Enumerating Oriented Triangle Graphs

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ABSTRACT. With Burnside's lemma as the main tool we derive a formula for the number of oriented triangle graphs and for the number of such graphs in which all largest cliques are transitively oriented.

1 Introduction

Let $n \geq 2$. D_n denotes the set of the unordered pairs $\{x, y\}$, with $x, y \in \{1, 2, \ldots, n\}$, $x \neq y$. We shall often write the elements of D_n as xy and call them *dyads*. The *triangle graph* T_n has D_n as its point set and two points are adjacent if and only if they are not disjoint. T_5, for instance, is the complement of the Petersen graph. T_n has $\binom{n}{2}$ points and $(n-2)\binom{n}{2}$ edges. For $i = 1, 2, \ldots, n$ the set $D_{n,i} = \{ij | j \neq i\}$ of order $n-1$ spans a complete subgraph $T_{n,i}$, and these subgraphs are edge-disjoint. If a subset of D_n consists of mutually non-disjoint dyads these dyads have a common element, unless the subset has the form $\{xy, yz, zx\}$. It follows that if $n > 4$, the $D_{n,i}$ are the cliques of maximal size in T_n.

Replacing the edge $\{ij, ik\}$ by either the arc (ij, ik) or the arc (ik, ij) for all $i \neq j \neq k \neq i$ we get an *oriented triangle graph*, in which the above subgraphs $T_{n,i}$ become tournaments (a *tournament* is a directed graph in

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which there is precisely one arc between every pair of points; [7] is a general reference on tournaments). By Ω_n we denote the set of all oriented triangle graphs with underlying graph T_n.

These graphs often arise in the social sciences in the study of the perception of similarity. For instance, in the *complete method of triads* ([8], p. 263; [4], p. 92) one successively presents to a person all triples $\{a, b, c\}$ from a set of, say, colors or shapes. He is asked to decide, for all three pairs $\{xy, xz\}$ of dyads with $\{x, y, z\} = \{a, b, c\}$, whether y or z is more similar to x, i.e., to produce the arc (xy, xz) or the arc (xz, xy). In the method of *multidimensional rank order* ([8], p. 263) a subject ranks, for every color x, the other colors in order of degree of similarity to x, thus producing transitive orientations of the $T_{n,i}$. Such orientations can also be derived from data known as *conditional proximities*, common in psychology ([2], p. 422; [3], p. 154)). These consist of a *proximity measure* $s(x, y)$ for each ordered pair (x, y), for instance the fraction of times a subject reports to have seen y while x was presented. For each x the $s(x, y)$ (provided they are mutually different) yield a linear order on the set of dyads xy. Circularity in the resulting oriented triangle graphs is studied in [1].

If G is a group, V a set, and ϕ a homomorphism of G into the symmetric group on V, we say that G *acts* on V, by ϕ, and that G is *represented* by ϕ as the permutation group $\phi(G)$ on V. Each permutation π in the symmetric group S_n induces a permutation ϕ_π of D_n, defined by $\phi_\pi\{i, j\} = \{\pi(i), \pi(j)\}$. Clearly S_n acts on D_n by ϕ. Moreover ϕ_π is an automorphism of T_n: its permutes the points of T_n (the dyads) preserving adjacency (non-disjointness). Thus $\phi(S_n)$ is a subgroup of $Aut(T_n)$, the subgroup of $\mathrm{Sym}(D_n)$ that consists of the automorphisms of T_n. Clearly the representation ϕ is *faithful*, i.e., injective, if $n > 2$. If $n \neq 2, 4$ it is also surjective, which we see as follows. An automorphism ψ of T_n permutes the cliques $D_{n,i}$, so there is a $\pi \in S_n$ such that ψ maps $T_{n,i}$ onto $T_{n,\pi(i)}$, and the dyad ij common to $T_{n,i}$ and $T_{n,j}$ must be mapped by ψ onto the dyad $\pi(i)\pi(j)$ common to $T_{n,\pi(i)}$ and $T_{n,\pi(j)}$, so $\psi = \phi_\pi$.

For $\pi \in S_n$ and $\Delta \in \Omega_n$ we define $\Phi_\pi(\Delta)$ to be the digraph in Ω_n having as arcs the $\big(\phi_\pi(xy), \phi_\pi(xz)\big) = \big(\pi(x)\pi(y), \pi(x)\pi(z)\big)$ for which (xy, xz) is an arc in Δ. Then S_n acts on Ω_n by Φ and Φ_π is a graph isomorphism from Δ to $\Phi_\pi(\Delta)$.

Since every isomorphism between elements of Ω_n is an automorphism of T_n, we have

Lemma 1. *Every isomorphism between two elements of Ω_n, $n \neq 4$, is induced by some permutation in S_n, which is unique if $n \neq 2$.*

Thus the isomorphism classes of oriented triangle graphs are the *orbits* $\{\Phi_\pi(\Delta) | \pi \in S_n\}$ in Ω_n under the action Φ of S_n, provided $n \neq 4$. (For the applications mentioned above this means that isomorphic 'patterns' of similarity arising from two sets of 'stimuli', both numbered $1, \dots, n$, can be made equal by suitably renumbering one of the sets.)

2 The number of oriented triangle graphs

We first state Burnside's Lemma, which gives a formula for the number of orbits:

Burnside's Lemma: Let G be a group that acts on a set V by Φ. For $g \in G$ let $V_g = \{v \in V | \Phi_g(v) = v\}$ be the set of fixed points of g. Then the number of orbits in V under the action Φ of G is

$$|G|^{-1} \sum_{g \in G} |V_g|.$$

For a proof we refer to [5] (Corollary 15.3(a)), [6] (Theorem 5-2) or [7] (Theorem 40). In fact in these references the lemma is stated for groups of permutations of V only (our $\Phi(G)$), but the proof is easily adapted for the slightly more general form above.

To be able to apply Burnside's Lemma we must determine the number of fixed points in Ω_n for each π in S_n.

Let $\pi \in S_n$ with $\pi(x) = x$, $\pi(y) = z$ and $\pi(z) = y$ for a certain triple $\{x, y, z\}$. If (xy, xz) is an arc of $\Delta \in \Omega_n$, then (xz, xy) is an arc in $\Phi_\pi(\Delta)$, so $\Phi_\pi(\Delta) \neq \Delta$; likewise if (xz, xy) is an arc of Δ. Using this fact we first show that only a few elements of S_n have a non-empty set of fixed points in Ω_n.

Lemma 2. *Let $\pi \in S_n$. There is a $\Delta \in \Omega_n$ with $\Phi_\pi(\Delta) = \Delta$ if and only if there is an i such that the length of every cycle of π is divisible by 2^i but not by 2^{i+1}.*

Proof: Let π induce an automorphism of $\Delta \in \Omega_n$. Suppose π has a cycle of length $2^i r$ and one of length $2^j s$ with r and s odd and $i < j$. Then $\gcd(2^i r, 2^j s) = 2^i \gcd(r, s)$ is a divisor of $2^{j-1}s$, so there are integers p and q with $2^{j-1}s = 2^i rp + 2^j sq$. Now, $\pi^{2^i rp}$ induces an automorphism of Δ and fixes the elements of the first cycle. However, since $2^i rp \equiv 2^{j-1}s \pmod{2^j s}$, it pairwise interchanges the elements of the second cycle. As we have seen above this is impossible. Conversely, let π be a permutation satisfying the condition in the lemma. The edge set of the undirected graph T_n is permuted by ϕ_π. Pick one edge out of each cycle of this permutation and orient it in an arbitrary way. If the edge $\{xy, xz\}$ in a cycle of length m is oriented as (xy, xz) we orient the other edges of the cycle as $(\pi^j(x)\pi^j(y), \pi^j(x)\pi^j(z))$, for $j = 1, 2, \dots, m-1$. We claim that we now have an oriented triangle graph on which π induces an automorphism. In fact it suffices to prove that $(\pi^m(x)\pi^m(y), \pi^m(x)\pi^m(z)) = (xy, xz)$.

Now $\{\pi^m(x)\pi^m(y), \pi^m(x)\pi^m(z)\} = \{xy, xz\}$ so $\pi^m(x) = x$ and either $\pi^m(y) = y$, $\pi^m(z) = z$ or $\pi^m(y) = z$, $\pi^m(z) = y$. The second, however, is impossible, since then the length $2^i s$ of the cycle of y and z in π is a divisor of $2m$ but not of m, so $2^{i-1}|m$ but not $2^i|m$, contradicting the assertion that m is a multiple of the length $2^i r$ of the cycle of x. □

We now determine the number of fixed points in Ω_n of a permutation π in S_n, assuming that π satisfies the condition in Lemma 2 (for all other π that number is 0). The proof of that lemma shows that the required number is $2^{\lambda(\pi)}$, where $\lambda(\pi)$ is the number of cycles of the permutation of the edges of T_n induced by π. We call these cycles e-cycles. Let $\Gamma(x, yz)$ denote the e-cycle that contains the edge $\{xy, xz\}$. Let l_x, l_y, l_z be the lengths of the (not necessarily different) cycles of π that contain x, y and z, respectively. Then the length of $\Gamma(x, yz)$ is lcm (l_x, l_y, l_z) (cf. the proof of Lemma 2). We note that $\Gamma(\pi(x), \pi(y)\pi(z)) = \Gamma(x, yz) = \Gamma(x, zy)$, so when determining the $\Gamma(x, yz)$ we need only take one fixed x from every cycle of π and combine it with unordered pairs $\{y, z\}$. We distinguish four cases, that are clearly disjoint.

1) Suppose that y and z are in the same cycle as x. We have $\binom{l_x - 1}{2}$ choices for $\{y, z\}$, each giving another e-cycle $\Gamma(x, yz)$, since $\pi^i(x) \neq x$ for $i = 1, 2, \ldots, l_x - 1$; so $\{xy, xz\}$ occurs in $\Gamma(x, vw)$ only if $\{v, w\}=\{y, z\}$.

2) Suppose that either y or z, say y, are in the same cycle as x and that z is in another cycle. We have $(l_x - 1)l_z$ ways of choosing $\{y, z\}$. $\Gamma(x, yz) = \Gamma(x, y'z')$ if and only if $\Gamma(x, yz)$ contains $\{xy', xz'\}$, and thus if and only if $y' = \pi^i(y)$ and $z' = \pi^i(z)$, where i is a multiple of l_x in $\{0, 1, \ldots, \text{lcm}\,(l_x, l_z) - 1\}$. There being $l_x^{-1}\,\text{lcm}(l_x, l_z)$ of such i, the number of e-cycles we find is

$$\frac{(l_x - 1)l_z}{l_x^{-1}\text{lcm}(l_x, l_z)} = (l_x - 1)\gcd(l_x, l_z).$$

3) Suppose that y and z are in the same cycle, other than that of x. We have $\binom{l_y}{2}$ ways of choosing $\{y, z\}$. As in the end of the proof of Lemma 2 it can be shown that $\Gamma(x, yz) = \Gamma(x, y'z')$ iff $y' = \pi^i(y)$ and $z' = \pi^i(z)$ with i a multiple of l_x in $\{0, 1, \ldots, \text{lcm}\,(l_x, l_y) - 1\}$. Hence the number of e-cycles is

$$\frac{1}{l_x^{-1}\text{lcm}(l_x, l_y)}\binom{l_y}{2} = \frac{1}{2}(l_y - 1)\gcd(l_x, l_y).$$

4) Suppose that x, y and z are all in different cycles. We have $l_y l_z$ ways of choosing $\{y, z\}$. We have $\Gamma(x, yz) = \Gamma(x, y'z')$ iff $y' = \pi^i(y)$ and

$z' = \pi^i(z)$ with i a multiple of l_x in $\{0, 1, \ldots, \text{lcm }(l_x, l_y, l_z) - 1\}$. Hence the number of e-cycles is

$$\frac{l_y l_z}{l_x^{-1}\text{lcm}(l_x, l_y, l_z)} = \frac{l_x l_y l_z}{\text{lcm}(l_x, l_y, l_z)}.$$

Note that $\Gamma(x, yz)$ determines only the cycle of x and the unordered pair of the cycles of y and z. This explains the $\frac{1}{2}$ in the following lemma, which collects the above results.

Lemma 3. *Let π be a permutation in S_n which satisfies the requirements of Lemma 2. Let t be the number of cycles of π and let $l_1, l_2, \ldots, l_t$ be their lengths. Then the number of $\Delta \in \Omega_n$ fixed by Φ_π is $2^{\lambda(\pi)}$ where $\lambda(\pi)$ equals*

$$\sum_i \binom{l_i - 1}{2} + \frac{3}{2}\sum_i \sum_{j \neq i} (l_i - 1)\gcd(l_i, l_j) + \frac{1}{2}\sum_i \sum_{j \neq i} \sum_{k \neq i,j} \frac{l_i l_j l_k}{\text{lcm}(l_i, l_j, l_k)}.$$

Remark: Leaving the proof to the reader we note that the formula for $\lambda(\pi)$ in the lemma can be rewritten as

$$t - \frac{3}{2}\sum_{i,j} \gcd(l_i, l_j) + \frac{1}{2}\sum_{i,j,k} \frac{l_i l_j l_k}{\text{lcm}(l_i, l_j, l_k)}.$$

Using Burnside's Lemma we now have:

Theorem 4. *For $n \neq 4$ the number of isomorphism classes of oriented triangle graphs on $\binom{n}{2}$ points is*

$$\frac{1}{n!}\sum_{\pi \in S'_n} 2^{\lambda(\pi)}$$

where S'_n is the subset of S_n consisting of the permutations in S_n that satisfy the condition of Lemma 2 and $\lambda(\pi)$ is defined as in Lemma 3.

Note that also for $n = 4$ the number in the theorem is the number of orbits in Ω_n under the action of S_n.

To illustrate the calculation we tabulate the results for $n = 5$. In the table s is the number of permutations of the particular type, λ_j is the number of e-cycles found as in case j) above for a permutation of that type, and λ is the total number of e-cycles.

$n = 5$							
type	s	λ_1	λ_2	λ_3	λ_4	λ	$s2^\lambda$
1^5	1	0	0	0	30	30	2^{30}
$1^2 3^1$	20	1	4	2	3	10	$5 \cdot 2^{12}$
5^1	24	6	0	0	0	6	$3 \cdot 2^9$
$\sum_{\pi \in S_5'} 2^{\lambda(\pi)} = 1,073,763,840$							
number of oriented triangle graphs on 10 points: $8,948,032$							

Table 1

For some other values of n the results can be found in table 2 in the next section.

3 The number of oriented triangle graphs in which all the largest cliques are transitively oriented.

For $\Delta \in \Omega_n$ let Δ_i be the subgraph spanned by the $n-1$ dyads ij, $j \neq i$. The Δ_i, having the $T_{n,i}$ as underlying graphs, are the largest tournaments in Δ if $n \neq 3, 4$. Let Ω_n' consist of those $\Delta \in \Omega_n$ for which all Δ_i are transitive (such a Δ would be the outcome of an experiment in which, for all x, the pairs xy of stimuli have been put into a linear order).

Lemma 5. *Let $\pi \in S_n$. There is a $\Delta \in \Omega_n'$ with $\Phi_\pi(\Delta) = \Delta$ if and only if all cycles of π have the same length.*

Proof: Suppose $\Delta \in \Omega_n'$ is fixed by $\pi \in S_n$. Let C_1 and C_2 be two cycles of π of lengths l_1 and l_2, respectively, with $l_1 \neq l_2$, say $l_1 < l_2$. Δ is also fixed by $\sigma = \pi^{l_1}$. Let x and y be elements of the cycles C_1 and C_2, respectively, and let $(y_1 y_2 \cdots y_k)$ be the cycle of σ containing y. Then $k > 1$. Either (xy_1, xy_2) or (xy_2, xy_1) is an arc in Δ. In the first case applying σ we see that also $(xy_2, xy_3), (xy_3, xy_4), \ldots, (xy_k, xy_1)$ are arcs; so we would have a circuit, contradicting the transitivity of Δ_x. Likewise in the second case. So $l_1 = l_2$. Conversely, suppose that all cycles of $\pi \in S_n$ have equal length l. Pick elements $x_1, x_2, \ldots, x_k$, one from every cycle ($k = n/l$). Orient each of the complete subgraphs T_{n,x_i}, $i = 1, 2, \ldots, k$ as a transitive tournament (there are $\left((n-1)!\right)^k$ ways to do so). For any other $T_{n,y}$ there is a unique pair $\langle j, r\rangle$ with $j = 1, 2, \ldots, l-1$ and $r = 1, 2, \ldots, k$ such that $y = \pi^j(x_r)$, and we take as arcs the $\left(y\pi^j(v), y\pi^j(w)\right)$ for which $(x_r v, x_r w)$ is an arc. Thus $T_{n,y}$ is transitively oriented and it is easily checked that π induces an automorphism of the resulting oriented triangle graph. □

Theorem 6. *For $n > 4$ the number of oriented triangle graphs on $\binom{n}{2}$*

points with transitively oriented maximum cliques is

$$\sum_{d|n} \frac{d^d((n-1)!)^d}{n^d d!}.$$

Proof: Again, by Burnside's Lemma, we have only to consider the $\pi \in S_n$ satisfying the condition in Lemma 5. For a given d which divides n, the permutations having all their cycles of length d can be constructed by first selecting n/d disjoint subsets of order d, which can be done in

$$\binom{n}{d}\binom{n-d}{d}\cdots\binom{d}{d}\Big/\left(\frac{n}{d}\right)! = n!\Big/\left((d!)^{\frac{n}{d}}\left(\frac{n}{d}\right)!\right)$$

ways. Each subset yields a cycle in $(d-1)!$ ways. So the number of permutations π with cycles of length d is

$$n!\Big/\left(d^{\frac{n}{d}}\left(\frac{n}{d}\right)!\right).$$

Since by the proof of the lemma each permutation leaves invariant $((n-1)!)^{n/d}$ elements of Ω'_n, by Burnside's Lemma the number of orbits is:

$$\sum_{d|n} \frac{((n-1)!)^{\frac{n}{d}}}{d^{\frac{n}{d}}(\frac{n}{d})!}.$$

Substituting $\frac{n}{d}$ for d we obtain the formula stated in the theorem. □

If n is a prime, the formula reduces to $n^{-1}\left((n-1)! + ((n-1)!)^{n-1}\right)$. As in section 3 the number in the theorem is the number of orbits in Ω'_n under S_n for $n = 4$ as well. We tabulate some values resulting from Theorems 4 and 6, including the values for the case $n = 4$, to be discussed in the next section.

n	oriented triangle graphs on $\binom{n}{2}$ points	oriented triangle graphs on $\binom{n}{2}$ points with trans. oriented maximum cliques
2	1	1
3	2	1
4	112	18
5	8,948,032	66,360
6	1,601,279,890,171,392	4,147,236,620
7	8,048,575,239,544,313,784,372,575,680	19,902,009,929,142,960

Table 2

4 The case $n = 4$

Ω_4 consists of the 2^{12} orientations of T_4, which is the graph of the octahedron. It contains 8 triangles of which 4 are spanned by triples $\{xy, xz, xw\}$ and 4 by triples $\{xy, yz, zx\}$. S_4 contains 1, 6, 8, 3, 6 permutations of type $1^4, 1^2 2^1, 1^1 3^1, 2^2, 4^1$, respectively. Except for $1^2 2^1$ they satisfy the requirements of Lemma 2, and by the formula of Theorem 4 (see the note just after it) we find that there are

$$24^{-1}(2^{12} + 8 \cdot 2^4 + 3 \cdot 2^6 + 6 \cdot 2^3) = 186$$

orbits in Ω_4 under S_4.

The automorphism group K of T_4 has order 48. The 24 automorphisms ϕ_π, $\pi \in S_4$, permute the triangles of type $\{xy, xz, xw\}$. The other 24 are the $\psi\phi_\pi$, where ψ is the permutation of D_4 mapping every dyad onto its complement in $\{1, 2, 3, 4\}$ (geometrically the reflection of the octahedron in its centre). They interchange the triangles of type $\{xy, xz, xw\}$ with those of type $\{xy, yz, zx\}$. Note that $\psi\phi_\pi = \phi_\pi\psi$.

We now apply Burnside's Lemma for K. The sum of the numbers of fixed points of the ϕ_π we know already: $24 \cdot 186$. For the $\psi\phi_\pi$ with π of type $1^4, 1^2 2^1, 1^1 3^1, 2^2, 4^1$, respectively, the number of cycles of the permutation induced on the edge-set of T_4 is easily checked to be 6, 7, 2, 8, 3, respectively. Only in the case of $1^2 2^1$ it turns out to be impossible to orient the edges in every cycle in such a way that $\psi\phi_\pi$ becomes an automorphism. (If, for instance, $\pi = (1\,2)(3)(4)$, then $\psi\phi_\pi$ interchanges the dyads 13 and 14, so it would map the arc (non-arc) (13, 14) onto the non-arc (arc) (14, 13).)

Thus the number of oriented triangle graphs on 6 points is

$$48^{-1}(24 \cdot 186 + 2^6 + 8 \cdot 2^2 + 3 \cdot 2^8 + 6 \cdot 2^3) = 112.$$

Let Ω_4'' be the set of all $\Delta \in \Omega_4$ having all 8 triangles transitively oriented. Ω_4'' is invariant under the action of K as well as under that of G. We want to count the number of orbits under K. Note that $\Omega_4'' \subset \Omega_4'$ but that Ω_4' is not invariant under K, since there are $\Delta \in \Omega_4'$ in which one or more triangles of type $\{xy, yz, zx\}$ are oriented cyclically. Ω_4' contains $6^4 = 1296$ elements, since each $T_{4,i}$ can be oriented as a transitive triangle in 6 ways. According to the formula in theorem 6 they form 60 orbits under S_4. Elements of Ω_4' that are not in the same of the 60 orbits could only be equivalent under the larger group if they belong to Ω_4'' .

We first determine the number of elements in Ω_4'', omitting details. By inspection (use a planar drawing of the octahedral graph) one can establish that there are $2 \cdot 3^3 \cdot 6 = 324$ ways to orient T_4 such that all 4 triangles $\{xy, xz, xw\}$ become transitive whereas the triangle $\{23, 34, 42\}$ becomes a cycle. For $2 \cdot 2^2 \cdot 3^2 + 2 \cdot 3^2 = 90$ of these also $\{13, 34, 41\}$ becomes a cycle, and for $2^4 + 2^3 = 24$ of the latter $\{12, 24, 41\}$ becomes a cycle too. Finally, in 6 cases all four triangles $\{xy, yz, zx\}$ become cycles. Using inclusion and exclusion we find that the cardinality of Ω_4'' is

$$1296 - \binom{4}{1} \cdot 324 + \binom{4}{2} \cdot 90 - \binom{4}{3} \cdot 24 + \binom{4}{4} \cdot 6 = 450$$

(which we confirmed by computer search).

We now determine how many of the 60 orbits under S_4 in Ω_4' are in Ω_4'', i.e. we determine the number of orbits in Ω_4'' under S_4. By Lemma 5 we have only to look at id., the 3 permutations of type 2^2 and the 6 of type 4^1. Careful inspection (again we omit details) shows that they leave invariant 450, 26 and 4 elements, respectively, so the number of orbits is

$$24^{-1}(450 + 3 \cdot 26 + 6 \cdot 4) = 23.$$

Another inspection shows that for π of type $1^4, 1^2 2^1, 1^1 3^1, 2^2, 4^1$ the permutation $\psi\phi_\pi$ leaves invariant 18, 0, 0, 82, 8, respectively, of the 450 elements of Ω_4''. So under K the number of orbits in Ω_4'', and therefore the number of oriented triangle graphs on 6 points with all maximal cliques transitively oriented, is

$$48^{-1}(24 \cdot 23 + 1 \cdot 18 + 3 \cdot 82 + 6 \cdot 8) = 18.$$

We have seen that the 23 orbits in Ω_4'' under S_4 reduce to 18 under K. We have also seen that, under K, there is no further equivalence between the 37 orbits in $\Omega_4' - \Omega_4''$. Each of them, however, is mapped by ψ onto one of the 126 orbits in $\Omega_4 - \Omega_4'$. Apparently the remaining 89 orbits (consisting of the elements in which a triangle of type $\{xy, xz, xw\}$ as well as one of type $\{xy, yz, zx\}$ is cyclically oriented), reduce to $112 - 18 - 37 = 57$.

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