The chromatic number of the union of two forests

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ABSTRACT. Let F and F' be two forests sharing the same vertex set V such that $E(F) \cap E(F') = \emptyset$. By $F \cup F'$ we denote the graph on V with edge set $E(F) \cup E(F')$. Since both F and F' are 2-colorable, we have $\chi(F \cup F') \leq 4$. In this paper we will investigate forests for which we can actually obtain this upper bound for the chromatic number. It will turn out that if neither F nor F' contain a path of length three then the chromatic number of $F \cup F'$ is at most three. We will characterize those pairs of forests F and F' which both contain a path of length three and for which the chromatic number of $F \cup F'$ is always at most three. In the case where one of the forests contains a path of length three and the other does not contain a path of length three we obtained only partial results. The problem then seems to be similar to a problem of Erdos which recently has been solved by Fleischner [2] using a theorem of Alon [3].

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1 Introduction

Let F and G be two graphs with the same number of vertices. A bijection $\sigma: V(F) \to V(G)$ is a packing from F into G if for every pair a and b of adjacent vertices of F the vertices $\sigma(a)$ and $\sigma(b)$ are not adjacent in G. Of course there is a packing from F into G if and only if there is a packing from G into F. Hence we are justified in saying there is a packing of F and G. The chromatic member of the packing σ from F into G is the chromatic number of the graph $\sigma(F) \cup G$ whose set of vertices is the set $V(G)$ and whose edge set is $\sigma(E(F)) \cup E(G)$ where $\sigma(E(F)) = \{\{\sigma(a), \sigma(b)\}: \{a, b\} \in E(F)\}$. We will say that the packing σ contains the complete graph K_4 as a subgraph if the graph $\sigma(F) \cup G$ contains K_4. Note that if both F and G are forests and the packing σ from F into G contains K_4 then σ maps a path of length three of F to a path of length three of G so that those two paths form the complete graph K_4. This is so because the only way to partition the edge set of K_4 into two forests is to partition the edge set of K_4 into two paths of length three.

A packing of two forests has chromatic number at most four because the chromatic number of a forest is at most two. The forests F for which there exists a 4-chromatic packing from F into F have been completely characterized in [1]. Namely, there exists a 4-chromatic packing from a forest F to F itself if and only if F contains a path of length three and F does not consist of exactly two connected components where one of the two components is a path of length three and the other is a star. In the case of two different forests F and F' the situation is much more difficult. We will prove, Theorem 1, that if neither F nor F' contains a path of length three then F and F' do not have a 4-chromatic packing. If both forests F and F' contain a path of length three then they have a 4-chromatic packing unless every connected component of F and F' is a star or a path of length three, none of them contains an isolated vertex and one of them consists of exactly two connected components, a star and a path of length three. (Theorem 2).

If every connected component of one of the two forests is a star and the other forest contains a path of length three we conjecture that they do not have a -chromatic packing. We can not prove this conjecture but can show that if it fails then it fails only in "large" instances. There is an interesting way of looking at this problem. Assume that G is a graph of chromatic number four. Is it possible to color some ofthe edges of G blue so that the blue graph consists of vertex disjoint stars and every circuit of G contains at least one blue edge?

If G is a graph and $A \subseteq V(G)$ a set of verffces of G then we denote by $G - A$ the graph with $V(G - A) = V(G) - A$ in which any two vertices are adjacent if and only if the are adjacent in G. If G is a graph and H a subgraph of G then $G-H = G-V(H)$. The length of a path is the number of its eyes. The diameter of a forest F is the length of a latest path which is a subgraph of F. We denote by P_n the path of length n. An endpoint of a graph G is any vertex of G whose valence is at most one. A vertex of G which is not an endpoint is called interior point.

We will need several special graphs. A star is a tree with at most one vertex whose valence is larger than one. A star contains at least one vertex. The tree M_1 having five edges is constructed from P_4 by adding an additional endpoint to P_4 which has distance two from one of the endpoints of P_4. The tree M_2 having five edges is constructed from P_4 by adding an additional endpoint to P_4 which has distance three from both endpoints of P_4. The tree M_3 having five edges is constructed from P_3 by adding two additional endpoints to P_3 in such a way that each one of them is adjacent to one of the interior points of P_3 and so that they have distance three from each other. A broom is any tree which has been constructed from a star having at least three vertices by adding an additional endpoint adjacent to one of the endpoints of the star. Note that every broom contains a path of length three.

The endpoints a and b are a reducing pair of endpoints of the forest F if they are either in different connected components of F or their distance is larger than two. The reducing pair a, b of endpoints of F preserves a property P if F and also $F-\{a,b\}$ have property P. The forest F is reduced under property P if F has property P but does not contain a reducing pair of endpoints which preserves property P.

We will use the following three packing lemmas of forests.

Lemma 1. *Let F be a forest with the reducing pair a, b of endpoints and F' a forest with the reducing pair a', b' of endpoints. Assume that σ is a packing from $F - \{a,b\}$ to $F' - \{a',b'\}$. Then there is an extension of σ to a packing from F to F'.*

Proof: Denote by c the vertex of F adjacent to a and by d the vertex of F adjacent to b. Note that $c \neq d$. Denote by c' the vertex of F' adjacent to a' and by d' the vertex of F' adjacent to b'. Note that $c' \neq d'$. Let σ be a packing from $F - \{a,b\}$ to $F' - \{a',b'\}$. After possible renaming of a and b and accordingly c and d we may assume that $\sigma(c) \neq d'$ and $\sigma(d) \neq c'$. We can then extend σ to a packing from F to F' by putting $\sigma(a) = b'$ and $\sigma(b) = a'$. □

We will use this Lemma as follows. Assume that we wish to prove that whenever F is a forest having some property P and G is a forest having property Q then there is a packing (4-chromatic packing) from F into G. We classify all reduced forests having property P and all reduced forests having property Q and prove that every forest reduced under property P has a packing (4-chromatic packing) with every forest having property

Q and every forest reduced under property Q has a packing (4-chromatic packing) with every forest having property P. The first examples of this are the following two Lemmas.

Lemma 2. *If G and F are two forests with the same number of vertices and F contains an isolated vertex then there exists a packing from G into F.*

Proof: Using Lemma 1 we may assume that either G is reduced, which implies that it is a star, or that F is reduced under the property that it has an isolated vertex. In the first case it is easy to see that G and F have a packing by mapping the center of the star G to the isolated vertex of F. In the second case F has exactly two connected components, one being the isolated point and the other a star, say S. We obtain a packing σ from G into F by mapping some endpoint e of G to the center of S and the vertex adjacent to e to the isolated point of F. All the other vertices of G are mapped to the endpoints of the star S. □

Lemma 3. *If F and G are two forests each having at least two connected components then there is a packing from F to G.*

Proof: Using Lemma 1 we may assume that one of the two forests, say F, is reduced under the property of having at least two connected components. The reduced forest F contains then an isolated vertex. The Lemma follows from Lemma 2. □

2 The forests do not contain a path of length three

Theorem 1. *If neither one of two forests contains a path of length three then they do not have a 4-chromatic packing.*

Proof: Assume for a contradiction that F and G are two forests which do not contain a path of length three and that they have a 4-chromatic packing σ from F into G. We will also assume that the number of vertices n of F and G is minimal under those conditions. If σ maps an endpoint of F to an endpoint of G then n is not minimal. (Note that if H is a graph with chromatic number four containing a vertex x of valence at most two then $G - x$ also has chromatic number four.) Every connected component of a forest which does not contain a path of length three is a star. Hence if such a forest has n vertices it contains at least $\frac{2}{3}n$ endpoints and hence at most $\frac{1}{3}n$ interior points. In order for n to be minimal the member of endpoints of F can be at most as large as the number of interior points of G. But this would imply the contradiction $\frac{2}{3}n \leq \frac{1}{3}n$ for $n \geq 1$. □

3 Both forests contain a path of length three

Definition: A P_3-star-forest is a forest which does not contain any isolated points and in which every connected component is a path of length three or a star. A P_3-star-forest is simple if it contains exactly two connected components one of which is a path of length three and the other a star.

Theorem 2. *Two forests having diameter at least three and with the same number of vertices do not have a 4-chromatic packing if and only if one of the two forests is a P_3-star-forest and the other is a simple P_3-star-forest.*

Proof: We will first establish a series of nine claims.

Claim 1: A forest F washes reduced under the property of containing the path P_3 as a subgraph is either a broom or P_4 or a forest with two components one of which is P_3 and the other is a star.

Proof of Claim: Assume that the forest F is reduced under the property of containing the path P_3. If F contains at least three connected components then there are two of them with endpoints, say a in one and b in the other, such that $F - \{a, b\}$ still contains a P_3. Hence we can assume that F contains at most two connected components.

If F contains exactly two components then one of them must be a star, otherwise we could remove two endpoints with distance larger than two from one of the connected components and the other one would still have diameter two. If the component which is not a star is not a path of length three we could remove an appropriate endpoint from it together with an endpoint from the component which is a star and retain the property of containing a path of length three.

We are left with the case that F is a tree of diameter at least three. Hence F contains two endpoints a and b which have distance larger than two and $F - \{a, b\}$ is a star. Denote the center of this star by c. Because F has diameter at least three not both of the vertices a and b can be adjacent to c. If one of them is adjacent to c and the other is not then F is a broom. If both of them are not adjacent to c then F contains a path, say a, a_1, c, b_1, b of length four. If the star $F - \{a, b\}$ would contain any endpoint d beside a_1 and b_1 then a, d would be a reducing pair of endpoints of F such that $F - \{a, d\}$ would still have diameter three. Hence F is a path of length four. □

Definition: A forest is coarse if it contains a connected component with at least four edges and diameter at least three.

Claim 2: If F is a forest, which is reduced under the property of being coarse, then it is one of the following:

a) A broom with at least four edges or a path of length four or five or one of M_1, M_2, M_3.

b) A forest with exactly two connected components one of them being a star and the other one a path of length four or a broom with four edges .

Proof of Claim: Assume that the forest F is reduced under the property of being coarse . As in the proof of Claim 1 we argue that F has at most two connected components. If F has exactly one connected component then F is either reduced under the property of having diameter at least three or it has two endpoints a and b with distance larger than two such that $F - \{a, b\}$ is a path of length three. In the first case it follows from Claim 1 that F is either a broom with at least four edges or a path of length four. In the second case F is either a path of length five or one of the graphs M_1, M_2, M_3.

If F has two connected components then as in the proof of Claim 1, one of the two connected components must be a star S. The other connected component is a tree T which is reduced under the property of having at least four edges and diameter at least three. We proved in the raves paragraph that T is one of the trees listed under a) in the statement of Claim 2. If T has five edges then F can be reduced by an appropriate endpoint of T and an endpoint of the star S preserving the property that F is coarse. Hence T is either a path of length four or the broom with four edges . □

Claim 3: If B is a broom and F a forest containing a path of length three and B and F have the same number of vertices then B has a 4-chromatic packing into F.

Proof of Claim: Let a_1, b_1, c_1, d_1 be a path of length three in F such that a_1 is an endpoint of F and A_1 the forest $F - \{a_1, b_1, c_1, d_1\}$. Let a, b, c, d be a path of length three in B and A be the forest $B - \{a, b, c, d\}$. Because B is a broom either all vertices of A are adjacent to B or all vertices of A are adjacent to c. We may assume that all vertices of A are adjacent to b. Note that A_1 and A have the same number of vertices. We let σ be a packing from B into F given by $\sigma(a) = c_1$, $\sigma(b) = a_1$, $\sigma(c) = d_1$, $(d) = b_1$ and such that σ is a bijection from the vertices of A to the vertices of A_1. Clearly σ is a packing and has chromatic number four because it contains the complete graph on four vertices. □

Claim 4: If D is a forest which contains a connected component P which is a path of length three, F is a forest which is coarse and D and F have the same number of vertices then D has a 4-chromahc embedding into F.

Proof of Claim: We observe first that T contains a path a, b, c, d of length three such that $T - \{a, b, c, d\}$ contains an isolated vertex x. If T has diameter at least four this can be achieved by choosing a path x, a, b, c, d of length four in which x is an endpoint of T. If T has diameter three then any path a, b, c, d of length three will have this property because T has at least four vertices. We choose the packing σ so that the path $\sigma(P)$ of $\sigma(D)$ and the path a, b, c, d of F form a K_4 and extend σ by a packing α from $D - P$ into $F - \{a, b, c, d\}$. Such a packing α exists because $F - \{a, b, c, d\}$ contains an isolated vertex. □

Claim 5: If F and G are two forests with the same number of vertices and if both are reduced under the property of containing a connected component with at least four edges and diameter at least three then there is a 4-chromatic packing from F into G.

Proof of Claim: We will investigate all pairs of forests described by Claim 2. If one of F or G is a broom then we are done by Claim 3. If one of them, say F is a path of length four then, excluding the broom, there is no other four edge forest described by Claim 2 besides the path of length four itself. It is easy to find a chromatic pacing from P_4 to P_4. Observe that each one of the four trees on five vertices described by Claim 2 can be reduced to a path of length three. Because P_3 has a 4-chromatic packing into P_3 it follows using Lemma 1 that if both forests F and G are one of the four trees with five edges described by point a) of Claim 2, which are non-broom trees, then F has a 4-chromatic packing into G.

If one of the forests, say F is one of the five edge trees and G is disconnected then G is a path of length four or a room with four edges together with an isolated vertex. Hence again both forests F and G can be reduced to a path of length three.

The last case is that both graphs F and G have exactly two connected components one of them being a star and the other a path or a broom with four edges . Assume that a, b, c, d is a path of length three of F and a_1, b_1, c_1, d_1 is a path of length three of G. $F - \{a, b, c, d\}$ contains then an isolated vertex, say x and $G - \{a_1, b_1, c_1, d_1\}$ contains an isolated vertex, say y. Let F be the center of the star component of F and G be the center of the star component of G. Let σ be a packing from F to G such that a_1, b_1, c_1, d_1 and $\sigma(a), \sigma(b), \sigma(c), \sigma(d)$ forms a complete graph on four vertices, $\sigma(x) = g$, $\sigma(f) = y$ and the endpoints of the star component of F are mapped to the endpoints of the star component of G. □

Claim 6: If F and G are two forests with the same number of vertices and both being coarse, then the two forests have a 4-chromatic packing

Proof of Claim: Using Lemma 1 and the remarks after that Lemma we may assume that the forests are reduced in such a way that both forests contain a path with at least three vertices and one of the following three cases occur:

a) One of the two forests is reduced under the property of containing a path of length three. The other forest is coarse.

b) Both forests are reduced under the property of being coarse.

c) One of the forests is reduced under the property of being coarse.

Every connected component of the other forest is either a star or a path of length three and it contains at least one connected component which is a path of length three. In the first case we use Claim 1 to deduce that one of the two forests, say F, is either a broom or P_4or a forest with two components one of which is P_3 and'the other is star. The other forest, say G, contains a connected component T with diameter at least three which also contains at least four edges . If F is a broom we are done using Claim 3. If F is a path of length four then G too must be a path of length four and any two paths of length four have a 4-chromatic packing. If F contains a connected component which is a path of length three we use Claim 4.

In case b) the present Claim follows from Claim 5 and in case c) from Claim 4. □

Claim 7: Assume that F and G are two forests having the same number of verses and that each contains a connected component which is a path of length three. If each one of the forests F and G has at least three connected components then they have a 4-chromatic pacing.

Proof of Claim: Let P be a path of F of length three and Q be a path of G of length three. Choose a packing σ from P to Q such that $\sigma(P) \cup Q$ is the complete graph on four vertices. The forests $F - P$ and $G - Q$ have both at least two components and hence by Lemma 3 the packing σ can be extended to a packing from F to G. □

Claim 8: Let G and F be forests with n vertices in which every connected component is either a star or a path of length three. The forest has almost two connected components of which not more than one is a path of length three. Then if σ is a 4-chromatic pacing from F into G, σ must contain the complete graph K_4.

Proof of Claim: Assume for a contradiction that F has a 4-chromatic packing σ into G which does not contain K_4. We also assume that n is minimal. If σ maps an endpoint a of F to an endpoint b of G then the restriction of σ to $F - a$ will be a four chromatic packing from $F - a$ to $G - B$ which also does not contain K_4. This will be in contradiction to the minimality of n. Hence every endpoint of F is mapped to some interior point of G which implies that the number of interior points of G is at least as big as the number of endpoints of F. The number of endpoints of F is at least $n-3$ and the number of interior points of G is at most $\frac{1}{2}n$. We get $n - 3 \leq \frac{1}{2}n$ and hence $n \leq 6$. If $n = 6$ the forest G contains at most two interior points and F at least four endpoints. If $n = 5$ the forest G contains at most two interior points and F contains at least three endpoints. This leaves $n = 4$ but the only four chromatic graph on four vertices is K_4. □

Claim 9: Let F and G be forests with n verses and diameter three in which every connected component is either a star or a path of length three. There exists a 4-chromahc packing from F into G unless one of the two forests consists of exactly two components one of which is a star and none of the two forests contains an isolated vertex. In this case the two forests do not have a 4-chromatic packing.

Proof of Claim: We prove first that, unless one of the two forests consists of exactly two components one of which is a star and none of the two forests contains an isolated vertex, there exists a 4-chromatic packing from F into G. If each of the two forests has at least three connected components then it follows from Claim 7 that they have a 4-chromatic packing. Hence we may assume that one of them, say F, has exactly two connected components. Assume that P is a path of length three in F and Q is a path of length three in G. If one of the two forests would contain an isolated vertex then we could map P to Q to form a K_4 and then continue this map to a packing using Lemma 2. So neither F nor G has an isolated vertex. If the second component, say S, of F is a star we are done. If S is also a path of length three then $G - Q$ is either a path of length three or it contains an isolated point or it consists of two nonadjacent edges or it is a star. It is easy to see that unless $G - Q$ is a star, in which case we are done, S has a packing into $G - Q$ and hence F has a 4chromatic packing into G.

For the other direction we have to show that if neither F nor G contains an isolated vertex and if F consists of exactly two connected components one of them being a path P of length three and the other a star S then F does not have a 4-chromatic packing into G. Assume for a contradiction that σ is a 4-chromatic packing from F into G. We deduce from Claim 8 that there is a path Q of length three in G such that $\sigma(F) \cup Q$ form a complete graph on four vertices. The restriction of the packing σ to $F - P = S$ is a packing from the star S to the forest $G - Q$. This is only possible if $G - Q$ contains an isolated vertex v. Because Q must be a connected component v will also be an isolated vertex of G. □

We are now in the position to finish the proof of Theorem 2. Let F and G be two forests of diameter outlast three with the sin number of vertices. Note that if a forest is not a P_3-star-forest then it contains a connected component of diameter at least three having at least four edges . Hence if both forests are not P_3-star-forests then the Theorem follows from Claim 6. If one of the forests is not a P_3-star-forest but the other one is a P_3-star-forest, hence contains a connected component which is a path of length three, then the theorem follows from claim 4. If both forests are P_3-star-forests the theorem follows from Claim 9.

4 One of the forests contains and the other does not contain a path of length three

Let F and G be two forests with the sin number of vertices. As mentioned earlier, if one of the forests, say F, contains no path of length three (that is, if F is a anion of stars), we know no example of a pair F, G that would have a 4-chromatic packing. In fact, we conjecture the following.

Conjecture 1: If F and G are forests with the same number of vertices such that every component of F is a star, then every packing of F into G is 3-colourable.

It is easy to check that this conjecture is equivalent to the following one.

Conjecture 2: Let F be a forest whose every component is a star, and let T be a tree edge-disjoint from F but on the same vertex set. Then, $\chi(F \cup T) \leq 3$.

If true, then the second conjecture is sharp in the following sense. If the tree T is replaced by a unicyclic graph, the chromatic number of its edge-disjoint union with a forest consisting only of stars can jump to four. This is witnessed on any wheel with an odd number of spokes, where the unicyclic graph is a hamiltonian cycle of the wheel.

We contribute to the second conjecture by showing that it is true for any tree and any forest that contains at most four nontrivial components, all of them stars. Also, we show that this conjecture is valid for all trees of diameter at most 11, irrespective of the number of stars in F.

4.1 Reduction Lemmas

Throughout, F will denote a forest whose each component is a star (note that we allow also trivial 1-vertex stars), and T will denote a tree; both F and T share the same vertex set V but are edge-disjoint. As usual, $F \cup T$ win denote the graph with vertex set V and edge set $E(F) \cup E(T)$. By $d_F(u)$ or $d_T(u)$ we denote the valency of the vertex $u \in V$ in F and T, respectively. The vertices in V that are central vertices of stars in F will be called *essential*, the vertices $v \in V$ with $d_F(v) \leq 1$ will be *peripheral* Thus, isolated vertices of F as well as endvertices of isolated edges in F are both essential and peripheral. This ambiguity will prove convenient in statements of our results.

For any graph H and $v \in V(H)$ we denote by $H - V$ the graph obtained from H by deleting the vertex v. If $e \in E(H)$ and $F \notin E(H)$, then $H - e$ (or $H + f$) denotes the graph that arises from H by deleting the edge e (or adding the new edge f) while keeping the vertex set unchanged. If $u, v \in V(H)$, the symbol H/uv stands for the graph obtained from H by identifying the vertices u and v (and suppressing parallel edges, if any, as well as the "loop" if $uv \in E(H)$).

In the subsequent reduction lemmas, we always construct a new forest F' from F and a new tree T' from T such that $V(F') = V(T')$ and $E(F') \cap E(T') = \emptyset$. To simplify the statements, the symbol $F' \cup T' \rightarrow_3 F \cup T$ means that every 3-coloring of $F' \cup T'$ extends to a 3-coloring of $F \cup T$.

Reduction Lemma 1: Let $v \in V$ be a peripheral vertex which is also an endpoint of T. Put $F' = F - v$ and $T' = T - v$. Then, $F' \cup T' \rightarrow_3 F \cup T$.

Proof: Since v has valence at most 2 in $F \cup T$, the extendibility of 3-colorings is obvious. □

Reduction Lemma 2: Let $v_1, v_2 \in V$ be peripheral verses, both adjacent to an essential vertex u in F. Assume that there is a vertex $w \in V$ such that $v_1w, v_2w \in E(T)$. Let $F' = F/v_1v_2$ and $T' = T/v_1v_2$. Then, $F' \cup T' \rightarrow_3 F \cup T$.

Proof: Let v denote the vertex obtained by identifying v_1 with v_2. Then, any 3-coloring of the smaller graph extends to one of $F \cup T$ obtained by assigning to both v_1 and v_2 the color of v and keeping the colors of other vertices unchanged. □

Reduction Lemma 3: Let $u_1, u_2, u_3 \in V$ be essential vertices such that $u_1u_2, u_2u_3, \in E(T)$ and $u_1u_3 \notin E(F)$. If $F' = F/u_1u_3$ and $T' = T/u_1u_3$, then $F' \cup T' \rightarrow_3 F \cup T$.

Proof: Similar to the above and therefore omitted. □

Reduction Lemma 4: Let $u_1, u_2 \in V$ be essenhal vertices, let $u_1v_1, u_2u_3 \in E(F)$, $(v_i \neq u_j)$ and $u_1, u_2 \notin E(T)$. Assume that $e = u_1v_2 \in E(T)$ and that v_2 and u_2 are in the same component in the graph $T - e$. Put $F' = F/u_1u_2$, $T' = (T - e)/u_1u_2$. Then, $F' \cup T' \rightarrow_3 F \cup T$.

Proof: Let u be the vertex obtained by identifying u_1 with u_2. Then, any 3-coloring of $F' \cup T'$ extends to one of $F \cup T$ by assigning the color of u to both u_1 and u_2. No conflict can occur, because $v_2u \in E(F' \cup T')$ and thus the colors of v_2 and u_1 in $F \cup T$ will be different (and the edge e can be safely inserted back to obtain $F \cup T$). □

Reduction Lemma 5: Let u_1v_1, u_2v_2 be two edges of F where both u_1 and u_2 are essenhal vertices and $u_1, u_2 \notin E(T)$. Let $e = u_1v_2 \in E(T)$, $f = u_2v_1 \in E(T)$, and let u_2 add v_2 belong to the same component of $T - e$. If F contains at least 3 components, then there always exists a new edge g such that $T' = ((T - e - f)/u_1u_2) + g$ is a tree. Moreover, $F/u_1u_2 \cup T' \rightarrow_3 F \cup T$.

Proof: Again, let u arise by identifying u_1 with u_2. Now, $(T - e - f)/u_1u_2$ has exactly 2 components T_1 and T_2. We have two possibilities: Either $v_1 \in V(T_1)$ (say) and $v_2, u \in V(T_2)$, or $v_1, v_2 \in V(T_1)$ and $u \in V(T_2)$. Due to the fact that F has at least 3 components, in either case there is a new edge g (not in F/u_1u_2) such that $T' = ((T-e-f)/u_1u_2)+g$ is a tree. The coloring extension procedure is the same as in the preceding lemma. □

4.2 Results

As in the preceding section, let F be a forest consisting of star components only and let T be a tree with $V(F) = V(T) = V$ and $E(F) \cap E(T) = \emptyset$. Denote by $\alpha(F)$ the number of non-trivial components of F (i.e., $\alpha((F)$ is the number of stars in F with at least one edge).

Proposition 1. *If $\alpha(F) = l$, then $\chi(F \cup T) \leq 3$.*

Proof: Let v be essential vertex of F. Any 2-coloring of T extends to a 3-coloring of $F \cup T$ by assigning the vertex v a third color. □

Proposition 2. *If $\alpha(F) = 2$, then $\chi(F \cup T) \leq 3$.*

Proof: The assertion is trivially true for small graphs. We proceed by induction on the cardinality of V. Let u_1 and u_2 be essential vertices of the two non-trivial stars in F. We first take care of vertices that are isolated in F. Suppose that $w \in V$ is isolated in F but both wu_1 and wu_2 are edges of T. Forming F' and T' from F and T, respectively, by identifying u_1 with u_2 and combining Proposition 1 with Reduction Lemma 3 (where w is considered as essential) we see that $\chi(F \cup T) \leq 3$. Thus, we may assume that for every vertex w with $d_F(w) = 0$ there exists a vertex $u_w \in \{u_1, u_2\}$ such that $wu_w \notin E(T)$. Let us add all the new edges wu_w into F, obtaining a forest F^* consisting of two stars and no isolated vertices on the same set V. If we show that $\chi(F^* \cup T) \leq 3$, then we are done. In what follows we thus assume that F contains no isolated vertices.

Let v be a peripheral vertex in F. If v is also an endpoint of T, then the graphs $T' = T - V$ and $F' = F - V$ either satisfy the induction hypothesis, or Proposition 1 applies (if the removal of v destroys a non-trivial star in F). In either case, by Reduction Lemma 1 we have $\chi(F \cup T) \leq 3$. It remains to deal with the case when no pendant vertex of T is peripheral. But since o (F) = 2, it follows that then T must have exactly 2 endpoints, i.e., T is a path with endvertices u_1 and u_2, and $u_1u_2 \notin E(T)$. The desired 3-coloring of $F \cup T$ is now obtained from a 2-coloring of $T - u_1 - u_2$ by assigning u_1 and u_2 a third color. □

Proposition 3. *If $\alpha(F) = 3$ then $\chi(F \cup T) \leq 3$.*

Proof: The assertion is easily seen to be true if the graph has no more than 6 vertices. Again, we proceed by induction on the number of vertices. By arguments similar to those in the preceding proof, we can restrict ourselves to the case when F contains no isolated vertices and no endpoint of T is a peripheral vertex of F. It follows that T has at most three pendant vertices that are, at the same time, essential vertices of F (of valence at least 2 in F).

In this situation, it is easy to check that there exists a pendant edge of T, say, $e = u_1v$, such that u_1 is essential and v is peripheral (and adjacent to another essential vertex in F). But then, it is possible to apply one of the Reduction Lemmas 4 or 5. Combining Proposition 2 with the induction hypothesis we conclude that $\chi(F \cup T) \leq 3$. □

Now we state and prove our first main result in this case.

Theorem 3.[1] *Let F be a forest and T a tree with the same number of vertices. If F contains at most four non-trivial components, all of them stars, then every packing of F and T is 3-chromatic.*

Proof: For the sake of simplicity we assume that F and T share the same vertex set but are edge-disjoint. In view of the preceding results and their proofs, we let $\alpha(F)$ be equal to 4. The case when F has 8 vertices can be settled easily (e.g., using Reduction Lemma 1 and Proposition 3). We proceed by induction on the number of vertices. As before, we may assume that F has no isolated vertices and that T has at most four endpoints that are essential (and of valence at least 2 in F). Now, if T has at least three such endpoints, then it necessarily contains a pendant edge (say) u_1v where u_1 is essential (and pendant in T) and v is peripheral. Applying the same method as in the end of the preceding proof, we deduce that Reduction Lemmas 4 or 5 combined with Proposition 3 yield $\chi(F \cup T) \leq 3$. We are therefore left with the situation when T is a path.

Let u_1, u_2, u_3, u_4 be essential vertices of the four stars of F. If one of the pendant edges of T joined an essential vertex with a peripheral one, then we would apply Reduction Lemma 4 or 5 again to conclude that $\chi(F \cup T) \leq 3$. Therefore, each of the two pendant edges of T must join essential vertices; without loss of generality we assume that u_1 and u_4 are endpoints of T and $u_1u_2, u_3u_4 \in E(T)$. So, $T = u_1u_2v_1v_2 \dots v_pu_3u_4$ is a path with essential vertices $u_1u_2u_3u_4$.

Colour u_1u_2 by two different colours, set $v_0 = u_2$, $u_{p+1} = u_3$ and let q be the least i in $\{1, 2, \dots, p-3\}$ such that v_i is joined by an edge of F to u_3 or u_4.

If i is an integer, $1 \leq i \leq q-1$, v_{-1} is and v_{i+1} is not colored, the coloring can be extended to v_i, since this vertex has exactly two colored neighbors – one in F and one (v_{i-1}) in T.

Now suppose that v_i are colored for all $i \leq q-1$ and $v_qu_j \in E(F)$ for some $j \in \{3, 4\}$. Colour u_j by the same color as v_{q-1} and u_{q-j} by any of the two remaining colours.

[1]We are grateful to the referee for supplying a substantial part of the proof of theorem 3

By analogous reasons as above if i is an integer, $q+1 \leq i \leq p$, v_{+1} is and v_{i-1} is not colored, the coloring can be extended to v_i.

Finally, if only v_q is not colored, its three neighbors v_{q-1}, v_{q+1} and u_j are colored by at most two colours and the coloring can be finished properly. □

As a counterpart to the above, we prove that $F \cup T$ is 3-colorable when T has "small" diameter, regardless of the number of stars in F.

Theorem 4. *Let F be a forest whose each component is a star and let T be a tree with the same number of vertices as F. If the diameter of T is at most 11, then each packing of F and T is 3-colorable.*

Proof: Again, let F and T be edge-disjoint and on the same vertex set V. We employ induction on n, the number of vertices in V, and concentrate only on the induction step. We can safely assume that every star in F is non-trivial. Also, we may assume that every pendant edge of T joins two essential vertices of F (otherwise we could use reductions as in the preceding proofs, noticing that they do not increase the diameter). Now, let u_1 be an endpoint and u_1u_2 a pendant edge of T. If $d_F(u_1) = 1$, then clearly $(F \cup T) - u_1 \to_3 F \cup T$. On the other hand, if $d_F(u_2) = 1$ and $u_2v_2 \in E(F)$, then we can for free interpret u_2 as a peripheral vertex and use Reduction Lemma 4 (and the induction hypothesis) to show that $\chi(F \cup T) \leq 3$. It therefore remains to consider the case when $d_F(u_1) \geq 2$ and $d_F(u_2) \geq 2$ for every pendant edge $u_1u_2 \in E(T)$.

Let t be the number of pendant edges in T. Since the diameter of T is at most 11, we have $n \leq 5t+2$. But every pendant edge gives rise to two stars in F on at least 3 vertices each; thus, $n \geq 6t$. We have $6t \leq n \leq 5t+2$, and hence $t = 2$. Consequently, T is a path on at most 12 vertices and F consists of at most 4 stars; but then $\chi(F \cup T) \leq 3$ by Theorem 1. The proof is complete. □

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