i\} \times \{\infty_j\} = \{\infty_j\}$ for $1 \le i \le p$, $1 \le j \le n$, $(\{p\} \times I_{v-n} \cup Y, \mathbf{D})$ be a $(v, k, 1)$-RPMD, and $(I_{v-n} \cup Y, Y, \mathbf{M} \cup \mathbf{N})$ be a $(v, n, k, 1)$-nearly-IRPMD where $\mathbf{M}$ can be partitioned into $(v - n)$ parallel classes of $I_{v-n} \cup Y$ and $\mathbf{N}$ can be partitioned into n partial parallel classes of I_{v-n}. Let $\mathbf{B}$ be a resolvable orthogonal array based on I_{v-n} with $\mathbf{B}_1 = \{(i, i, \ldots, i) : 1 \le i \le (v - n)\}$.

Since $\langle \mathbf{A}_p, \mathbf{B}_1 \rangle \cup I_{p-1} \times \mathbf{N} = \mathbf{A}_p \times I_{v-n} \cup I_{p-1} \times \mathbf{N}$ can be partitioned into n parallel classes of $I_{p-1} \times I_{v-n}$ by Lemma 2.1, so it is easy to see that $\langle \mathbf{A}, \mathbf{B} \rangle \cup \mathbf{D} \cup I_{p-1} \times (\mathbf{M} \cup \mathbf{N}) = \{((\langle \mathbf{A}_p, \mathbf{B}_1 \rangle \cup I_{p-1} \times \mathbf{N}) \cup (\langle \mathbf{A}_p, \mathbf{B} \setminus \mathbf{B}_1 \rangle) \cup \mathbf{D}\} \cup \{\cup_{1 \le j \le p-1}(\langle \mathbf{A}_j, \mathbf{B} \rangle \cup \{j\} \times \mathbf{M})\}$ can be partitioned into $(v - 1) + (p - 1)(v - n) = p(v - n) + n - 1$ parallel classes of $I_p \times I_{v-n} \cup Y$. Hence there exists a $(p(v - n) + n, k, 1)$-RPMD.

Theorem 2.6. *Let $v \equiv 0$, $n, p \equiv 1 \pmod k$. suppose that.*

(1) There exists a $(v, n, k, 1)$-nearly-IRPMD with the complement type $(c_1, c_2, \ldots, c_n)$;

(2) There exists a $(p, k, 1)$-RPMD;

(3) $N(v - n) \ge k - 1$.

Then there exists a $((v - n)p + n, n, k, 1)$-nearly-IRPMD with the complement type $(pc_1, pc_2, \ldots, pc_n)$.

Proof: We adopt the notation of the proof in Theorem 2.5. Since $\langle \mathbf{A}, \mathbf{B} \rangle \cup I_p \times \mathbf{M}$ can be partitioned into some parallel classes and $I_p \times \mathbf{N}$ can be partitioned into n partial parallel classes.

Therefore $(I_p \times I_{v-n} \cup Y, Y, \langle \mathbf{A}, \mathbf{B} \rangle \cup I_p \times (\mathbf{M} \cup \mathbf{N})$ is a $((v-n)p+n, n, k, 1)$-nearly-IRPMD.

Theorem 2.7. *Let $v \equiv 0 \pmod k$, $p \equiv 1 \pmod k$. Suppose that*

(1) There exists a $(v, k, 1)$-nearly-RPMD;

(2) There exists a $(p, k, 1)$-RPMD;

(3) $N(v-1) \geq k-1$.

Then there exists a $(p(v-1)+1, k, 1)$-nearly-RPMD.

Proof: We adopt the notation of the proof in Theorem 2.5.

Let $(I_{v-1} \cup \{\infty\}, \mathbf{N})$ be a $(v, k, 1)$-nearly-RPMD where $\mathbf{N}$ can be partitioned into v partial parallel classes : $\mathbf{N}_j$, $1 \leq j \leq v$. It is easy to see that $\langle \mathbf{A}, \mathbf{B} \rangle \cup I_p \times \mathbf{N} = \{\cup_{1 \leq i \leq p, 1 \leq j \leq v-1}(\langle \mathbf{A}_i, \mathbf{B}_j \rangle \cup \{i\} \times \mathbf{N}_j)\} \cup (I_p \times \mathbf{N}_v)$ is the union of $p(v-1)+1$ partial parallel classes of $I_p \times I_{v-1} \cup \{\infty\}$. This completes the proof.

3 Recursive construction for $(v, 4, 1)$-RPMDs with $v \equiv 0 \pmod 4$

We establish construction for $(v, 4, 1)$-RPMDs with $v \equiv 0 \pmod 4$ by using nearly-IRPMDs or nearly-RPMDs.

Let

$$
\begin{aligned}
H_i &= \{h_{ij} \colon 1 \leq j \leq 5\} \text{ for } n-m+1 \leq i \leq n, \\
P_i &= \{p_{ij} \colon 1 \leq j \leq 5\} \text{ for } n-t+1 \leq i \leq n, \\
G_j &= \{j\} \times I_n \text{ for } 1 \leq j \leq 15, \\
G_{16} &= \{16\} \times I_{n-t} \cup (\cup_{n-t+1 \leq i \leq n} P_i), \\
G_{17} &= \{17\} \times I_{n-m} \cup (\cup_{n-m+1 \leq i \leq n} H_i), \\
\mathbf{G} &= \{G_i \colon 1 \leq i \leq 17\}, \ X = \cup_{1 \leq i \leq 17} G_i, \\
G &= \{n, n+4t, n+4m\} \\
K &= \{5, 17\}, \mathbf{B}(x) = \{B \colon x \in B \in \mathbf{B}\}, v = 17n + 4m + 4t \\
Q_i &= I_{16} \times \{i\} \text{ for } 1 \leq i \leq n-t, \\
Q_j &= I_{15} \times \{j\} \cup P_j \text{ for } n-t+1 \leq j \leq n.
\end{aligned}
$$

In this section, the following figure is very helpful to readers.

	Q_1	$Q_2 \cdots\cdots$				Q_{n-1}	Q_n
G_1	*	*.........*	*	*	...	*	*
G_2	*	*.........*	*	*	...	*	*
G_3	*	*.........*	*	*	...	*	*
G_{15}	*	*.........*	*	*	...	*	*
G_{16}	*	*.........*	*	—	...	—	—
G_{17}	*	*.........*	—	—	...	—	—

Lemma 3.1. *If $N(n) \geq 15$ then there exists a $GDD[K, G, v]$ of $(X, \mathbf{G}, \mathbf{B})$ satisfying the following condition: (B) For $x \in H_n$, $\mathbf{B}(x)$ can be partitioned into n parts: $\mathbf{B}(x)_i$, $1 \leq i \leq n$ such that $\cup_{B \in \mathbf{B}(x)_i}(B \setminus \{x\}) = Q_i$ for $1 \leq i \leq n$.*

Proof: Since $N(n) \geq 15$ we can let $(I_{17} \times I_n, \{\{i\} \times I_n : i \in I_{17}\}, \mathbf{D})$ be a $TD[17, 1; n]$ and there is no loss of generality by assuming that (A) $\{(17, n)\} \cup I_{16} \times \{i\} \in \mathbf{D}$ for $1 \leq i \leq n$. Select $(m+t)$ points: $(17, n)$, $(17, n-1), \ldots, (17, n-m+1)$, $(16, n)$, $(16, n-1), \ldots, (16, n-t+1)$. A $GDD[K, G, v]$ is constructed as follows : First replace $(17, i)$ with H_i and $(16, i)$ with P_i. For any block which formerly contained one point which is now replaced form new blocks from a $GDD[\{5\}, \{1, 5\}, 21]$ which has as groups the set of new points and singleton sets for all old points still in the block. For any block of the $TD[17, 1; n]$ which formerly contained two points which are now replaced new blocks are constructed from a $[GDD[\{5\}, \{1, 5\}, 25]$ which has as groups the two sets of new points and singleton sets for all old points still in the block. It is easy to see that the $GDD[K, G, v]$ satisfies the condition (B) from the condition (A).

Since there exist a $(5, 4, 1)$-RPMD and a $(17, 4, 1)$-RPMD, we can let $(B, \mathbf{A}(B))$ be an $(h, 4, 1)$-RPMD for $B \in \mathbf{B}$, provided $|B| = h$, which has h parallel classes: $\mathbf{A}(B)_x$, $x \in B$ where $\mathbf{A}(B)_x$ is a partition of $B \setminus \{x\}$. It is easy to see that $\mathbf{P}_x = \cup_{B \in \mathbf{B}(x)} \mathbf{A}(B)_x$ is a partition of $X \setminus G_i$, provided $x \in G_i$, for $1 \leq i \leq 17$, and $(X, \mathbf{G}, \mathbf{A})$ is a $(v, 4, 1)$-HPMD where $\mathbf{A} = \cup_{B \in \mathbf{B}} \mathbf{A}(B) = \cup_{x \in X} \mathbf{P}_x$. Moreover from the condition (B) it satisfies the following condition: (C) For $x \in H_n$, $\mathbf{P}_x$ can be partitioned into n parts: $\mathbf{P}_x^i$, $1 \leq i \leq n$ such that $\mathbf{P}_x^i$ is a partition of Q_i for $1 \leq i \leq n$.

Let $Y = \{\infty_1, \infty_2, \ldots, \infty_w\}$, $w \equiv 1 \pmod 4$, $n \equiv 3 \pmod k$. Suppose that

(1) $(G_{17} \cup Y, \mathbf{D})$ is an $(n + 4m + w, 4, 1)$-RPMD,

(2) $(G_{16} \cup Y, Y, \mathbf{M} \cup \mathbf{N})$ is an $(n + 4t + w, w, 4, 1)$-nearly-IRPMD with the complement type $(c_1, c_2, \ldots, c_w)$,

(3) $(I_n \cup Y, Y, \mathbf{K} \cup \mathbf{E})$ is an $(n + w, w, 4, 1)$-nearly-IRPMD with the complement type $(f_1, f_2, \ldots, f_n)$,

(4) $f_i \geq t_i \geq 0$ for $1 \leq i \leq w$ where $t_i = (c_i - f_i)/4$.

We are to prove that $(X \cup Y, \mathbf{A} \cup \mathbf{D} \cup \mathbf{M} \cup \mathbf{N} \cup (I_{15} \times (\mathbf{K} \cup \mathbf{E})))$ is a $(v + w, 4, 1)$-RPMD where $\{i\} \times \{\infty_j\} = \{\infty_j\}$.

Since the union of $\mathbf{P}_x$ and a parallel class of $G_i \cup Y$, provided $x \in G_i$, is a parallel class of $X \cup Y$, so we only need to show that $\mathbf{P}_x \cup \mathbf{N} \cup I_{15} \times \mathbf{E}$ can be partitioned into w partitions of $X \setminus G_{17}$ where $x = h_{n1}$.

Since the complements N_j of $\mathbf{N}_j$, $1 \le j \le w$, partition G_{16} and the complements E_j of $\mathbf{E}_j$, $1 \le j \le w$, partition I_n, and $f_i \ge t_i \ge 0$, for $1 \le i \le w$, so there is no loss of generality by assuming that

$$N_1 = \{\{16\} \times \{j\} : 1 \le j \le (f_1 - t)\} \cup \{P_j : n - t + 1 \le j \le n - t + t_1\}$$

$$N_i = \{\{16\} \times \{j\} : \sum_{1 \le s \le i-1} (f_s - t_s) + 1 \le j \le \sum_{1 \le s \le i} (f_s - t_s)\}$$

$$\cup \{P_j : n - t + \sum_{1 \le s \le i-1} t_s + 1 \le j \le n - t + \sum_{1 \le s \le i} t_s\} \text{ for } 2 \le i \le w$$

$$E_1 = \{j : 1 \le j \le f_1 - t_1\} \cup \{j : n - t + 1 \le j \le n - t + t_1\}$$

$$E_i = \{j : 1 + \sum_{1 \le s \le i-1} (f_s - t_s) \le j \le \sum_{1 \le s \le i} (f_s - t_s)\}$$

$$\cup \{j : n - t + \sum_{1 \le s \le i-1} t_s + 1 \le j \le n - t + \sum_{1 \le s \le i} t_s\} \text{ for } 2 \le i \le w.$$

It is easy to see that $\mathbf{N}_j \cup I_{15} \times \mathbf{E}_j$ is a partition of $\cup_{i \in I_n \setminus E_i} Q_i$. Therefore $\mathbf{P}_x \cup \mathbf{N} \cup I_{15} \times \mathbf{E}$ can be partitioned into w partitions of $X \setminus G_{17}$ for $x = h_{n1}$, from the condition (C). That is

Theorem 3.2. *Suppose that*

(1) $N(n) \ge 15$, $n \equiv 3 \pmod 4$, $w \equiv 1 \pmod 4$;

(2) There exists an $(n + 4m + w, 4, 1)$*-RPMD;*

(3) There exists an $(n + 4t + w, w, 4, 1)$*-nearly-IRPMD with the complement type* $(c_1, c_2, \ldots, c_w)$

(4) There exists an $(n + w, w, 4, 1)$*-nearly-IRPMD with the complement type* $(f_1, f_2, \ldots, f_w)$*;*

(5) $f_i \ge t_i \ge 0$ *for* $t_i = (c_i - f_i)/4$ *and* $1 \le i \le w$.

Then there exists $(v + w, 4, 1)$*-RPMD.*

Let

$$\begin{aligned}
V &= \{v_{1j}: 1 \le j \le 5\}, \\
G'_j &= G_j \text{ for } 1 \le j \le 17 \text{ and } j \ne 15 \\
G'_{15} &= \{(15, j): 2 \le j \le n\} \cup V \\
\mathbf{G}' &= \{G'_i: 1 \le i \le 17\} \\
X' &= \cup_{1 \le i \le 17} G'_i, \\
G' &= \{n, n+4, n+4t, n+4m\} \\
Q'_1 &= I_{15} \times \{1\} \cup V, \\
Q'_i &= Q_i \text{ for } 2 \le i \le n \\
\mathbf{B}'(x) &= \{B: x \in B \in \mathbf{B}'\}
\end{aligned}$$

Lemma 3.3. *If $N(n) \ge 15$ and $m+t \le n$ then there exists a $GDD[K, G', v+4]$ of $(X', \mathbf{G}', \mathbf{B}')$ satisfying the following cnndition (B').*

For $x \in H_n$, $\mathbf{B}'(x)$ can be partitioned into n parts: $\mathbf{B}'(x)_i$, $1 \le i \le n$ such that $\cup_{B \in \mathbf{B}'(x)_i} B \setminus \{x\} = Q'_i$ for $1 \le i \le n$.

Proof: Since $N(n) \ge 15$, let $\mathbf{D}$ be blocks of a $TD[17, 1; n]$ based on $I_{17} \times I_n$. Let $U = \{(17, j): n - m + 1 \le j \le n\} \cup \{16, j): n - t + 1 \le j \le n\}$ and $\mathbf{D}(x) = \{D: x \in D \in \mathbf{D}\}$. Since $m + t \le n$ so there is no loss of generality by assuming that (A') $\{(17, n)\} \cup I_{16} \times \{i\} \in \mathbf{D}$ for $1 \le i \le n$ and $|D \cap U| \le 1$ for $D \in \mathbf{D}((15, 1))$.

Select $(m+t+1)$ points: $(17, n), (17, n-1), \ldots, (17, n-m+1), (16, n), (1p, n-1), \ldots, (16, n-t+1), (15, 1)$. The construction of a $GDD[K, G', v+4]$ satisfying the condition (B') is similar to that of a $GDD[K, G, v]$ in Lemma 3.1.

Theorem 3.4. *Suppose that*

(1) $N(n) \ge 15$, $n \equiv 3 \pmod 4$, $w \equiv 1 \pmod 4$, $m + t \le n$;

(2) There exists an $(n + 4m + w, 4, 1)$-RPMD;

(3) There exists an $(n + 4t + w, w, 4, 1)$-nearly-IRPMD with the complement type $(c_1, c_2, \ldots, c_w)$;

(4) There exists an $(n + 4 + w, w, 4, 1)$-nearly-IRPMD with the complement type $(g_1, g_2, \ldots, g_w)$;

(5) There exists an $(n + w, w, 4, 1)$-nearly-IRPMD with the complement type $(f_1, f_2, \ldots, f_w)$;

(6) $f_1 > t_1 \ge 0$, $g_1 - f_1 = 4$, $f_i \ge t_i \ge 0$, $g_i = f_i$ for $2 \le i \le w$, where $t_i = (c_i - f_i)/4$, for $1 \le i \le w$.

Then there exists a $(v+4+w,4,1)$*-RPMD where* $v = 17n+4m+4t$.

Proof: The proof is similar to that of Theorem 3.2.

It is easy to see that the existence of a $(4u,4,1)$-RPMD is equivalent to that of a $(4u,1,4,1)$-nearly-RPMD with the complement type $(4u-1)$. From Theorem 3.2 we have

Theorem 3.5. *Suppose that*

(1) $N(n) \geq 15$, $n \equiv 3 \pmod 4$, $0 \leq m, t \leq n$;

(2) There exists a $(4u,4,1)$*-RPMD for* $4u = n+4m+1$, $n+4t+1$ *and* $n+1$.

Then there exists a $(v+1,4,1)$*-RPMD, where* $v = 17n+4m+4t$.

4 $(v,4,1)$-RPMD for $v \equiv 0 \pmod 4$

Lemma 4.1. *There exists a* $(v,n,k,1)$*-nearly-IRPMD with standard complement type for* $(v,n) \in T$ *from Appendix in Section 6 of this paper, where*

$$T = \{(20,5),(24,5),(28,5),(32,5),(36,5),(36,9),(40,9),$$
$$(44,9),(52,13),(56,13),(60,13),(68,17),(132,33)\}$$

The following result is due to Baker and Wilson [8].

Lemma 4.2. *There exist a* $(40,4,1)$*-RPMD and a* $(40,5,4,1)$*-nearly-IRPMD with standard complement type.*

Proof: Let $X = (Z_7 \cup \{\infty\}) \times Z_5$ $Y = Z_7 \cup \{\infty\}$,

$$\mathbf{B}_1 = \{((\infty,0),(1,2),(2,1),(4,2)),((0,0),(3,1),(6,0),(5,0))\}$$
$$\mathbf{B}_2 = \{((\infty,0),(2,3),(4,0),(1,4)),((0,0),(6,2),(5,3),(3,4))\}$$
$$\mathbf{B}_3 = \{((\infty,0),(4,1),(1,3),(2,3)),((0,0),(5,3),(3,2),(6,4))\}$$
$$\mathbf{B}_4 = \{((\infty,0),(1,4),(2,2),(4,0)),((0,0),(3,0),(6,3),(5,1))\}$$
$$\mathbf{B}_5 = \{((\infty,0),(6,0),(5,4),(3,1)),((0,0),(4,0),(1,3),(2,0))\}$$

Let $(Z_5, \mathbf{A})$ be a $(5,4,1)$-RPMD and $\mathbf{B} = \cup_{1 \leq i \leq 5} \mathbf{B}_i$. It is readily checked that $(X, (dev\mathbf{B}) \cup Y \times \mathbf{A})$ is a $(40,4,1)$-PMD, and it is easy to see that $\mathbf{B}_i$ can be partitioned into 7 parallel classes of X for $1 \leq i \leq 5$. Taking one parallel class of them together with $Y \times \mathbf{A}$ it is not difficult to see that the union can be partitioned into 5 parallel classes. Therefore the $(40,4,1)$-PMD is resolvable.

Since $\{x\}\times \mathbf{A}$ can be partitioned into 5 parallel classes of $\{x\}\times Z_5$: $\{x\}\times \mathbf{A}_i$, $1\le i\le 5$ where $\{x\}\times \mathbf{A}_i$ is a partition of $\{x\}\times Z_5\setminus(x,i)$ for $1\le i\le 5$. So $Z_7\times \mathbf{A}$ be partitioned into 5 partial parallel classes: $Z_7\times \mathbf{A}_i$, $1\le i\le 5$. Hence $(X,\{\infty\}\times Z_5,(Z_7\times \mathbf{A})\cup dev\mathbf{B})$ is a $(40,5,4,1)$-nearly-IRPMD with standard complement type.

Lemma 4.3. *There exist a $(v,4,1)$-RPMD for $v=60$ and 72, and a $(v,n,4,1)$-nearly-IRPMD with standard complement type for $(v,n)=(60,5)$ and $(72,9)$.*

Proof: The proof is similar to that of Lemma 4.2. We only need to present a collection of base blocks for each of the two cases:

For $(v,n)=(60,5)$, $X=(Z_{11}\cup\{\infty\})\times Z_5$.

$$\mathbf{B}_1=\{((\infty,0),(0,1),(3,1),(-1,1)),((1,0),(5,2),(-2,3),(4,1))\\ ((2,0),(-4,3),(-5,4),(-3,1))\}$$
$$\mathbf{B}_2=\{((\infty,0),(0,2),(1,2),(-4,2)),((4,0),(-2,2),(3,3),(5,1))\\ ((-3,0),(-5,3),(2,4),(-1,1))\}$$
$$\mathbf{B}_3=\{((\infty,0),(0,3),(4,3),(-5,3)),((5,0),(3,2),(1,3),(-2,1))\\ ((-1,0),(2,3),(-3,4),(-4,1))\}$$
$$\mathbf{B}_4=\{((\infty,0),(0,4),(5,4),(2,4)),((-2,0),(1,2),(4,3),(3,1))\\ ((-4,0),(-3,3),(-1,4),(-5,1))\}$$
$$\mathbf{B}_5=\{((\infty,0),(0,0),(-2,0),(-3,0)),((3,0),(4,2),(5,3),(1,1))\\ ((-5,0),(-1,3),(-4,4),(2,1))\}$$

For $(v,n)=(72,9)$, $X=(Z_7\cup\{\infty\})\times Z_9$.

$$\mathbf{B}_1=\{((\infty,0),(0,0),(6,3),(2,0)),((4,0),(5,8),(1,8),(3,3))\}$$
$$\mathbf{B}_2=\{((\infty,0),(0,3),(5,6),(4,3)),((1,0),(3,8),(2,8),(6,3))\}$$
$$\mathbf{B}_3=\{((\infty,0),(0,6),(3,0),(1,6)),((2,0)(6,8),(4,8),(5,3))\}$$
$$\mathbf{B}_4=\{((\infty,0),(0,1),(1,2),(6,7)),((2,0),(4,0),(5,7),(3,5))\}$$
$$\mathbf{B}_5=\{((\infty,0),(0,4),(2,5),(5,1)),((4,0),(1,0),(3,7),(6,5))\}$$
$$\mathbf{B}_6=\{((\infty,0),(0,7),(4,8),(3,4)),((1,0),(2,0),(6,7),(5,5))\}$$
$$\mathbf{B}_7=\{((\infty,0),(0,2),(1,4),(6,5)),((4,0),(5,3),(3,2),(2,4))\}$$
$$\mathbf{B}_8=\{((\infty,0),(0,5),(2,7),(5,8)),((1,0),(3,3),(6,2),(4,4))\}$$
$$\mathbf{B}_9=\{((\infty,0),(0,8),(4,1),(3,2)),((2,0),(6,3),(5,2),(1,4))\}$$

Lemma 4.4. *There is a $(16,4,1)$-HRPMD of type 4^4 (see Lemma 3.16 in [15].*

Lemma 4.5. *There is a $(32,4,1)$-HRPMD of type 4^8.*

Proof: Let $X_i = \{[j,i]: 1 \le j \le 4\}$ for $1 \le i \le 8$ and $X = \cup_{1\le i\le 8} X_i$.
Let

$$\begin{aligned}
&E_1 = (2,2,4,4),\ E_2 = (1,1,3,3),\ E_3 = (1,3,2,4),\ E_4 = (3,1,4,2)\\
&E_5 = (1,1,4,4),\ E_6 = (2,2,3,3,),\ E_7 = (1,2,1,2),\ E_8 = (3,4,3,4)\\
&\mathbf{A}_1 = \{(5,1,3,2),(2,3,1,5),(7,4,6,8),(8,6,4,7)\\
&\quad (6,1,4,.3),(3,4,1,6),(7,2,8,5),(5,8,2,7)\\
&\quad (7,1,5,4),(4,5,1,7),(6,2,3,8),(8,3,2,6)\\
&\quad (8,1,6,5),(5,6,1,8),(4,2,7,3),(3,7,2,4)\\
&\quad (2,1,7,6),(6,7,1,2),(5,3,8,4),(4,8,3,5)\\
&\quad (3,1,8,7),(7,8,1,3),(5,2,4,6),(6,4,2,5)\\
&\quad (4,1,2,8),(8,2,1,4),(6,3,5,7),(7,5,3,6)\}\\
&\mathbf{A}_2 = \{(b,a,d,c): (a,b,c.,d) \in \mathbf{A}_1\}\ \mathbf{A}_3 = \{(b,c.,a,d): (a,b,c,d) \in \mathbf{A}_1\}\\
&\mathbf{A}_4 = \{(d,a,c,b): (a,b,c,d) \in \mathbf{A}_1\}\ \mathbf{A}_5 = \{(a,b,d,c): (a,b,c,d) \in \mathbf{A}_1\}\\
&\mathbf{A}_6 = \{(b,a,c,d): (a,b,c,d) \in \mathbf{A}_1\}\ \mathbf{A}_7 = \mathbf{A}_8 = \mathbf{A}_3
\end{aligned}$$

Let $\mathbf{B}_i = \langle E_i, \mathbf{A}_i\rangle$ for $1 \le i \le 8$. It is readily checked that $(X, \mathbf{G}, \mathbf{B})$ is a $(32,4,1)$-HRPMD where $\mathbf{B} = \cup_{1\le i\le 8}\mathbf{B}_i$ and $\mathbf{B}_i \cup \mathbf{B}_{i+1}$ can be partitioned into 7 parallel classes of X for $i = 1,3,5,7$.

Lemma 4.6. *There exists a $(24,4,1)$-HRPMD of type 3^8.*

Proof: The proof is similar to that of Lemma 4.2. We only need to present a collection of base blocks. Let $X = (Z_7 \cup \{\infty\}) \times Z_3$

$$\begin{aligned}
\mathbf{B}_1 &= \{((\infty,0),(0,0),(1,2),(4,0)),((2,0),(3,1),(6,1),(5,0))\}\\
\mathbf{B}_2 &= \{((\infty,0),(0,1),(2,0),(1,1)),((4,0),(6,1),(5,1),(3,0))\}\\
\mathbf{B}_3 &= \{((\infty,0),(0,2),(4,1),(2,2)),((1,0),(5,1),(3,1),(6,0))\}
\end{aligned}$$

Lemma 4.7. *There exists a $(4s,4,1)$-HRPMD of type m^{4t} for $(s,m,t) = (4,4,1),(8,4,2),(6,3,2)$ satisfying the condition (1) in Theorem 2.2*

Proof: For $(s,m,t.) = (4,4,1)$, from Lemma 4.4 there is no loss of generality by assuming that there exists a $(16,4,1)$-HRPMD of type 4^4 satisfying the condition (1). For $(s,m,t.) = (8,4,2)$. We adopt the notation of the proof in Lemma 4.5. Let

$$\begin{aligned}
\mathbf{A}_{11} &= \{(5,1,3,2),(2,3,1,5),(7,4,6,8),(8,6,4,7)\} \subset \mathbf{A}_1.\\
\mathbf{A}_{21} &= \{(b,a,d,c): (a,b,c,d) \in \mathbf{A}_{11}\}
\end{aligned}$$

Since $\cup_{i=1,2}\langle E_i, \mathbf{A}_{i1}\rangle$ is a parallel class of X and can be partitioned into a parallel class of $X_5 \cup (\cup_{i=1,2,3} X_i)$ and a parallel class of $X_4 \cup (\cup_{i=6,7,8} X_i)$, so we can say there is a $(32,4,1)$-HRPMD of type 4^8 satisfying the condition (1) of Theorem 2.2 from Lemma 4.5. For $(s,m,t.) = (6,3,2)$, it is easy to see that there exists a $(24,4,1)$-HRPMD of type 3^8 satisfying the condition (1) from Lemma 4.6.

Lemma 4.8. *There exists a $(4us,4,1)$-HRPMD of type mu^{4t} for $(s,m,t.) = (4,4,1),(8,4,2),(6,3,2)$ satisfying the condition (1) in Theorem 2.2 where $N(u) \geq 3$.*

Proof: The proof is similar to that of Lemma 2.13 in [15] by using Lemma 4.7.

Theorem 4.9. *There exists a $(4t,4,1)$-RPMD for $t = 42$, 66 and 186.*

Proof: applying Lemma 4.8 with $u = 7, 11, 31$ we can obtain a construction of a $(4t,4,1)$-RPMD for $t = 42$, 66, 186 which is similar to that of Lemma 4.2.

Theorem 4.10. *There exists a $(4t,4,1)$-RPMD for $t \in \{20,28,32,36,44,48,52,56,60,64,68,132\}$.*

Proof: From Theorem 1.2 and Lemma 4.1 we have that there is a $(4t.,4,1)$-nearly-RPMD for $t \in \{5,6,7,8,9,10,11,13,14,15,17,33\}$. Apply Theorem 2.2 with $h = 4$, $v = 4u$ to obtain a $(4 \cdot 4u,4,1)$-RPMD for $u = 5,7,8,9,11,13,14,15,17$ and 33 where the required $(16u,4,1)$-HRPMD of type $4u^4$ comes from Lemma 4.8. Apply Theorem 2.2 with $h = 8$, $v = 24,32$ to obtain a $(4t,4,1)$-RPMD for $t = 48,64$ where the required HRMDs come from Lemma 4.8.

Theorem 4.11. *There exists a $(4t,4,1)$-RPMD for t = 25,35,40,45,50,55, 63,65,70,72,75,81,85,147,168,169,170,175,182, 185,195,203,231,261.*

Proof: From Theorem 1.2 and Lemma 4.1 and 4.2 we have that there is $(4t,n,4,1)$-nearly-RPMD for $(t,n) = (5,5),(7,5),(8,5),(9,5),(10,5),(11,9),(13,13),(14,13),(15,13),(17,17)$. Apply Theorem 2.4 to obtain a $(4mp,4,1)$-RPMD for (m,p) = (5,5),(7,5),(8,5),(5,9),(11,5) (7,9),(5,13),(14,5),(8,9),(15,5),(9,9),(5,17),(7,21),(8,21), (13,13),(7,25),(14,13),(5,37),(15,13),(7,29),(7,33),(9,29) and a $(8mp,4,1)$-RPMD for $(m,p) = (5,5)$ and (5,17).

Lemma 4.12. *There is a $(v,n,4,1)$-nearly-IRPMD with the complement type $(c_1,c_2,\ldots,c_n)$ where (v,n) and $(c_1,c_2,\ldots,c_n)$ are shown in Table A*

Proof: For $(v,n) = (4\cdot 20,5),(4\cdot 28,5),(4\cdot 32,5)$, $(4\cdot 36,9),(4\cdot 44,9),(4{\cdot}52,13),(4{\cdot}56,13),(4{\cdot}60,13)$, we apply Theorem 2.3 to obtain a $(v,n,4,1)$-nearly-IRPMD where the required nearly-IRPMDs come from Lemma 4.1. For $(v,n) = (4\cdot 30,5),(4\cdot 41,9),(4\cdot 46,9),(4\cdot 57,13)$, since

$$4\cdot 30 = 5\cdot 23+5, \quad 4\cdot 41 = 5\cdot 31+9,$$
$$4\cdot 46 = 5\cdot 35+9, \quad 4\cdot 57 = 5\cdot 43+13$$

we apply Theorem 2.6 to obtain a $(v, n, 4, 1)$-nearly-IRPMD where the required nearly-IRPMDs come from Lemma 4.1.

(v, n)	$(c_1, c_2, \dots, c_n)$
(4· 20,5)	(15,15,15,15,15)
(4· 28,5)	(15,15,15,31,31)
(4· 30,5)	(15,15,15,35,35)
(4· 32,5)	(15,15,31,31,31)
(4· 36,9)	(15,15,... ,15)
(4· 41,9)	(15,15,... ,15,35)
(4· 44,9)	(15,15,... ,15,31,31)
(4· 46,9)	(15,15,... ,15,35,35)
(4· 52,13)	(15,15,... ,15)
(4· 56,13)	(15,15,... ,15,31)
(4· 57,13)	(15,15,... ,15,35)
(4· 60,13)	(15,15,... ,15,31,31)

Table A

Theorem 4.13. *There exists* $(4t., 4, 1)$*-RPMD for* $t = 41, 49, 62, 74, 80, 171, 172, 174, 176, 178, 179, 180, 184, 196, 211, 241, 244, 270$.

Proof: Apply Theorem 2.2 with $h = 4$, $v = 80, 172, 176, 184$ to obtain a $(4t, 4, 1)$-RPMD for $t = 80, 172, 176, 184$. Here the required $(v, 4, 1)$-nearly-RPMD for $v = 80, 176, 184$ comes from Lemma 4.12 and Theorem 1.2 and the required $(172, 4, 1)$-nearly-RPMD comes from Theorem 2.7 since $172 = 9 \cdot 19 + 1$ and there is a $(20, 4, 1)$-nearly-RPMD. Since

$4 \cdot 41 = 5 \cdot 31 + 9$,	$4 \cdot 49 = 5 \cdot 39 + 1$,	$4 \cdot 62 = 5 \cdot 47 + 13$
$4 \cdot 74 = 6 \cdot 59 + 1$,	$4 \cdot 171 = 5 \cdot 135 + 9$,	$4 \cdot 174 = 5 \cdot 139 + 1$
$4 \cdot 178 = 9 \cdot 79 + 1$,	$4 \cdot 179 = 5 \cdot 143 + 1$,	$4 \cdot 180 = 13 \cdot 55 + 5$
$4 \cdot 196 = 25 \cdot 31 + 9$,	$4 \cdot 211 = 5 \cdot 167 + 9$,	$4 \cdot 244 = 25 \cdot 39 + 1$

we can apply Theorem 2.5 to obtain a $(4t, 4, 1)$-RPMD for $t = 41, 71, 180, 196, 211$ and apply Theorem 2.6 with $n = 1$ to obtain a $(4t, 4, 1)$-RPMD for $t = 49, 74, 174, 178, 179, 244$ since the existence of a $(4u, 4, 1)$-RPMD is equivalent to that of a $(4u, 1, 4, 1)$-nearly-RPMD, and apply Theorem 2.4 with $p = 9$, $u = 4$ and $m = 30$ to obtain a $(4 \cdot 270, 4, 1)$-RPMD. Here the required nearly-IRPMDs come from Lemma 4.1, 4.2 and 4.12 and RPMDs come from Theorem 1.2 and Lemma 4.2 and 4.3 and Theorem 4.10 and 4.11

Apply Theorem 2.5 with $p = 5$, $v = 200$, $n = 9$ to obtain a $(4 \cdot 241, 4, 1)$-RPMD where the required $(4 \cdot 50, 4, 1)$-RPMD comes from Theorem 4.11 and the required $(4 \cdot 50, 9, 4, 1)$-nearly-IRPMD is from Theorem 2.4.

Summarizing the ahove results, we have

Theorem 4.14. *There exists a* $(4t, 4, 1)$*-RPMD for* $t =$ *10, 15, 18, 20, 25, 28, 32, 35, 36, 40, 41, 42, 44, 45, 48, 49, 50, 52, 55, 56, 60, 62, 63, 64, 65, 66, 68, 70, 72, 74, 75, 80, 81, 85, 132, 147, 168, 169, 170, 171, 172, 174, 175, 176, 178, 179, 180, 182, 184, 185, 186, 195, 196, 203, 211, 231, 241, 244, 261, 270.*

Theorem 4.15. *There exists a* $(4u, 4, 1)$*-RPMD for* $86 \le u \le 266$.

Proof: Apply Theorem 3.2 with $n = 19$, $w = 5$, $t = 0, 1, 2, 3, 4$, $m = 4, 9, 12, 14, 19$ to obtain a $(17n+4m+4t+w, 4, 1)$-RPMD, that is, $4(82+m+t) \in$ RPMD for $t = 0, 1, 2, 3, 4$, $m = 4, 9, 12, 14, 19$. So we have $\{4u\colon 86 \le u \le 105\} \subset$ RPMD. Here the required conditions come from Lemma 4.1 and 4.2 and Theorem 4.14.

Apply Theorem 3.2 with $n = 23$, $w = 5$, $t = 0, 1, 2, 3$, $m = 3, 8, 11, 13, 18, 21$, to obtain a $(4(99+m+t), 4, 1)$-RPMD, that is, $\{4u\colon 107 \le u \le 115, 117 \le u \le 123\} \subset$ RPMD.

Apply Theorem 3.4 with $n = 23$, $w = 5$, $t = 3$, $m = 3, 13$ to obtain $(4u, 4, 1)$-RPMD for $u = 106$ and 116.

Apply Theorem 3.2 with $n = 27$, $w = 5$, $t = 0, 1, 2$, $m = 7, 10, 12, 17, 20, 24, 27$ to obtain that $\{4u\colon 124 \le u \le 130, 133 \le u \le 138, 140 \le u \le 145\} \subset$ RPMD.

Apply Theorem 3.4 with $n = 27$, $w = 5$, $t = 2$, $m = 12, 20$ to obtain $4u \in$ RPMD for $u = 131$ and 139.

Apply Theorem 3.2 with $n = 27$, $w = p$, $t = 2$, $m = 27$ to obtain $4 \cdot 146 \in$ RPMD.

Apply Theorem 3.2 with $n = 31$, $w = 9$, $t = 0, 1$, $m = 15, 18, 22, 25, 26, 30, 31$ to obtain that $\{4u\colon 149, 150, 152, 153, 156, 157, 159, 160, 161, 164, 165, 166\} \subset$ RPMD.

Apply Theorem 3.2 with $n = 27$, $w = 5$, $t = 12, 20, 24$, $m = 20, 27$ to obtain that $4u \in$ RPMD for $u = 148, 155, 163, 167$ where the reqired $(32+4t, 5, 4, 1)$-nearly-IRPMD for $t = 12, 20, 24$ comes from Lemma 4.12.

Taking $n = 31$, $w = 9$, $t = 1$, $m = 15, 18, 22, 26$ we have that $4u \in$ RPMD for $u = 151, 154, 158, 162$ by Theorem 3.4.

Taking $n = 43$, $w = 13$, $t = 0, 1, 38, 42, 43$, $m = 1, 4, 6, 11, 14, 18, 21, 22, 26, 27, 28, 30, 31, 34, 35, 36, 38, 41, 42$ we have that $\{4u\colon 187 \le u \le 266\} \setminus \{4u\colon u = 195, 196, 203, 211, 231, 236, 237, 241, 244, 248, 253, 261\} \subset$ RPMD by using Theorem 4.9 and Theorem 4.11.

Apply Theorem 3.2 with $n = 31$, $w = 9$, $t = 8$, $m = 31$ to obtain a $(4 \cdot 173, 4, 1)$-RPMD where the required $(72, 9, 4, 1)$-nearly-IRPMD comes from Lemma 4.3.

Apply Theorem 3.2 with $n = 31$, $w = 5$, $t = 21$, $m = 23$ and 27 to obtain $4u \in$ RPMD for $u = 177$ and 181.

Apply Theorem 3.2 with $n = 31$, $w = 9$, $t = 31$, $m = 18$ to obtain $4 \cdot 183 \in$ RPMD where $(4 \cdot 41, 9, 4, 1)$-nearly-IRPMD is from Lemma 4.12

Taking $n = 47$, $w = 13$, $t = 0, 37, 45$, $m = 0, 13, 33, 34$ we have $4u \in$ RPMD for $u = 236, 237, 248, 253$ by Theorem 3.2.

Combining the above results and Theorem 4.14, we have $\{86 \leq u \leq 266\} \subset$ RPMD.

Theorem 4.16. *There exists a* $(4u, 4, 1)$*-RPMD for* $u \geq 266$.

Proof: Taking $n = 59$, $m, t \in A = \{0, 3, 5, 10, 13, 17, 20, 21, 25, 26, 27, 29, 30, 33, 34, 35, 37, 40, 41, 45, 47, 48, 49, 50, 51, 53, 55, 57, 59\}$ we have $\{4u\colon 266 \leq u \leq 361\} \setminus \{270\} \subset$ RPMD by applying Theorem 3.5.

Taking $n = 79$, $m, t \in A = \{0, 5, 8, 12, 15, 16, 20, 21, 22, 24, 25, 28, 29, 30, 32, 35, 36, 40, 42, 43, 44, 45, 46, 48, 50, 52, 54, 55, 60, 61, 65, 66, \ldots, 78, 79\}$ we have $\{4u\colon 361 \leq u \leq 494\} \subset$ RPMD by using Theorem 3.5

Taking $n = 107$, $w = 5$, $t = 0$, $m \in A = \{4, 7, 8, 12, 13, 14, 16, 17, 20, 21, 22, 24, 27, 28, 32, 34, 35, 36, 37, 38, 40, 42, 44, 46, 47, 52, 53, 57, 58, \ldots, 106, 107\}$ we have $\{4u\colon 494 \leq u \leq 563\} \subset$ RPMD by Theorem 3.2 where the required $(4 \cdot 28, 5, 4, 1)$-nearly-IRPMD comes from Lemma 4.12.

Taking $n = 127$, $t, m \in \{0, 3, 4, 8, 9, 10, 12, 13, 16, 17, 18, 20, 23, 24, 28, 30, 31, 32, 33, 34, 36, 38, 40, 42, 43, 48, 49, 53, 54, \ldots, 126, 127\}$ we have $\{4u\colon 563 \leq u \leq 794\} \subset$ RPMD bv applying Theorem 3.5.

Taking $n = 163, 199, 239, 347, 383, 503, 719, 1019, 1427, 1831$ we have $\{4u\colon 794 \leq u \leq 11444\} \subset$ RPMD.

Similarly taking $n = 3^s$, $3^{s-3} \cdot 31$, $3^{s-4} \cdot 113$, $3^{s-3} \cdot 47$, $3^{s-3} \cdot 59$, $3^{s-3} \cdot 79$, $3^{s-3} \cdot 87$, $3^{s-3} \cdot 127$, $3^{s-1} \cdot 19$, $3^{s-1} \cdot 23$, 3^{s+2} for $s = 7, 9, 11, \ldots$ we have $\{4u\colon u \geq 11444\} \subset$ RPMD.

Since $270 \in$ RPMD from Theorem 4.14, so we have $\{4u\colon u \geq 266\} \subset$ RPMD.

From Theorem 1.3, summarizing the ahove results we have

4·3	4·4	4·5	4·6	4·7	4·8	4·9	4·11	4· 12	4·13
4·14	4·16	4·17	4·19	4·21	4·22	4·23	4·24	4·26	4·27
4·29	4·30	4·31	4·33	4·34	4·37	4·38	4·39	4·43	4·46
4·47	4·51	4·53	4·54	4·57	4·58	4·59	4·61	4·67	4·69
4·71	4·73	4·76	4·77	4·78	4·79	4·82	4·83	4·84.	

Table B

Theorem 4.17. *A $(v,4,1)$-RPMD exists for all integers $v \geq 4$ where $v \equiv 0 \pmod 4$, except for $v = 4, 8$, and with 49 possible exceptions shown in Table B.*

5 Asymptotic results about $(v,k,1)$-RPMDs and (v,k,λ)-PMDs

Remark 5.1: Let $(X, \mathbf{A})$ be a $(v,k,1)$-RPMD where $X = \{x_1, x_2, \ldots, x_v\}$ By Definition 1.1 and 1.2, it is easy to see that for $v \equiv 1 \pmod k$ $\mathbf{A}$ can be partitioned into v parallel classes $\mathbf{A}_i$ such that $\mathbf{A}_i$ is a partition of $X \setminus \{x_i\}$ where $1 \leq i \leq v$, and for $v \equiv 0 \pmod k$ $\mathbf{A}$ can be partitioned into $(v-1)$ parallel classes $\mathbf{A}_i$ such that $\mathbf{A}_i$ is a partition of X where $1 \leq i \leq (v-1)$.

Definition 5.2: Let $(X, \mathbf{A})$ be a $(v,k,1)$-RPMD having parallel classes $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r$. Let $(X, \mathbf{B})$ be a $(u,k,1)$-RPMD having parallel classes $\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_t$. If $X \supset Y$ and $\mathbf{A}_i \supset \mathbf{B}_i$, $1 \leq i \leq t$, we say the first design contains the second as a subdesign.

Theorem 5.3. *A $(v,k,1)$-RPMD exists for all sufficiently large v with $k \geq 3$ and $v \equiv 1 \pmod k$ (see [2]).*

Theorem 5.4. *Suppose there exists a $(v,K;\lambda)$-PBD and for each $m \in K$ there exists an $(m,k,1)$-PMD. Then there exists a (v,k,λ)-PMD (see Theorem 4.1 in [10]).*

Theorem 5.5. *A $(v,k,1)$-PMD exists with $v(v-1) \equiv 0 \pmod k$ for the case when k is an odd prime and v is sufficiently large (see Theorem 3.3 in [11]).*

Theorem 5.6. *Suppose there exists a $(v,K;1)$-PBD and for each $m \in K$ there exists an $(m,k,1)$-RPMD and $m \equiv 1 \pmod k$. Then there exists a $(v,k,1)$-RPMD and there exists an $(m,k,1)$-RPMD as a subdesign, for $m \in K$ (see Corollary 2.19 in [15]).*

Theorem 5.7. *Suppose there exist a $(u,k,1)$-RPMD and a $(v,k,1)$-RPMD where $u, v \equiv 0$ or $1 \pmod k$. Then there exists a $(uv,k,1)$-RPMD which contains a $(u,k,1)$-RPMD and a $(v,k,1)$-RPMD, respectively, as a subdesign (see Theorem 2.1 and Remark 2.2 in [15]).*

Theorem 5.8. *Suppose there exist a $(u,k,1)$-RPMD and a $(v+1,k,1)$-RPMD, where $u \equiv 1 \pmod k$ and $v+1 \equiv 0$ or $1 \pmod k$. If there is an $RTD[k,1;v]$. Then there exists a $(uv+1,k,1)$-RPMD which contains a $(u,k,1)$-RPMD and a $(v+1,k,1)$-RPMD, respectively, as a subdesign (see Theorem 2.14 and Remark 2.15 in [15]).*

Theorem 5.9. *If there exists a $GDD[K,1,M;v]$ satisfying for each $h \in K$ there exists an $(h,k,1)$-RPMD, $h \equiv 1 \pmod k$ and for each $n \in M$ there exists an $(n+w,k,1)$-RPMD having a $(w,k,1)$-RPMD as a subdesign, $n \equiv 0$*

(mod k). *Then there exists a* $(v+w,k,1)$*-RPMD where* $w \equiv 0 \pmod{k}$ *(see Theorem 2.12 and Theorem 2.18 in [15]).*

Given a set K of positive integers, we define $\alpha(K)$ as the greatest common divisor of $\{k-1 \colon k \in K\}$ and $\beta(K)$ as the greatest common divisor of $\{k(k-1) \colon k \in K\}$. The following theorem is essantially Theorem 1 in [13].

Theorem 5.10. *Given a set* K *of positive integers,* $B_\lambda(K) = \{v\colon$ *there exists a* $(v, K; \lambda)$*-PBD*$\}$ *contains all sufficiently large integers* v *satisfying the congruences:*

$$\lambda(v-1) \equiv 0 \pmod{\alpha(K)} \text{ and}$$
$$\lambda v(v-1) \equiv 0 \pmod{\beta(K)}$$

Let $q = en+1$ be an odd prime power. The cyclic multiplicative group of the field $GF(q)$ has a unique subgroup H_0^e of index e and order n. The multiplicative cosets $H_0^e, H_1^e, \ldots, H_{e-1}^e$ of H_0^e are the cyclotomic classes of index e. They evidently partition $GF(q) \setminus \{0\}$. The class of cosets $\{H_0^e, H_1^e, \ldots, H_{e-1}^e\}$ will be denoted $\mathbf{H}^e$.

Let P_r be the set of ordered pairs $\{(i,j)\colon 1 \le i < j \le r\}$. We define a choice to be any map $C\colon P_r \to \mathbf{H}^e$, assigning to each pair $(i,j) \in P_r$ a coset $C(i,j)$ modulo the e-th powers in $GF(q)$. An r-vector $(a_1, a_2, \ldots, a_r)$ of elements of $GF(q)$ is consistent with the choice C if and only if $a_j - a_i \in C(i,j)$ for all $1 \le i < j \le r$.

Wilson [12] proved in his cyclotomic paper the following.

Theorem 5.11. *Let* $q \equiv 1 \pmod{e}$ *is a prime power and* $q > e^{r(r-1)}$, *then for any choice* $C\colon P_r \to \mathbf{H}^e$, *there exists an* r*-vector* $(a_1, a_2, \ldots, a_r)$ *of elements of* $GF(q)$ *consistent with* C.

Consider the arithmetic progression $\{mk(k-2)+k-1 \colon m = 1,2,3,\ldots\}$. Since $(k-1, k(k-2)) = (k-1, (k-1)(k-2)+k-2) = (k-1, k-2) = 1$, so, by Dirichlet's theorem and Theorem 5.3 there is an m_0 satisfying

(a) $q = m_0 k(k-2) + k - 1 = (k-2)(m_0 k + 1) + 1 > e^{r(r-1)}$, where $r = k-1$, $e = k-2$;

(b) $q = (k-2)(m_0 k + 1) + 1$ is odd prime power;

(c) there exists an $(m_0 k + 1, k, 1)$-RPMD;

(d) there is an $RTD[k, 1; v_0 - 1]$ where $v_0 = (m_0 k + 1)q + (m_0 k + 1)$.

In this section we always let $n = m_0 k + 1$, $e = k-2$, $r = k-1$, $q = en+1$, $v_0 = nq+n$, $k \ge 3$, and Θ ba generator of $GF(q) \setminus \{0\}$ and $H_i^e = \{\Theta^t \colon t \equiv i \pmod{e}\}$ for $0 \le i \le e-1$.

Let $A = (\infty, a_2, a_3, \ldots, a_k)$ be a cyclic k-tuple, where $a_i \in GF(q)$ for $2 \leq i \leq k$. We define

$$D_t(A) = \{a_{i+t} - a_i : 2 \leq i \leq k, i \neq 1 - t\}$$

where $i + t$ and $1 - t$ are taken modulo k.

From Theorem 5.11, we have that

Lemma 5.12. *There exists an ordered set of r distinct elements $(a_1, a_2, \ldots, a_r)$ satisfying the following conditions:*

(1) $(a_{i+j} - a_i) \in H^e_{i+2j-2}$ for $1 \leq j \leq r/2$, $1 \leq i \leq r - j$ and $(a_{\frac{r}{2}+i+j} - a_i) \in -H^e_{r+1-i-2j}$ for $1 \leq j \leq (r/2) - 1$, $1 \leq i \leq (r/2) - j$, when r is even;

(2) $(a_{i+j} - a_i) \in H^e_{i+j-1}$ for $1 \leq j \leq (r-1)/2$, $1 \leq i \leq r - j$ and $(a_{\frac{r+1}{2}+i+j} - a_i) \in -H^e_{\frac{r+1}{2}-1-j}$ for $0 \leq j \leq (r-1)/2 - 1$, $1 \leq i \leq (r-1)/2 - j$, when r is odd.

Since q is odd, then n is even when r is even and n is odd when r is odd. Hence we have $H^e_i = -H^e_i$ when r is even and $H^e_i = H^e_{i+\frac{e}{2}}$ when r is odd. Let $b_{j-1} = (a_{j+1} - a_i)/(a_2 - a_1)$ for $1 \leq j \leq r - 1$. From Lemma 5.12 we have

Lemma 5.13. *There exists an ordered set of $r - 1$ distinct elements $(b_0, b_1, \ldots, b_{r-2})$ satisfying the following conditions:*

(1) the set $\{b_0, b_1, \ldots, b_{r-2}\}$ forms a system of representatives for the cyclotomic classes of index e in $GF(q)$.

(2) for $A = (\infty, 0, b_0, b_1, \ldots, b_{r-2})$, the set $D_t(A)$ forms a system of representatives for the cyclotomic classes of index e in $GF(q)$, where $1 \leq t \leq r/2$ when r is even , or $1 \leq t \leq (r+1)/2$ when r is odd.

Lemma 5.14. *There exists an $(n + q, k, 1)$-PMD.*

Proof: Let $(Y, \mathbf{B})$ be an $(n, k, 1)$-RPMD where $Y = \{\infty_i : 0 \leq i \leq n - 1\}$ and $X = GF(q)$. Let $A_i = \Theta^{ie} A = (\infty_i, 0, b_0\Theta^{ie}, \ldots, b_{r-2}\Theta^{ie})$ for $0 \leq i \leq n - 1$, be base blocks, where A comes from Lemma 5.13.

From the condition (2) in Lemma 5.13, it is easy to see that

$$\cup_{0 \leq i \leq n-1} D_t(A_i) = GF(q) \setminus \{0\}$$

for $1 \leq t \leq r/2$ when r is even, or for $1 \leq t \leq (r+1)/2$ when r is odd. Hence it is easy to see that $(X \cup Y, (dev\mathbf{A}) \cup \mathbf{B})$ is an $(n + q, k, 1)$-PMD where $dev\mathbf{A} = \{A_i + g : g \in GF(q), 0 \leq i \leq n - 1\}$.

Theorem 5.15. *There exists an $(nq+n,k,1)$-RPMD.*

Proof: We adopt the notation of the proof in Lemma 5.13 and 5.14. Let

$$\{j\} \times A_i = (\infty_{i+j}, (j,0), (j, b_0\Theta^{ie}), \ldots, (j, b_{r-2}\Theta^{ie})),$$
$$N_\ell = \{(0,\ell), (1,\ell\Theta^e), \ldots, (i,\ell\Theta^{ie}), \ldots, (n-1, \ell\Theta^{(n-1)e})\},$$
$$\mathbf{A} = \{A_i : 0 \le i \le n-1\}, \mathbf{N} = \{N_\ell : \ell \in GF(q)\},$$

and $(N, \mathbf{D}(N))$ be an $(n,k,1)$-RPMD having n parallel classes : $\mathbf{D}(N)_i$, $0 \le i \le (n-1)$, where $\mathbf{D}(N)_i$ is a partition of $N \setminus \{(i,a_i)\}$ for $N = \{(0,a_0),(1,a_1),\ldots,(n-1,a_{n-1})\}$.

Since N_ℓ, $\ell \in GF(q)$, are base blocks of $TD[n,1;q]$, it is easy to see that $((I_n \times GF(q)) \cup Y, dev(\mathbf{D}(\mathbf{N}) \cup I_n \times \mathbf{A}) \cup \mathbf{B})$ is an $(nq+n,k,1)$-PMD where $\mathbf{D}(\mathbf{N}) = \cup_{\ell \in GF(q)} \mathbf{D}(N_\ell)$, and $dev(\mathbf{D})(\mathbf{N}) \cup I_n \times \mathbf{A}) = \{B + (0,g) : g \in GF(q), B \in \mathbf{D}(\mathbf{N}) \cup I_n \times \mathbf{A}\}$ and $(GF(q) \cup Y, (dev\mathbf{A}) \cup \mathbf{B})$ is an $n+q,k,1)$-PMD from Lemma 5.14.

We are to prove that the blocks can be partitioned into $(nq+n-1)$ parallel classes. We denote N_ℓ for $\ell = b_s\Theta^{te}$ by $N_{s,t}$, it is easy to see that

$$\mathbf{E}_i = I_n \times A_i \cup (\cup_{\substack{0 \le s \le r-2 \\ 0 \le t \le n-1}} \mathbf{D}(N_{s,t})_{i-t}) \text{ where } 0 \le i \le n-1$$

is a parallel class, so

$$I_n \times \mathbf{A} \cup \mathbf{D}(\mathbf{N} \setminus N_0) = \cup_{0 \le i \le n-1} \mathbf{E}_i$$

is the union of n parallel classes. Hence

$$dev((I_n \times \mathbf{A} \cup \mathbf{D}(\mathbf{N} \setminus N_0) \setminus \mathbf{E}_0$$

is the union of $(nq-1)$ parallel classes. Finally we are to prove that

$$\mathbf{E}_0 \cup \mathbf{B} \cup dev\mathbf{D}(N_0)$$

is the union of n parallel classes. Let

$$\mathbf{F}_j = \mathbf{B}_j \cup \{i\} \times A \cup (\cup_{0 \le s \le r-2} \mathbf{D}(N_{s,n-j})_j),$$

where $0 \le j \le n-1$. It. is easy to see that

$$\mathbf{E}_0 \cup \mathbf{B} = \cup_{0 \le j \le n-1} \mathbf{F}_j.$$

Since each $\mathbf{F}_j$ contains one point of $I_n \times \{\ell\}$ for all $\ell \in GF(q)$, so $dev\mathbf{D}(N_0)$ can be partitioned into n parts $\mathbf{G}_j$, $0 \le j \le n-1$, such that each $\mathbf{F}_j \cup \mathbf{G}_j$ is a parallel classes. That is $\mathbf{E}_0 \cup \mathbf{B} \cup dev\mathbf{D}(N_0)$ is the union of n parallel classes. This completes the proof.

From Theorem 5.3 there exists a $(u, k, 1)$-RPMD for all $u > u_0$ and $u \equiv 1 \pmod k$ where u_0 is sufficiently large integer.

Let $K = \{v_0(u-1)+1, (u-1)(v_0-1)+1 : u > u_0, u \equiv 1 \pmod k\}$, $K_0 = \{v_0(u-1)+1 : u = v_0k^2+1, (v_0k+1)k+1\} \cup \{(v_0-1)(u-1)+1 : u = (v_0-1)k^2+1, (v_0-1)k^2+k+1\} \cup K \subset \{u : u > u_0, u \equiv 1 \pmod k\}$, it is not difficult to see that $\alpha(K_0) = \alpha(K) = k$, $\beta(K_0) = \beta(K) = k$ when k is even and $\beta(K_0) = \beta(K) = 2k$ when k is odd. Hence, from Theorem 5.10 we have

Theorem 5.16. *There is a u_1 such that $\{u : u > u_1, u \equiv 1 \pmod k\} \subset B_1(K)$.*

Theorem 5.17. *There exists a $(h + v_0 - 1, k, 1)$-RPMD for $h \in B_1(K)$.*

Proof: It is easy to see that there exists a $GDD[K, 1, M; h-1]$ where $M = \{g - 1 : g \in K\}$ for $h \in B_1(K)$. Since there exists a $(g, k, 1)$-RPMD for $g \in K$, and there exists a $(v_0u, k, 1)$-RPMD which contains a $(v_0, k, 1)$-RPMD as a subdesign from Theorem 5.7, and there exists a $(u(v_0-1)+1, k, 1)$-RPMD which contains a $(v_0, k, 1)$-RPMD as a subdesign from Theorem 5.8. Apply Theorem 5.9 with $v = h - 1$ and $w = v_0$ to obtain a $(h+v_0-1, k, 1)$-RPMD.

From Theorem 5.16 and Theorem 5.17, we have

Theorem 5.18. *There exists a $(v, k, 1)$-RPMD for all sufficiently large v with $v \equiv 0 \pmod k$.*

Theorem 5.19. *There exists a (v, k, λ)-PMD for all sufficiently large integers v with $\lambda v(v-1) \equiv 0 \pmod k$.*

Proof: Let $K = \{v :$ there exists a $(v, k, 1)$-PMD$\}$. From Theorem 5.3 and Theorem 5.14, there is u_0 such that $K_0 = \{v : v \equiv 1 \pmod k, v \geq u_0\} \cup \{q + n\} \subset K$. Since $q + n \equiv 0 \pmod k$, it is easy to see that $\alpha(K) = \alpha(K_0) = 1$, $\beta(K) = \beta(K_0) = k$ when k is even and $\beta(K) = \beta(K_0) = 2k$ when k is odd. Therefore, by Theorem 5.4 and Theorem 5.10, we complete the proof.

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6 Appendix $(v, 4, 1)$-nearly-IRPMD for $v \equiv 0 \pmod 4$

Let $T = \{(20, 5), (24, 5), (28, 5), (32, 5), (36, 5), (36, 9), (40, 9), (44, 9), (52, 13), (56, 13), (60, 13), (68, 17), (132, 33)\}$.

Lemma 4.1. *There exists a $(v, n, 4, 1)$-nearly-IRPMD with standard complement type for $(v, n) \in T$.*

Proof: In each of the following cases for $(v,n) \in T$, we let $\mathbf{G} = Z_{v-n}$, $X = Z_{v-n}$ and $Y = \{\infty_i : 1 \le i \le n\}$. We then present a collection of base blocks $\mathbf{B}$ and n partial parallel classes of X namely, $\mathbf{D}_i$, $1 \le i \le n$, as defined. Since $\mathbf{B}$ is a parallel class of $X \cup Y$, it is easily checked that $(X \cup Y, Y, (dev\mathbf{B}) \cup \mathbf{D})$ is a $(v,n,4,1)$-nearly-IRPMD with standard complement type where $\mathbf{D} = \cup_{1 \le i \le n} \mathbf{D}_i$.

The case $(v,n) = (20,5)$ Let

$$\mathbf{B} = \{(\infty_1,-5,-1,6),(\infty_2,1,7,2),(\infty_3,3,-7,5),(\infty_4,4,-3,-4)\\ (\infty_5,0,-2,-6)\}$$
$$\mathbf{D}_j = \{(0,1,3,6)+4i+12(j-1)\colon i=0,1,2\} \text{ for } 1 \le j \le 5$$

The case $(v,n) = (24,5)$ Let

$$\mathbf{B} = \{(5,-9,-2,8),(\infty,-7,-1,-8),(\infty_2,0,4,-4),(\infty_3,1,9,7)\\ (\infty_4,-5,-6,3),(\infty_5,6,2,-3)\}$$
$$\mathbf{D}_j = \{(0,1,3,6)+4i+16(j-1)\colon i=0,1,2,3\} \text{ for } 1 \le j \le 4$$
$$\mathbf{D}_5 = \{(0,1,3,6)+4i+7\colon i=0,1,2\}$$

The case $(v,n) = (28,5)$ Let

$$\mathbf{B} = \{(-4,4,9,-10),(-11,7,-7,0),(\infty,-2,11,-1),\\ (\infty_2,-5,5,-3),(\infty_3,-9,10,8),(\infty_4,2,1,-6),(\infty_5,-8,6,3)\}$$
$$\mathbf{D}_j = \{(0,1,3,6)+4i+20(j-1)\colon i=0,1,2,3,4\} \text{ for } 1 \le j \le 3$$
$$\mathbf{D}_4 = \{(0,1,3,6)+4i-9\colon i=0,1,2,3\}$$
$$\mathbf{D}_5 = \{(0,1,3,6)+4i+7\colon i=0,1,2,3\}$$

The case $(v,n) = (32,5)$ Let

$$\mathbf{B} = \{(11,-5,-12,-13),(-10,-6,2,0),(10,-11,-4,5),\\ (\infty_1,8,-3,7),(\infty_2,13,-2,-7),(\infty_3,3,-1,12),\\ (\infty_4,-9,9,1),(\infty_5,4,-8,6)\}$$
$$\mathbf{D}_j = \{(0,1,3,6)+4i+24(j-1)\colon 0 \le i \le 5\} \text{ for } 1 \le j \le 2$$
$$\mathbf{D}_k = \{(0,1,3,6)+4i+20(k-3)-6\colon 0 \le i \le 4\} \text{ for } 3 \le k \le 5$$

The case $(v,n) = (36,5)$ Let

$$\mathbf{B} = \{(-9,-5,3,15),(-1,-4,14,13),(-2,-11,-6,10),\\ (5,-13,-3,12),(\infty_1,2,-8,1),(\infty_2,11,-14,9)\\ (\infty_3,-12,8,6),(\infty_4,7,-10,-15),(\infty_5,-7,4,0)\}$$
$$\mathbf{D}_1 = \{(0,1,3,6)+4i\colon 0 \le i \le 6\}$$
$$\mathbf{D}_j = \{(0,1,3,6)+4i+24(j-2)\colon 7 \le i \le 12\} \text{ for } 2 \le j \le 5$$

The case $(v,n) = (36,9)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(\infty_1, 4, 1, 3), (\infty_2, -4, -11, -6), (\infty_3, 11, 10, 8) \\
&(\infty_4, -10, 5, -5), (\infty_5, 13, -3, -7), (\infty_6, -1, 6, 7) \\
&(\infty_7, 2, -12, -8), (\infty_8, -2, -13, 9), (\infty_9, 0, -9, 12)\} \\
\mathbf{D}_j = \{&(8, 11, 17, 25) + i + 3(j-1) \colon i = 0, 1, 2\} \\
&\{(5, 14, 1, 20) + i + 3(j-1) \colon i = 0, 1, 2\} \text{ for } 1 \le j \le 9
\end{aligned}$$

The case $(v,n) = (40,9)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(-13, 9, -9, 0), (\infty_1, 12, 2, 10), (\infty_2, -1, -12, 4), \\
&(\infty_3, 5, -3, 11), (\infty_4, -14, -4, -7), (\infty_5, 6, -8, -15), \\
&(\infty_6, -6, -11, 14), (\infty_7, 3, 8, 15), (\infty_8, -5, -2, 13), \\
&(\infty_9, -10, 1, 7)\} \\
\mathbf{D}_j = \{&(0, 4, 16, 12) + i + 4(j-1) \colon i = 0, 1, 2, 3\} \cup \\
&\{(0, 1, 3, 2) + i + 4(j-1) \colon i = 8, 21, 26\} \text{ for } 1 \le j \le 7 \\
\mathbf{D}_8 = \{&(0, 1, 3, 2) + i \colon i = 0, 4, 9, 13, 17, 22, 27\} \\
\mathbf{D}_9 = \{&(0, 4, 16, 12) + i + 28 \colon i = 0, 1, 2\} \cup \\
&(0, 1, 3, 2) + i + 28 \colon i = 8, 21, 26\}
\end{aligned}$$

The case $(v,n) = (44,9)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(6, 3, -4, 8,), (9, -15, 15, -11), (\infty_1, 11, -3, -7), \\
&(\infty_2, 7, -13, -9), (\infty_3, -5, 17, 16), (\infty_4, -1, 5, -14) \\
&(\infty_5, 2, -10, 14), (\infty_6, -17, 0, -6), (\infty_7, 4, -12, 1), \\
&(\infty_8, -2, 12, 13), (\infty_9, -16, -8, 10)\} \\
\mathbf{D}_j = \{&(0, 3, 8, 10) + j \colon j \in A_j\} \cup \{(11, 2, 9, 1) + j \colon j \in B_j\} \\
&\text{for } 1 \le j \le 9, \text{ where} \\
A_1 = B_1 &= \{1, 5, 19, 23\}, \quad A_2 = B_2 = \{6, 10, 22, 26\} \\
A_3 = B_3 &= \{8, 12, 25, 29\}, \quad A_4 = B_4 = \{9, 13, 27, 31\} \\
A_5 = B_5 &= \{11, 15, 28, 32\}, \quad A_6 = B_6 = \{14, 18, 30, 34\} \\
A_7 = B_7 &= \{17, 21, 33; 2\}, \quad A_8 = \{3, 7, 20, 24\}, \\
B_8 = \{3&, 7, 24\} \quad A_q = \{0, 4, 16\}, \quad B_q = \{0, 4, 16, 20\}
\end{aligned}$$

The case $(v,n)=(52,13)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(\infty_1,0,-9,-2),(\infty_2,17,4,-19),(\infty_3,-16,-7,-12)\\
&(\infty_4,10,2,5),(\infty_5,3,9,-3),(\infty_6,-11,-17,-4)\\
&(\infty_7,14,13,6),(\infty_8,-6,16,-15),(\infty_9,11,-18,-13)\\
&(\infty_{10},-8,12,8),(\infty_{11},7,-14,-10),(\infty_{12},19,-1,1)\\
&(\infty_{13},-5,18,15)\}\\
\mathbf{D}_j &= \{\mathbf{B}'+i+12(j-1)\colon i=0,4,8\} \text{ for } 1\le j\le 13 \text{ where}\\
\mathbf{B}' &= \{(0,14,12,-12),(-15,2,13,3),(-14,-13,15,1)\}
\end{aligned}$$

The case $(v,n)=(56,13)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(0,-3,-10,10),(\infty_1,4,-13,6),(\infty_2,-12,12,-6)\\
&(\infty_3,-21,-9,-14),(\infty_4,-7,-20,1),(\infty_5,20,-18,11)\\
&(\infty_6,13,-15,2),(\infty_7,19,-17,5),(\infty_8,14-19,-1),\\
&(\infty_9,-4,-16,21),(\infty_{10},16,7,-8),(\infty_{11},-11,3,9)\\
&(\infty_{12},-5,8,17),(\infty_{13},15,18,-2)\}\\
\mathbf{D}_i &= \{16(0,1,3,2)+16i\colon i\in A_i\} \text{ for } 1\le i\le 4\\
\mathbf{D}_5 &= \{16(0,1,3,2)+16j\colon j=10,2,37\}\cup\\
&\quad (0,1,3,2)+i\colon i\in C_0\}\\
\mathbf{D}_{5+j} &= \{4(0,1,3,2)+j\colon j\in B_j\}\cup\{(0,1,3,2)+j\colon j\in C_j\}\\
&\quad \text{for } 1\le j\le 8 \text{ where}\\
A_1 &= 16\{14,18,22,26,30,34,38,42,3,7\}\\
A_2 &= 16\{11,15,19,23,27,31,35,39,0,6\}\\
A_3 &= 16\{4,8,12,16,20,24,28,32,36,41\}\\
A_4 &= 16\{40,1,5,9,13,17,21,25,29,33\}\\
B_1 &= \{1,2,3,4\}\quad B_2=\{5,6,7,24,25,26,27\}\\
B_3 &= \{8,9,10,11,28,29,30,31\}\quad B_4=B_3+4, B_5=B_3+8,\\
B_6 &= B_3+12\quad B_7=B_8=\phi\\
C_0 &= \{39,0,8,12,16,23,27\}\quad C_1=\{17,21,25,29,33,37\}\\
C_2 &= \{20,40,1\}\quad C_3=\{24,4\}\quad C_4=\{28,5\}\quad C_5=\{32,9\}\\
C_6 &= \{36,13\}\quad C_7=\{41,2,6,10,14,18,22,26,31,35\}\\
C_8 &= \{30,34,38,42,3,7,11,15,19\}
\end{aligned}$$

The case $(v,n) = (60,13)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(-8,-21,1,-5),(9,-23,14,-9),(\infty_1,-16,3,-14)\\
&(\infty_2,8,11,2),(\infty_3,-22,10,-12),(\infty_4,-4,22,-15)\\
&(\infty_5,-1,-17,12),(\infty_6,-13,7,19),(\infty_7,-3,6,-20)\\
&(\infty_8,5,-19,23),(\infty_9,-11,0,16),(\infty_{10},15,20,-10)\\
&(\infty_{11},-6,-18,18)(\infty_{12},-2,4,17),(\infty_{13},13,-7,21)\}\\
\mathbf{D}_j &= \{(0,1,3,2)+i\colon i\in A_j\} \quad \mathbf{D}_{4+j} = \{(0,7,21,14)+7i\colon i\in B_j\}\\
\mathbf{D}_{8+j} &= \{(0,4,12,8)+4i\colon i\in C_j\} \text{ for } 1\le j\le 4\\
\mathbf{D}_{13} &= \{(0,1,3,2)+i\colon i=3,10,17\}\cup\\
&\{(0,4,12,8)+4i\colon i=6,18,30,42\}\cup\\
&\{(0,7,21,14)+7i\colon i=0,7,27\}\\
A_1 &= \{21,25,29,33,37,41,45,2,7,11,15\}\\
A_2 &= \{14,18,22,26,30,34,38,42,46,5,9\}\\
A_3 &= \{19,23,27,31,35,39,43,0,4,8,13\}\\
A_4 &= \{16,20,24,28,32,36,40,44,1,6,12\}\\
B_1 &= \{4i\colon 1\le i\le 11\} \quad B_2 = \{31+4i\colon 0\le i\le 3\}\cup\\
&\{1+4i\colon 0\le i\le 6\}\\
B_3 &= \{29+4i\colon 0\le i\le 4\}\cup\{2,6,11,15,19,23\}\\
B_4 &= \{10+4i\colon 0\le i\le 10\}\\
C_1 &= \{44,1,5,10,14\},\{19+4i\colon 0\le i\le 5\}\\
C_2 &= \{46,3,7,11,15\},\{21+4i\colon 0\le i\le 5\}\\
C_3 &= \{45,2\}\cup\{4i\colon 2\le i\le 10\}\\
C_4 &= \{0,4,9,13,17,22,26,34,38,43\}
\end{aligned}$$

The case $(v,n) = (68,17)$ Let

$$\begin{aligned}
\mathbf{B} = \{&(\infty_1,-21,-17,-20)(\infty_2,-25,4,24)(\infty_3,-16,-24,-19)\\
&(\infty_4,25,14,21)(\infty_5,-15,8,-10)(\infty_6,3,15,9)\\
&(\infty_7,-13,-14,-6)(\infty_8,20,6,12)(\infty_9,16,-4,7)\\
&(\infty_{10},11,2,-5)(\infty_{11},-7,-9,10)(\infty_{12},5,-12,23)\\
&(\infty_{13},17,19,-2)(\infty_{14},-18,18,13)(\infty_{15},22,-11,-8)\\
&(\infty_{16},-1,0,-23)(\infty_{17},1,-3,-22)\}\\
\mathbf{D}_j &= \{\mathbf{B}'+i+12(j-1)\colon i=0,4,8\} \text{ for } 1\le j\le 17 \text{ where}\\
\mathbf{B}' &= \{(0,13,-12,-25),(1,15,-14,25),(2,12,-24,-15)\\
&(3,24,14,-13)\}
\end{aligned}$$

The case $(v, n) = (132, 33)$ Let

$$\mathbf{B} = \{(\infty_1, 33, -19, 32), (\infty_2, 18, 49, -5), (\infty_3, 24, -47, -15)$$
$$(\infty_4, -39, 19, -36), (\infty_5, -44, -10, 30), (\infty_6, 16, -12, -43)$$
$$(\infty_7, 10, -28, 5), (\infty_8, -21, 31, 6), (\infty_9, -34, -16, 8)$$
$$(\infty_{10}, -11, 27, -18), (\infty_{11}, 37, 3, -33), (\infty_{12}, -49, -29, -6)$$
$$(\infty_{13}, 17, -40, 26), (\infty_{14}, 44, 40, -24), (\infty_{15}, -37, 22, 7)$$
$$(\infty_{16}, -25, 25, -14), (\infty_{17}, -32, -3, 34), (\infty_{18}, 14, 29, -31)$$
$$(\infty_{19}, -48, 9, -35), (\infty_{20}, -30, -46, 35), (\infty_{21}, 2, 21, 48)$$
$$(\infty_{22}, -2, 41, -17), (\infty_{23}, 47, 1, 12), (\infty_{24}, 39, 15, -8)$$
$$(\infty_{25}, -26, -45, -9), (\infty_{26}, 43, 0, -20), (\infty_{27}, -13, 13, 38)$$
$$(\infty_{28}, -42, 28, -23), (\infty_{29}, -41, 23, -4), (\infty_{30}, 42, 46, -7)$$
$$(\infty_{31}, 20, 36, -1), (\infty_{32}, -27, -38, 11), (\infty_{33}, 4, -22, 45)\}$$

$\mathbf{D}_j = \{F + j + i\colon F \in \mathbf{F}, i = 0, 33, 66\}$ for $1 \le j \le 33$ where

$$\mathbf{F} = \{(1, 13, -1, -13), (2, 8, -2, -8), (3, 6, -3, -6)$$
$$(4, 9, -4, -9), (5, 12, -5, -12), (7, 15, -7, -15)$$
$$(10, 11, -10, -11), (14, 16, -14, -16)\}$$

References

[1] F.E. Bennett, Conjugate orthogonal latin squares and Mendelsohn designs, *Ars Combinatoria* **19**(1985),51–62.

[2] F.E. Bennett, E.Mendelsohn and N.S. Mendelsohn, Resolvable perfect cyclic designs, *J. Combinatorial Theorem Ser. A* **29**(1980),142–150.

[3] F.E. Bennett and D. Sotteau, Almost resolvable decomposition of K_n^*, *J. Combin. Theory Ser. B* **30**(1981), 228–232.

[4] F.E. Bennett and Zhang Xuebin, Resolvable Mendelsohn designs with block size 4, *Aequationes Math.* **40**(1990),248–260.

[5] F.E. Bennett., Zhang Xuebin and L. Zhu, Perfect Mendelsohn designs with block size four, *Ars Combinatoria* **29**(1990), 65–72.

[6] D.F. Hsu and A.D. Keedwell, Generalized complete mappings, neofields, sequenceable groups pnd block designs, *Pacific J. of Math.* **117**(1985), 291–312.

[7] J.C. Bermond, A. Germa, and D. Sotteau, Resolvable decomposition of K_n^*, *J. Combin. Theorey Ser. A* **26**(1979),179–185.

[8] R.D. Baker and R.M. Wilson, The Whist tournament problem of E.H. Moore, preprint.

[9] E.R. Lamken, W.H. Mills, and R.M. Wilson. Four pairwise balanced designs. *Designs, Codes and Crypt* **1**(1991), 63–68.

[10] L. Zhu, Perfect Mendelsohn designs, *J. Combin. Math. Combin. Computing* **5**(1989),43–54.

[11] N.S. Mendelsohn, Perfect cyclic designs, *Discrete Math.* **20**(1977), 63–68.

[12] R.M. Wilson, Cyclotomy and different families in elementary Abelian groups, *Journal of Number theory* **4**(1972),17–47.

[13] R.M. Wilson, An existence theory for pairwise balanced designs, *J. Combinatorial Theory Ser. A* **18**(1975),71–79.

[14] Zhang Xuebin, On the existence of $(v, 4, 1)$-PMD, *Ars Combinatoria* **29**(1990), 3–12.

[15] Zhang Xuebin, Constructions of Resolvable Mendelsohn designs, *Ars Combinatoria* **34**(1992), 225–250.