On (A, D)-Antimagic Parachutes

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ABSTRACT. This paper deals with (a,d)-antimagic labelings of special graphs called parachutes. After the introduction of the concept of a parachute the authors succeed in proving the finiteness of two very interesting subsets of the set of all (a,d)-antimagic parachutes by means of a method using the theory of Diophantine equations and other number-theoretical results.

1 Introduction

In [3], G. Ringel and N. Hartsfield introduce the concept of an antimagic graph. In [2], the authors define the concept of an (a, d)-antimagic graph as a special antimagic graph where $a, d \in \mathbb{N}$. While G. Ringel and N. Hartsfield conjecture that every connected Graph $G = (V, E)$ of order $|V| \geq 3$ is antimagic it turns out that the property of a graph $G = (V, E)$ to be (a, d)-antimagic actually depends on the shape of G for one can show that both every tree $T = (V, E)$ of even order $|V| \geq 4$ and every cycle $C = (V, E)$ of even order $|V| \geq 4$ are not (a, d)-antimagic. So it makes really sense to ask for the set of all (a, d)-antimagic graphs. In order to determine this set we consider the infinite subset of parachutes of the set of all connected graphs of order ≥ 3. In [4], the concept of an (a, d)-antimagic graph is introduced in the following way:

Definition 1: A connected graph $G = (V, E)$ of order $|V| \geq 3$ is said to be (a, d)-antimagic iff there exist positive integers $a, d \in \mathbb{N}$ and a bijective mapping f defined by

$$f: \begin{cases} E & \to \{1, 2, \ldots, |E|\} \\ e & \to f(e), e \in E \end{cases}$$

ARS COMBINATORIA 42(1996), pp. 129-149

such that the mapping g_f induced by f and defined by

$$g_f : \begin{cases} V & \to \mathbf{N} \\ v & \to g_f(v) = \sum_{e \in I(v)} f(e), v \in V \end{cases}$$

is injective and $g_f(V) = \{a, a+d, \ldots, a+(|V|-1)d\}$ where $I(v) = \{e \in E | e$ is incident to $v\}$ for $v \in V$.

If $G = (V, E)$ is (a, d)-antimagic and $f: E \to \{1, 2, \ldots, |E|\}$ is a corresponding bijective mapping of G then f is said to be an (a, d)-antimagic labeling of G.

2 Definitions and Notations

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and the edge set of a graph $G \in \Gamma =$ set of all finite, undirected, simple graphs. In order to introduce the notion of a parachute let $g, b \in \mathbf{N}$, $g \geq 3$, be two positive integers, and let P_g denote a path of order g with the vertex set $V(P_g) = \{v_1, v_2, \ldots, v_g\}$ and the edge set $E(P_g) = \{\{v_i, v_{i+1}\} | i \in \{1, 2, \ldots, g-1\}\}$. Then the graph $1 * P_g$ has got the vertex set $V(1 * P_g) = \{v\} \cup V(P_g)$ and the edge set $E(1*P_g) = E(P_g) \cup \{\{v, v_i\} | i \in \{1, 2, \ldots, g\}\}$. Finally, C_{g+b} denotes the cycle of order $g+b$ with the vertex set $V(C_{g+b}) = \{v_1, v_2, \ldots, v_g, v_1', v_2', \ldots, v_b'\}$ and the edge set $E(C_{b+g}) = E(P_g) \cup \{\{v_1, v_1'\}, \{v_g, v_b'\}\} \cup \{\{v_i', v_{i+1}'\} | i = 1, 2, \ldots, b-1\}$. Now the concept of a parachute will be defined in the following way:

Definition 2: A graph $P_{g,b} = (V_{g,b}, E_{g,b})$, $g, b \in \mathbf{N}$, $g \geq 3$, is said to be a parachute iff P is the amalgamation $(1 * P_g) \cup^{P_g} C_{g+b}$ obtained from the union of $1*P_g$ and C_{g+b} by pasting them along P_g such that the intersection $(1 * P_g) \cap C_{g+b}$ is equal to P_g.

A parachute $P_{g,b}$ is a connected graph on $|V_{g,b}| = g + b + 1$ vertices and $|E_{g,b}| = 2g + b$ edges and has the minimum degree 2 and the maximum degree g. Figure 1 shows the parachutes $P_{3,2}$, $P_{5,4}$ and $P_{6,3}$. If $\overline{\Gamma}(P) \subseteq \Gamma$ denotes the set of all parachutes $P_{g,b}$, $g, b \in \mathbf{N}$, $g \geq 3$, it makes sense to ask the question for the set $\Gamma(P) \subseteq \overline{\Gamma}(P)$ of all (a, d)-antimagic parachutes in $\overline{\Gamma}(P)$, $a, d \in \mathbf{N}$, as already mentioned above.

3 Parachutes and their Diophantine equations

In order to determine $\Gamma(P)$ we consider an arbitrary parachute $P_{g,b} \in \Gamma(P)$. Then there exist positive integers $a, d \in \mathbf{N}$, $a \geq 3$ because of minimum degree in $P_{g,b}$ is 2, such that the following equation holds:

(i) $2(1 + 2 + \cdots + 2g + b) = a + (a + d) + \cdots + (a + (g + b)d)$.

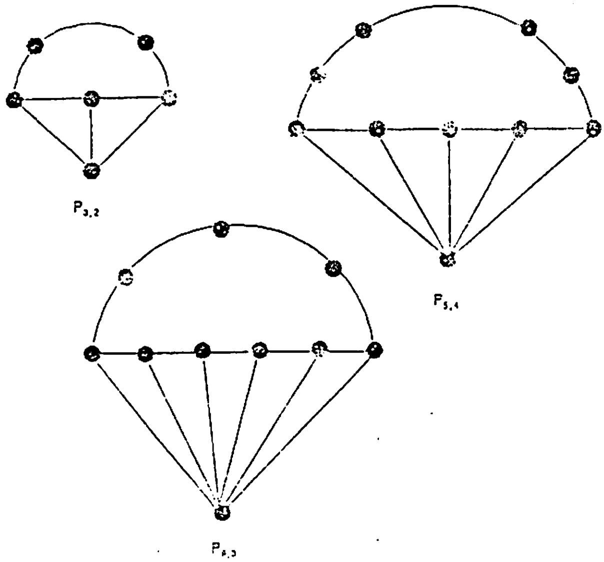


Figure 1

(i) is equivalent to the Diophantine equation

(ii) $(2g+b)(2g+b+1) = (g+b+1)a + \frac{d}{2}(g+b)(g+b+1)$

to be solved in positive integers $a \geq 3$ and $d \in \mathbf{N}$. Then we obtain a first proposition helping find $\Gamma(P)$ in the following way:

Proposition 1. *If $P_{g,b} \in \Gamma(P)$ then there exist at most 7 mutually different solutions (a,d) of the Diophantine equation (ii). That means that, for every parachute $P_{g,b} \in \Gamma(P)$ there exist at most 7 different values of d yielding (a,d)-antimagic labelings of $P_{g,b}$.*

Proof: Putting $d = 8$ (ii) becomes

(iii) $(g+b+1)a = -4h - 2g - 3b^2 - 3b$

such that $a < 0$ contradicting $a \geq 3$. It is obvious that one also obtains a contradiction for each $d \geq 9$.

Applying the solution criterion for Diophantine equations we know that the necessary and sufficient condition for the existence of integer solutions a, d of (ii) is that the greatest common divisor $gcd(g+b+1, \frac{1}{2}(g+b)(g+b+1))$ divides the product $(2g+b)(2g+b+1)$ such that

(iv) $gcd(g+b+1, \frac{1}{2}(g+b)(g+b+1)) | (2g+b)(2g+b+1)$

is true. Since

$$gcd(g+b+1, \frac{1}{2}(g+b)(g+b+1)) = \begin{cases} g+b+1 & \text{for } g+b \text{ even} \\ \frac{1}{2}(g+b+1) & \text{for } g+b \text{ odd} \end{cases}$$

(iv) is equivalent to

(v) (a) $(g+b+1)|(2g+b)(2g+b+1)$ or (b) $(g+b+1)|2(2g+b)(2g+b+1)$.

Let $g+b+1=t$. Thus, t is an integer $\geq g+2$ such that we deduce from (v) the conditions

(vi) (a) $t|(t+g)(t+g-1)$ or (b) $t|2(t+g)(t+g-1)$

which are equivalent to

(vii) (a) $t|g(g-1)$ or (b) $t|2g(g-1)$.

Since

$$\{t \in \mathbf{N} | t \geq g+2 \wedge t|g(g-1)\} \subseteq \{t \in \mathbf{N} | t \geq g+2 \wedge t|2g(g-1)\}$$

(vii) is equivalent to (viii) $t|2g(g-1)$.

That means that if $P_{g,b} \in \Gamma(P)$ then t satisfies (viii) and $b = t-g-1$. Since there are only finitely many different divisors $t \geq g+2$ satisfying (viii) we have proved a further step for finding $\Gamma(P)$. It holds:

Proposition 2. *For every $g \in \mathbf{N}$, $g \geq 3$, $\Gamma(P)$ contains at most finitely many parachutes $P_{g,b}$, where $b = t-g-1 \leq 2g^2-3g-1$ and $t|2g(g-1)$.*

Proof: It only remains to show that $b \leq 2g^2-3g-1$. Since $t = 2 \cdot g(g-1)$ is the greatest divisor satisfying (viii) the corresponding value of b is precisely $b = 2g^2-3g-1$ and obviously the greatest possible number in $\mathbf{N}$ such that $P_{g,b} \in \Gamma(P)$ for any $g \geq 3$.

The exact number of different integers $b = t-g-1$, t satisfies (viii), generally depends on the choice of g. Table 1 gives an impression of the set of divisors satisfying (viii) and shows that for each $g \geq 6$, there are at least five different values of b such that the corresponding parachute $P_{g,b}$ could belong to $\Gamma(P)$.

Furthermore, Table 1 shows that $\Gamma(P)$ has the following five subsets $\Gamma_i(P)$, $i = 1,2,3,4,5$, where

$$\Gamma_1(P) = \{P_{g,2g^2-3g-1} \mid g \geq 3\},$$
$$\Gamma_2(P) = \{P_{g,g^2-2g-1} \mid g \geq 4\},$$
$$\Gamma_3(P) = \{P_{g,g-1} \mid g \geq 3\},$$
$$\Gamma_4(P) = \{P_{g,g-3} \mid g \geq 4\},$$

and

$$\Gamma_5(P) = \{P_{g,\frac{1}{2}(g^2-3g-2)} \mid g \geq 4\}.$$

g	$2g(g-1)$	$t \geq g+2$	$b = t - g - 1$	number of different $t \mid 2g(g-1)$
3	12	6,12	2,8	2
4	24	6,8,12, 24	1,3,7,19	4
5	40	8,10,20, 40	2,4,14,34	4
6	60	10,12,15, 20,30,60	3,5,8,13, 23,53	6
7	84	12,14,21, 28,42,84	4,6,13,20, 34,76	6
8	112	14,16,28, 56,112	5,7,19,45, 103	5
$g \geq 9$	$2g(g-1)$	$t_1 = 2g(g-1)$ $t_2 = g(g-1)$ $t_3 = 2g$ $t_4 = 2(g-1)$ $t_5 = \frac{g(g-1)}{2}$	$b_1 = 2g^2 - 3g - 1$ $b_2 = g^2 - 2g - 1$ $b_3 = g - 1$ $b_4 = g - 3$ $b_5 = \frac{1}{2}(g^2 - 3g - 2)$	≥ 5

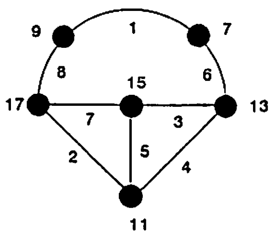
Table 1

4 Finiteness of $\Gamma_3(P)$

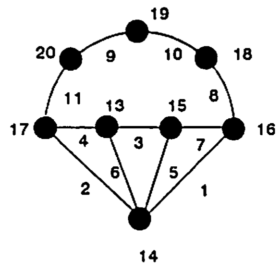
First of all we shall turn towards $\Gamma_3(P)$. Putting $b = g - 1$ Proposition 1 becomes

Proposition 3.

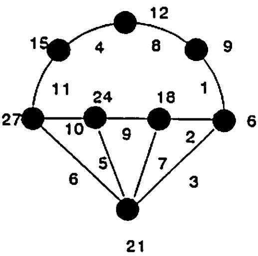
$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 3}} [P_{g,g-1} \in \Gamma(P) \rightarrow d \leq 4]$$



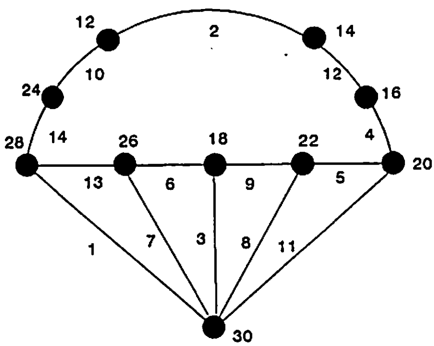
$P_{3,2}$ (7,2)-antimagic



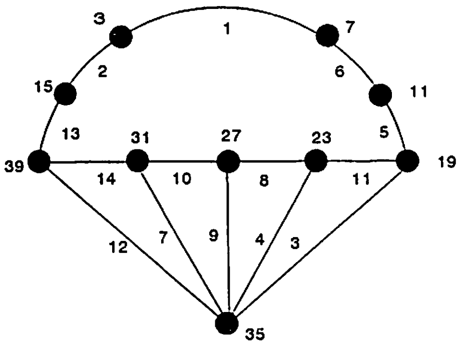
$P_{4,3}$ (13,1)-antimagic



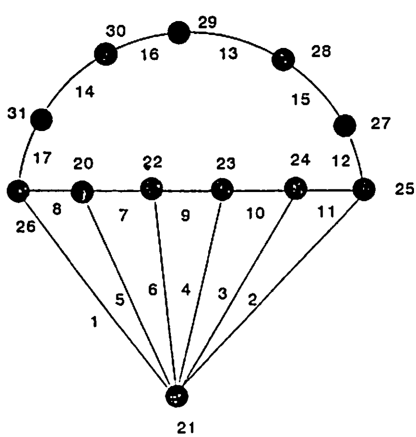
$P_{4,3}$ (6,3)-antimagic



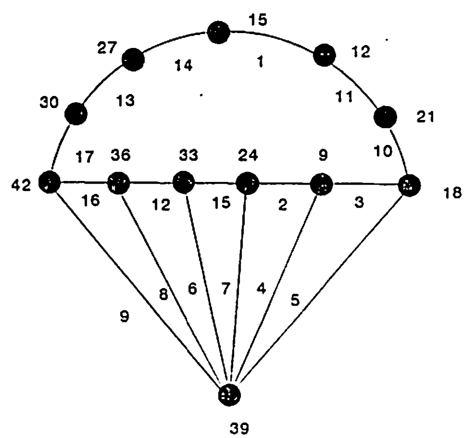
$P_{5,4}$ (12,2)-antimagic



$P_{5,4}$ (3,4)-antimagic

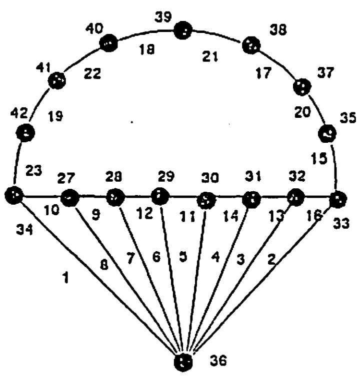


$P_{6,5}$ (20,1)-antimagic

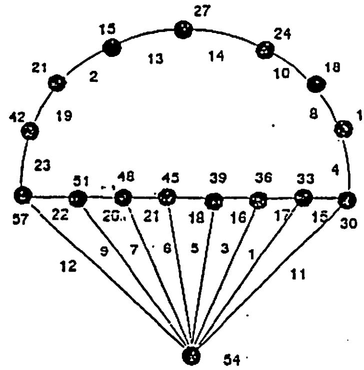


$P_{6,5}$ (9,3)-antimagic

Figure 2



$P_{8,7}$ (27,1)-antimagic



$P_{8,7}$ (12,3)-antimagic

Figure 2 con't

Since its proof is an immediate consequence of the proof of Proposition 1 we omit details.

The next step is to illustrate that $\Gamma_3(P) \neq \emptyset$. Figure 2 depicts parachutes $P_{g,g-1} \in \Gamma_3(P)$ and corresponding (a,d)-antimagic labelings for $g \leq 8$ and each possible d.

Applying Proposition 3 we obtain

Proposition 4. *Let $P_{g,g-1} \in \Gamma_3(P)$, $g \geq 3$. The corresponding Diophantine equation of $P_{g,g-1}$ is*

(ix) $3(3g-1) = 2a + (2g-1)d$

and has the two different solutions $(7n-1,1)$ and $(3n,3)$ if $g = 2n$, $n \geq 2$, or $(5n-3,2)$ and $(n,4)$ if $g = 2n-1$, $n \geq 2$.

Proof: Putting $b = g - 1$ the equation (ii) becomes (ix). According to Proposition 3, we know that $d \leq 4$, such that we obtain the two solutions by distinguishing the cases g odd and g even.

Now we are able to show that $\Gamma_3(P)$ is finite for it holds:

Theorem 1. *Let $a \in \mathbb{N}$ be an arbitrary integer ≥ 3.*

(1) $\bigwedge_{g \geq 10} P_{g,g-1}$ *is not $(a,1)$-antimagic,*

(2) $\bigwedge_{g \geq 12} P_{g,g-1}$ *is not $(a,2)$-antimagic,*

(3) $\bigwedge_{g \geq 14} P_{g,g-1}$ *is not $(a,3)$-antimagic,*

(4) $\bigwedge_{g \geq 16} P_{g,g-1}$ *is not $(a,4)$-antimagic.*

Proof: If $P_{g,g-1}$ is an arbitrary parachute from $\Gamma_3(P)$ then we know that the maximum vertex label

(x) $a_{\max} = a + (2g-1)d = \frac{(9g-3)-(2g-1)d}{2} + (2g-1)d$

when $P_{g,g-1}$ is (a,d)-antimagic. Since the maximum degree in $P_{g,g-1}$ is g the inequality $a_{\max} \geq \frac{g(g+1)}{2}$ is true.

Applying the contraposition, we obtain the assertion that the inequality $a_{\max} < \frac{g(g+1)}{2}$ implies that $P_{g,g-1}$ is not (a,d)-antimagic. Substituting the expression (x) for $a_{\max}$ we obtain

(xi) $\frac{g(g+1)}{2} > \frac{(9g-3)+(2g-1)d}{2}$ which is equivalent to

(xii) $(g-(4+d))^2 > d^2 + 7d + 13$ or

(xiii) $g > 4 + d + \sqrt{d^2 + 7d + 13}$ or $g < (4+d) - \sqrt{d^2 + 7d + 13}$.

While the second expression gives a contradiction we obtain

(xiv) $g > 4 + d + \sqrt{d^2 + 7d + 13}$.

Putting $g_o(d) = \lfloor 4 + d + \sqrt{d^2 + 7d + 13} \rfloor + 1$ where $\lfloor 4 + d + \sqrt{d^2 + 7d + 13} \rfloor$ means the greatest integer less than or equal to $4 + d + \sqrt{d^2 + 7d + 13}$ and putting $d = 1, 2, 3, 4$ we obtain $g_o(1) = 10$, $g_o(2) = 12$, $g_o(3) = 14$ and $g_o(4) = 16$. This proves Theorem 1 and the finiteness of $\Gamma_3(P)$.

Since Figure 2 shows all (a,d)-antimagic labelings of $P_{g,g-1}$, $3 \leq g \leq 8$ we only need to investigate the parachutes $P_{g,g-1}$ for $g \leq g \leq 15$.

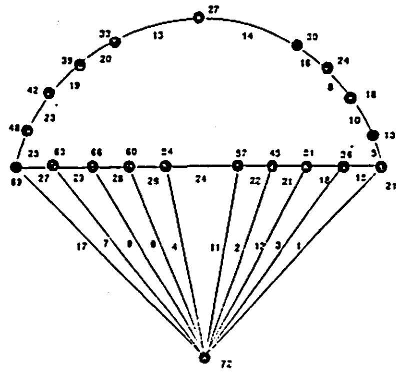
Theorem 2.

(a) $P_{10,9}$ *and* $P_{11,10}$ *belong to* $\Gamma_3(P)$,

(b) $P_{9,8}, P_{12,11}, P_{13,12}, P_{14,13}, P_{15,14} \notin \Gamma_3(P)$.

Proof: Ad (a) Assume $P_{10,9}$ is (a,d)-antimagic. Then we know because of Proposition 4 that (a,d) is either $(34,1)$-antimagic or $(15,3)$-antimagic. By means of Theorem 1 it follows that $P_{10,9}$ is not $(34,1)$-antimagic. Figure 3 gives an $(15,3)$-antimagic labeling of $P_{10,9}$ such that $P_{10,9} \in \Gamma_3(P)$.

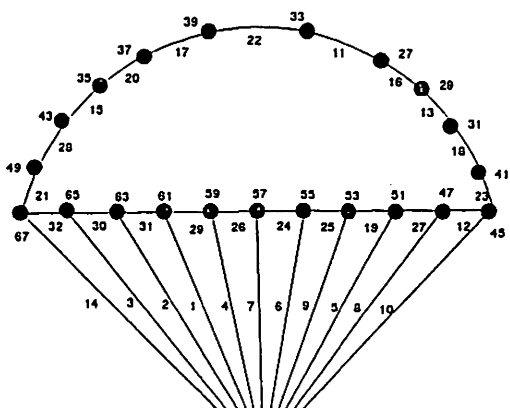
The procedure in case of $P_{11,10}$ is more complicated than in case of $P_{10,9}$ for we cannot cancel a solution by applying Theorem 1. Figure 4 depicts a $(27,2)$-antimagic labeling of $P_{11,10}$ such that $P_{11,10} \in \Gamma_3(P)$. For sake of completeness it remains to check whether $P_{11,10} = (1 * P_{11}) \cup^{P_{11}} C_{21}$ is $(6,4)$-antimagic. Assume there is a $(6,4)$-antimagic labeling f of $P_{11,10}$ such that $a_{\max} = 90$. Since each vertex value of f is even one has to label - by means of f - the 11 edges of $C_{21} \setminus P_{11}$ either by 11 even or by 11 odd numbers of the set $\{1, 2, \ldots, 32\}$. Assume the 11 edges of $C_{21} \setminus P_{11}$ are assigned by 11 odd numbers in $\{1, 2, \ldots, 32\}$. Since each vertex value is congruent 2 modulo 4 we have to take either 11 odd numbers in $\{1, 2, \ldots, 32\}$ congruent 1 modulo 4 or 11 odd numbers in $\{1, 2, \ldots, 32\}$ congruent 3 modulo 4.



$P_{16,3}$ (15,3)-antimagic

Figure 3

This is a contradiction for there are only 8 odd numbers congruent 1 modulo 4 or 8 odd numbers congruent 3 modulo 4, respectively.



$P_{11,10}$ (27,2)-antimagic

Figure 4

Assume the 11 edges of $C_{21} \setminus P_{11}$ are labeled by 11 even integers in $\{1, 2, \ldots, 32\}$. Denoting the two end vertices of P_{11} by x and y we have to distinguish the following three cases. 1) The two unlabeled edges incident to x are assigned by two odd integers in $\{1, 2, \ldots, 32\}$ and the two unlabeled edges incident to y are also labeled by two odd integers from $\{1, 2, \ldots, 32\}$. 2) The two unlabeled edges incident to x are labeled by two odd integers in $\{1,2,\ldots,32\}$ and the two unlabeled edges incident to y are assigned by two even numbers in $\{1,2,\ldots,32\}$. 3) The four unlabeled edges incident either to x or y are labeled by four even numbers in $\{1,2,\ldots,32\}$.

In case 1), Figure 5 shows three distributions of odd and even numbers on the edges of $P_{11,10}$. All these distributions of Figure 5 are characterized by the fact that the vertex value $f(v)$ of the vertex v of degree 11 is the sum of 10 odd numbers and one even number in $\{1,2,\ldots,32\}$ for each $(6,4)$-antimagic labeling f of $P_{11,10}$. That implies that $f(v) \geq (1+3+\cdots+19)+2 = 102$, contradiction to $a_{\max} = 90$. Thus at least two edges incident to v must be labeled by even numbers in $\{1,2,\ldots,32\}$. It turns out that such a distribution is not possible such that each vertex value is congruent 2 modulo 4. Observing the arguments excluding case 1) it also turns out that all the distributions possible in the cases 2) and 3) lead to contradictions. Details must be omitted.

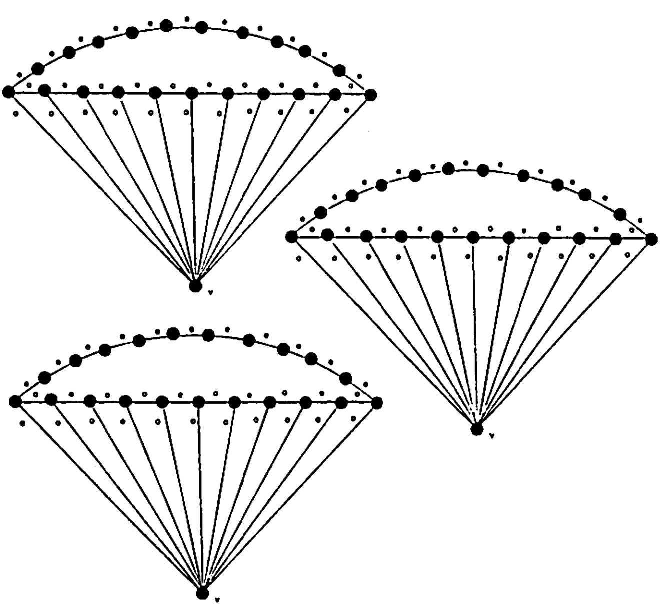


Figure 5

Ad (b) Firstly we assume the parachute $P_{9,8} = (1 * P_9) \cup^{P_9} C_{17}$ is an element in $\Gamma_3(P)$ and has an (a,d)-antimagic labeling. Then (a,d) is either (22,2) or (5,4). Assume there exists a (22,2)-antimagic labeling f of $P_{9,8}$ such that $a_{\max} = 56$. As each vertex value of f is even the 9 edges of $C_{17} \backslash P_9$ are labeled either by 9 even or by 9 odd numbers of the set $\{1,2,\ldots,26\}$. Figure 6 shows the two possible distributions (I) and (II) of even and odd numbers necessarily satisfied by f. In case (I) the smallest possible value

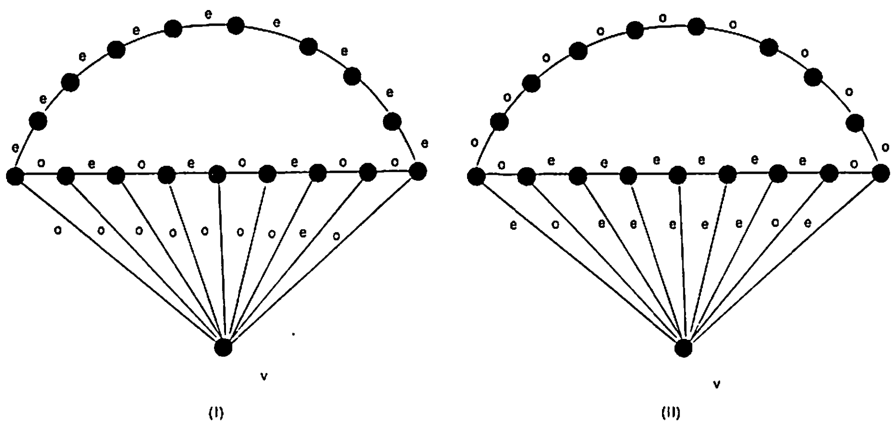


Figure 6

$f(v)$ at the vertex v of degree 9 is equal to the sum

$$f(v) = 2 + 1 + 3 + \cdots + 15 = 2 + 82 = 66$$

contradicting the property $a_{\max} = 56$. That means that f necessarily satisfies the distribution (II). In this case the smallest possible value $f(v)$ at v is equal to the sum

$$f(v) = 2 + 4 + \cdots + 14 + 1 + 3 = 56 + 4 = 60$$

also contradicting $a_{\max} = 56$. Hence $P_{9,8}$ is not (22,2)-antimagic. It remains to investigate whether $P_{9,8}$ is (5,4)-antimagic. Assume $P_{9,8}$ is (5,4)-antimagic and has the corresponding (5,4)-labeling f such that $a_{\max} = 73$. Since each vertex value of f belongs to the residue class 1 modulo 4 - the elements in 1 have got the principal remainder 1 modulo 4 - f labels the 9 edges of $C_{17} - P_9$ in precisely four different ways (I) - (IV) depicted in Figure 7 where $\overline{0} = \{4, 8, 12, 16, 20, 24\}$, $\overline{3} = \{3, 7, 11, 15, 19, 23\}$, $\overline{2} = \{2, 6, 10, 14, 18, 22, 26\}$ and $\overline{1} = \{1, 5, 9, 13, 17, 21, 25\}$. In case (I) f permits the following 14 mutually different distributions at the vertex v of degree 9 depicted in Table 2. For sake of shortness we have to restrict ourselves to investigating case 1 in Table 2 as a paradigm. Assume that f satisfies case 1. Then the smallest possible value $f(v)$ at the vertex v is equal to the sum

$$f(v) = 2 + 6 + 10 + 14 + 18 + 22 + 3 + 7 + 11 = 93$$

contradicting $a_{\max} = 73$. The proof that all the other cases of (I) - (IV) contradict the fact that f is a (5,4)-antimagic labeling of $P_{9,8}$ is a matter

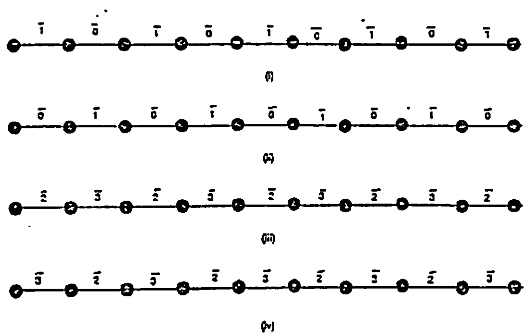


Figure 7

of routine checking such that we are allowed to omit further details. This means that $P_{9,8}$ is not (5,4)-antimagic. This completes the proof that the statement $P_{9,8} \notin \Gamma_3(P)$ is true.

Case	Number of different integers of $\overline{0}$	Number of different integers of $\overline{1}$	Number of different integers of $\overline{2}$	Number of different integers of $\overline{3}$
1	0	0	6	3
2	1	0	7	1
3	2	0	4	3
4	2	0	4	3
5	0	1	4	4
6	0	2	6	1
7	0	2	1	6
8	1	1	5	2
9	1	1	1	6
10	2	1	6	0
11	2	1	2	4
12	2	2	4	1
13	2	2	5	0
14	1	2	3	3

Table 2

The statement $P_{14,13} \notin \Gamma_3(P)$ is true. Its proof is an immediate consequence of Proposition 4 and Theorem 1. In order to show the truth of the statement $P_{15,14} \notin \Gamma_3(P)$ we assume $P_{15,14} = (1 * P_{15}) \cup^{P_{15}} C_{29}$ is (a, d)-antimagic. According to Proposition 4, (a, d) is either (37,2) or (8,4). The first case is not possible because of Theorem 1. Therefore we only have to investigate if it is possible that there exists a (8, 4)-antimagic labeling f of

$P_{15,14}$ such that $a_{max} = 124$. As each vertex value of f is even the 15 edges of $C_{29} - P_{15}$ are labeled either by 15 even numbers or by 15 odd numbers of the set $\{1, 2, \ldots, 44\}$. Then Figure 8 shows the two possible distributions of

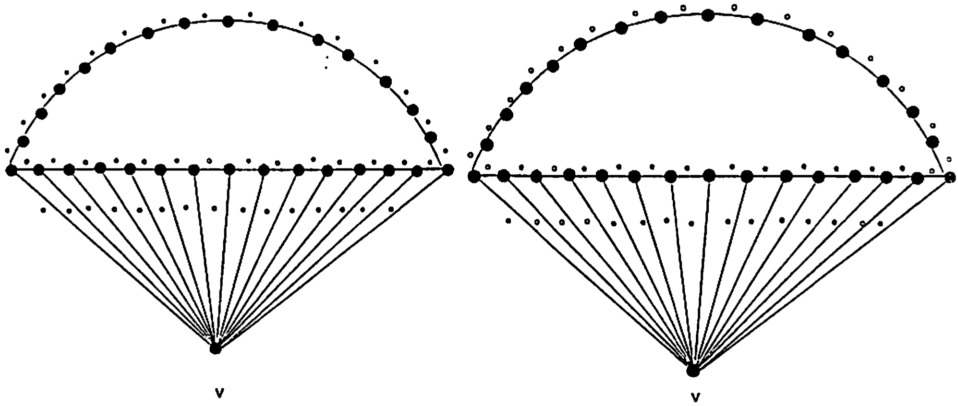


Figure 8

even and odd numbers of $\{1, 2, \ldots, 44\}$ for $P_{15,14}$. In both cases f implies the smallest possible value $f(v)$ at the vertex v as the sum

$$f(v) = 2 + 4 + \cdots + 28 + 1 = 211$$

or

$$f(v) = 2 + 4 + \cdots + 22 + 1 + 3 + 5 + 7 = 148$$

contradicting $a_{max} = 124$. This completes the proof of $P_{15,14} \notin \Gamma_3(P)$.

In case of $P_{13,12}$, the assumption $P_{13,12} \in \Gamma_3(P)$ implies because of Proposition 4 that (a, d) is either (32,2) or (7,4) where the first case is not possible according to Theorem 1. Consequently it remains to prove that the existence of a (7,4)-antimagic labeling f of $P_{13,12}$ with $a_{max} = 107$ leads to a contradiction. This proof is very lengthy and similar to the proof that the parachute $P_{9,8}$ is not (5,4)-antimagic. Therefore, we omit details and state that $P_{13,12} \notin \Gamma_3(P)$ is true.

The last statement of (b) is $P_{12,11} \notin \Gamma_3(P)$. Assume $P_{12,11} = (1 * P_{12}) \cup^{P_{12}} C_{23}$ is (a, d)-antimagic. Then it follows from Proposition 4 that (a, d) is either (41,1) or (18,3). While the first case can be omitted because of Theorem 1 we have to investigate whether the existence of a (18, 3)-antimagic labeling f of $P_{12,11}$ with $a_{max} = 87$ leads to a contradiction. In order to do this we start from the fact that each vertex value of f in $P_{12,11}$ is a multiple of 3 such that the distribution of the numbers of the residue classes $\overline{0}$, $\overline{1}$, $\overline{2}$ modulo 3 for the 12 edges of $C_{23} - P_{12}$ is unique and shown in Figure 9.

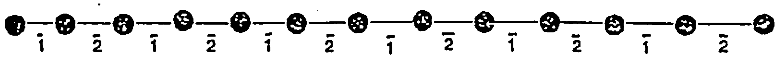


Figure 9

If the value $f(v)$ at the vertex v of degree 12 has 10 or 8 or 6 summands from $\overline{0}$ then $f(v)$ is greater than 165 or 108 or 90, respectively, and we obtain the contradiction to $a_{\max} = 87$. Therefore, it remains to check the case where $f(v)$ has 4 summands from $\overline{0}$, $\overline{1}$ and $\overline{2}$, each. It is necessary that $f(v)$ is equal to $a_{\max} = 87$.

Case	4 possible summands from $\overline{0}$	4 possible summands from $\overline{1}$	4 possible summands from $\overline{2}$
1)	3+6+9+12	1+4+7+10	2+5+8+20
2)			2+5+11+17
3)			2+8+11+14
4)	3+6+9+12	1+4+7+19	2+5+8+11
5)		1+4+10+16	
6)		1+7+10+13	
7)	3+6+9+21	1+4+7+10	2+5+8+11
8)	3+6+12+18		
9)	3+9+12+15		
10)	3+6+9+12	1+4+7+13	2+5+8+17
11)			2+5+11+14
12)	3+6+9+12	1+4+7+16	2+5+8+14
13)		1+4+10+13	
14)	3+6+9+18	1+4+7+10	2+5+8+14
15)	3+6+12+15		
16)	3+6+9+15	1+4+7+10	2+5+8+17
17)			2+5+11+14
18)	3+6+9+18	1+4+7+13	2+5+8+11
19)	3+6+12+15		
20)	3+6+9+15	1+4+7+16	2+5+8+11
21)		1+4+10+13	
22)	3+6+9+15	1+4+7+13	2+5+8+14

Table 3

Table 3 shows the 22 representations of 87 as sum with 12 summands such that four are from $\overline{0}$, $\overline{1}$ or $\overline{2}$, respectively. As the integers 84, 81, 78, 75, 72 and 69 must necessarily appear at vertices of degree 3 and as 35 is the greatest number available for edge labeling of $P_{12,11}$ by means of f it turns out to be a matter of routine checking of finding the contradiction by proving that it is not possible to represent the six numbers 84, 81, ... ,69 by means of one of the 22 representations of 87 given in Table 3. For sake of shortness we omit details. This completes the proof of Theorem 2.

An immediate consequence of Theorem 2 is

Corollary 1. *$\Gamma_3(P)$ is finite and consists of exactly 8 parachutes by* $\Gamma_3(P) = \{P_{3,2}, P_{4,3}, P_{5,4}, P_{6,5}, P_{7,6}, P_{8,7}, P_{10,9}, P_{11,10}\}$

5 Finiteness of $\Gamma_4(P)$

Now we turn towards $\Gamma_4(P)$. Putting $b = g - 3$ Proposition 1 becomes

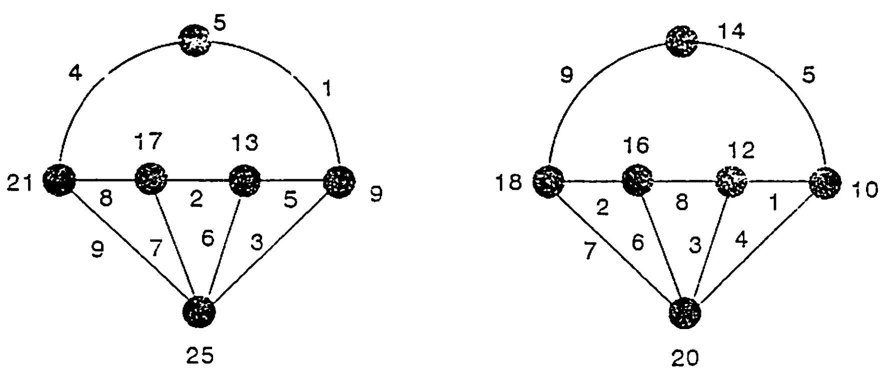
Proposition 5.

$$\bigwedge_{\substack{g \in \mathbb{N} \\ g \geq 4}} [P_{g,g-3} \in \Gamma(p) \rightarrow d \leq 4]$$

Since its proof is an immediate consequence of the proof of Proposition 1 we are allowed to omit details.

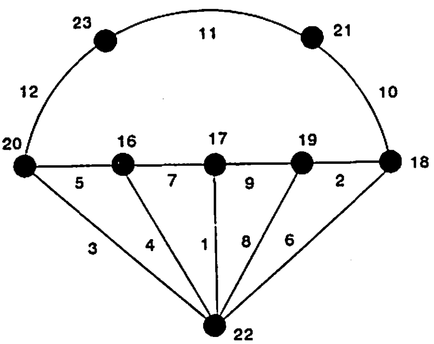
In order to illustrate that $\Gamma_4(P)$ is not empty we depict parachutes $P_{g,g-3} \in \Gamma_4(P)$ and corresponding (a,d)-antimagic labelings for $4 \leq g \leq 9$ in Figure 10.

Applying Proposition 5 we obtain

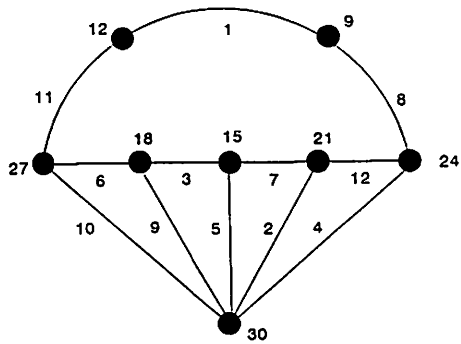


$P_{4,1}$ (5,4)-antimagic $P_{4,1}$ (10,2) -antimagic

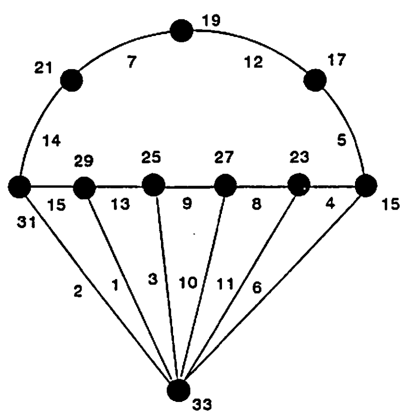
Figure 10



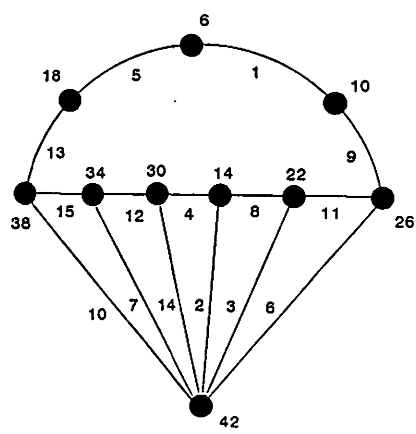
$P_{5,2}$ (16,1)-antimagic



$P_{5,2}$ (9,3)-antimagic

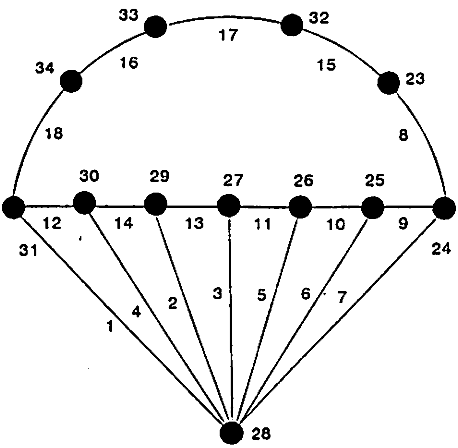


$P_{6,3}$ (15,2)-antimagic

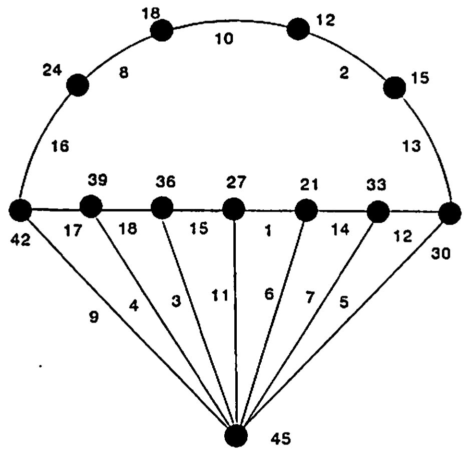


$P_{6,3}$ (6,4)-antimagic

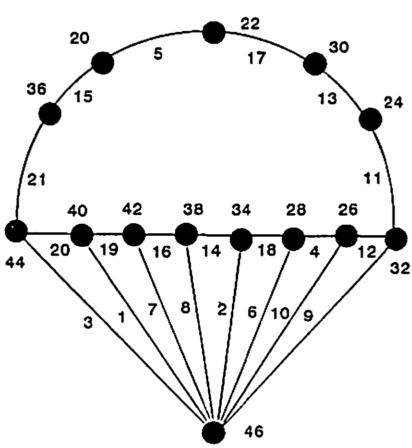
Figure 10 con't



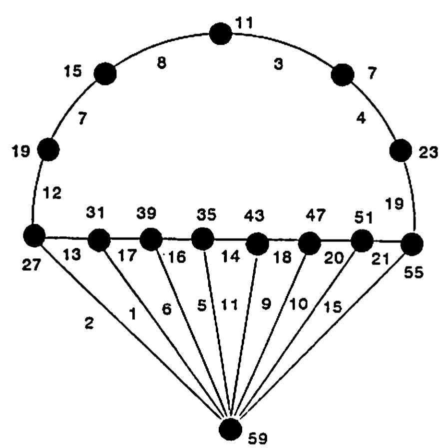
$P_{7,4}$ (23,1)-antimagic



$P_{7,4}$ (12,3)-antimagic

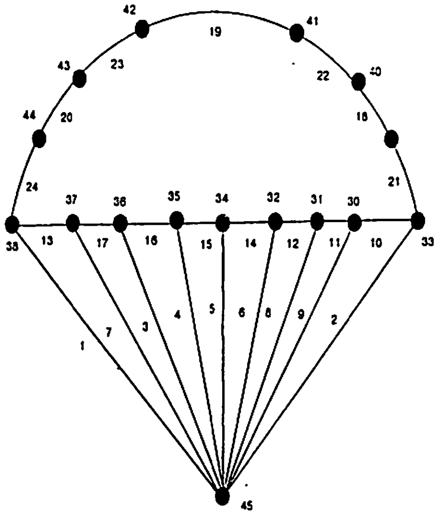


$P_{8,5}$ (20,2)-antimagic

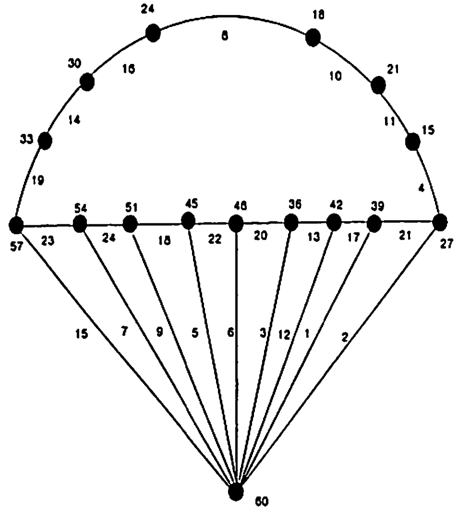


$P_{8,5}$ (7,4)-antimagic

Figure 10 con't



$P_{9,6}$ (30,1)-antimagic



$P_{9,6}$ (15,3)-antimagic

Figure 10 con't

Proposition 6. *Let $P_{g,g-3} \in \Gamma_4(P)$, $g \geq 4$. The corresponding Diophantine equation of $P_{g,g-3}$ is*

(xv) $3(3g-2) = 2a + d(2g-3)$

and has the two different solutions $(7n-5, 1)$ and $(3n, 3)$ if $g = 2n-1$, $n \geq 3$, or $(5n, 2)$ and $(n+3, 4)$ if $g = 2n$, $n \geq 2$.

Proof: Putting $b = g - 3$ the equation (ii) becomes (xv). According to Proposition 5 we know that $d \leq 4$ and we obtain the two solutions of the statement of Proposition 6 by distinguishing the cases g odd and g even.

Now it is possible to point out that $\Gamma_4(P)$ is finite.

Theorem 3. *Let $a \in \mathbf{N}$ be an arbitrary integer ≥ 3.*

(1) $\bigwedge_{g \geq 10} P_{g,g-3}$ is not $(a,1)$-antimagic.

(2) $\bigwedge_{g \geq 11} P_{g,g-3}$ is not $(a,2)$-antimagic.

(3) $\bigwedge_{g \geq 13} P_{g,g-3}$ is not $(a,3)$-antimagic.

(4) $\bigwedge_{g \geq 15} P_{g,g-3}$ is not $(a,4)$-antimagic.

Proof: If $P_{g,g-3}$ denotes an arbitrary parachute in $\Gamma_4(P)$, the maximum vertex value of each (a,d)-antimagic labeling f of $P_{g,g-3}$ is equal to

$$a_{\max} = \frac{9g + 2gd - 3d - 6}{2}.$$

Solving the inequality

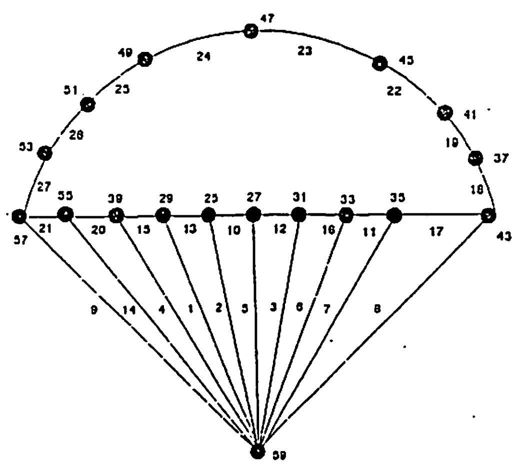
(xvi) $\frac{g(g+1)}{2} > \frac{9g+2gd-3d-6}{2}$

we obtain the statements of Theorem 3. (xvi) is equivalent to

(xvii) $g > d + 4 + \sqrt{d^2 + 5d + 10}$.

Putting $g_0(d) = [4 + d + \sqrt{d^2 + 5d + 10}] + 1$ where $[4 + d + \sqrt{d^2 + 5d + 10}]$ is the greatest integer less than or equal to $4 + d + \sqrt{d^2 + 5d + 10}$. Putting $d = 1, 2, 3$ or 4 we obtain $g_0(1) = 10$, $g_0(2) = 11$, $g_0(3) = 13$ and $g_0(4) = 15$ proving Theorem 3.

As an immediate consequence of Theorem 3 we formulate the following theorem expressing the finiteness of $\Gamma_4(P)$ in the following way:



$P_{10,7}$ (25,2)-antimagic

Figure 11

Theorem 4. *$\Gamma_4(P)$ is finite and consists of $P_{4,1}$, $P_{5,2}$, $P_{6,3}$, $P_{7,4}$, $P_{8,5}$, $P_{9,6}$ and $P_{10,7}$.*

Proof: Because of Figure 10 and Theorem 3 we only need to show that there exists an (a, d)-antimagic labeling of $P_{10,7}$ and that $P_{g,g-3} \notin \Gamma_4(P)$ for $g = 11, \ldots, 14$. The first part is proved by Figure 11 showing a (25,2)-antimagic labeling of $P_{10,7}$. For the sake of completeness it is mentioned that $P_{10,7}$ is not (8,4)-antimagic. As the proof is similar to the one given above in case of $P_{15,14}$ we omit details.

Assume $P_{13,10} \in \Gamma_4(P)$. According to Proposition 6 (a, d) is either (44,1) or (21,3) contradicting Theorem 3 such that $P_{13,10} \notin \Gamma_4(P)$ is true.

Assume $P_{11,8} \in \Gamma_4(P)$. Then (a,d) is equal to (18,3) because the second case $(a,d) = (37,1)$ is not possible due to Theorem 3. Since each vertex label of the (18,3)-antimagic labeling f is a multiple of 3 there are only two essentially different distributions of the residue classes $\overline{0}$, $\overline{1}$, $\overline{2}$ modulo 3 depicted in Figure 12. Then in both distributions,

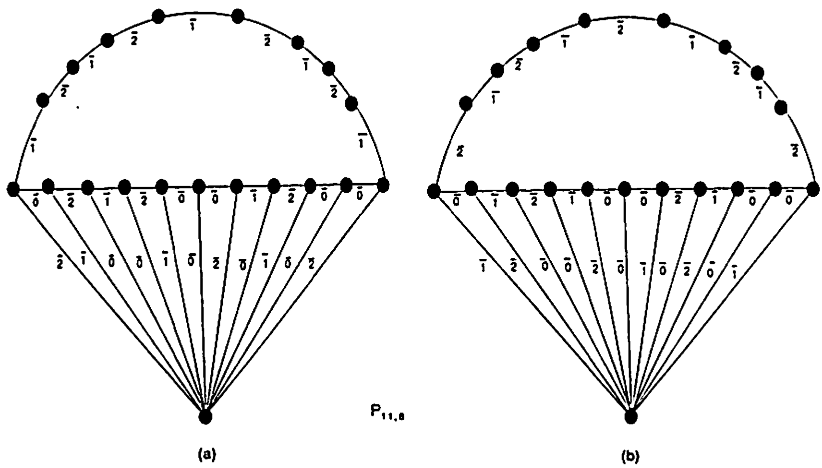


Figure 12

it is necessary that the vertex label $f(v)$ of the vertex v of maximum degree in $P_{11,8}$ has the value 75 and that the 10 vertex labels 72, 69, 66, 63, 60, 57, 54, 51, 48 and 45 have to appear at 10 of the 11 vertices of degree 3 such that the sum

$$72 + 69 + \cdots + 45 = 585.$$

Then the best possible sum of edge labels of f is either

$$73 + 22 + 18 + 42 + 2(30 + 29 + \cdots + 23) = 579$$

in case (a) or

$$73 + 23 + 18 + 2 - (30 + 29 + \cdots + 24) + 2(22 + 21) = 578.$$

This contradiction means that the assumption $P_{11,8} \in \Gamma_4(P)$ is not true.

Now assume that $P_{14,11}$ is an element of $\Gamma_4(P)$. According to Theorem 3 and Proposition 6, (a,d) is necessarily equal to (10,4) such that $a_{\max} = 110$. Since each vertex value of f in $P_{14,11}$ is even the 12 edges incident to at least one vertex of degree 2 must be labeled either by 12 odd or by 12 even numbers in $\{1, 2, \ldots, 39\}$. Discussing all possible cases of distributions one obtains contradictions in any case such that the assumption $P_{14,11} \in \Gamma_4(P)$ is not true.

Finally we assume $P_{12,9} \in \Gamma_4(P)$. According to Theorem 3 and Proposition 6 (a, d) is equal to (9,4) such that $a_{max} = 93$. It is not very difficult to see that it is not possible to obtain the vertex values 89,85, 81, 77, 73, 69, 65, 61, 57 as sums of three summands from $\{1, 2, \ldots, 33\}$ for because of $89 = 33+32+24$ and $85 = 33+31+21$ the sum $24+21+(1+\cdots+10) > 93$. This completes the proof of Theorem 4.

After this finiteness proof for $\Gamma_4(P)$ it remains to investigate the set $\Gamma(P) \setminus (\Gamma_3(P) \cup \Gamma_4(P))$ where we conjecture that this set is infinite. It turns out that there are very nice parachutes in $\Gamma(P) \setminus (\Gamma_3(P) \cup \Gamma_4(P))$. For sake of length we will give this infiniteness proof in a further paper.

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