Simple Proofs to three parity theorems

Yair Caro
Department of Mathematics
School of Education
University of Haifa – ORANIM
Tivon 36-910
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ABSTRACT. Using linear algebra over GF(2) we supply simple proofs to three parity theorems: the Gallai's partition theorem, the odd-parity cover theorem of Sutner, and generalize the "odd-cycle property" theorem of Manber and Shao to binary matroids.

1 Introduction

Gallai proved, more than 30 years ago, the following partition theorem.

Theorem 1. *Let G be a graph, then:*

(1) *there exists a partition $V(G) = A \cup B$, $A \cap B = \phi$, such that in the induced subgraphs $\langle A \rangle$ and $\langle B \rangle$ all degrees are even.*

(2) *there exists a partition $V(G) = A \cup B$, $A \cap B = \phi$ such that in the induced subgraph $\langle A \rangle$ all degrees are even and in $\langle B \rangle$ all degrees are odd.*

□

A simple graph theoretic proof was given by Posa, while a more complicated proof using linear algebra was given by Chen. For both proofs we refer to the book of Lovasz [4].

A seemingly unrelated game called "The Garden of Eden", has been considered by Sutner [6]. The original problem goes as follows: Suppose each of the squares of the n by n grid is equipped with an indicator light and a button. If the button of a square is pressed, the light indicator of the square changes from off to on and vice versa. The same change happens

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to the light of all the adjacent squares. Initially all the lights are off. Is it possible to press a sequence of buttons in such a way that in the end all the lights are on?

In graph theoretic notation this problem can be generalized and reformulated as follows: Let G be a simple graph (without loops and multiple edges), and denote, for every vertex $v \in V(G)$, by $N(v)$ the set of all neighbors of v in G including v itself. A subset $Q \subseteq V(G)$ is called odd-parity cover iff for every vertex $v \in V(G)$, $|N(G) \cap Q| \equiv 1 \pmod 2$. It is easy to see that a graph G has an odd-parity cover iff the "Garden of Eden" is solvable in G. Sutner [6] proved:

Theorem 2. *Every simple graph G has an odd-parity cover.* □

The proof is based upon unnecessary complicated linear algebra arguments, similar to those of Chen in his proof of Gallai's partition theorem.

Lastly, Manber and Shao [5] considered the following problem: A graph G is said to have the "odd-cycle property" if there exists a subset $Q \subseteq E(G)$, such that for every cycle C in G, $|E(C) \cap Q| \equiv 1 \pmod 2$. They [5] proved:

Theorem 3. *A graph G has the "odd-cycle property" iff every block in G is either a cycle or an edge.* □

The proof is graph theoretical, using an ad-hoc argument.

Our main goal here is to supply very simple proofs to those three parity theorems, with an emphasis on linear algebra over $GF(2)$, on the algorithmic aspects, and on possible extensions, and relations between those theorems.

We now introduce some definitions and notation.

Let $H(V, E)$ be a hypergraph with vertex set V and edge set E. The incidence matrix $B = B(H)$, of H, is an $|E| \times |V|$ matrix defined by

$$b_{ij} = \begin{cases} 1 & v_i \in e_j \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic linear system of H is the linear system, over $GF(2)$, $BX = 1$, where 1 is the "all one" vector.

A set of vectors, A, in a linear space over $GF(2)$ is called odd-zero family (OZF in short) if $|A| \equiv 1 \pmod 2$ and $\Sigma_{u \in A} = 0$, over $GF(2)$.

Our main tool is the following elementary result from linear algebra.

Theorem 4 [1]. *Let $BX = 1$ be a linear system over $GF(2)$. Then there exists no solution iff there is an odd number of row vectors in B whose sum (in $GF(2)$) is the zero-vector O.* □

In our notation, theorem 4 implies the following proposition.

Proposition 1. *Let $H(V,E)$ be a hypergraph with an incidence matrix B. Then $BX = 1$ has no solution, over $GF(2)$, iff $E(H)$ contains an OZF, namely there exists a subfamily $A \subseteq E(H)$ such that $|A| \equiv 1 \pmod 2$ and $\Sigma_{e \in A} = 0$ (in $GF(2)$) where the sum of the edges is actually the sum of their characteristic vectors.*

2 Simple Proofs

We start with a simple proof of theorem 2.

Theorem 2: [6]. *Every simple graph G has an odd-parity cover.*

Proof: Consider the linear system $BX = 1$ over $GF(2)$, where $B = A + I$, A is the adjacency matrix of G, (see [3]) and I is the unit matrix of order $|V(G)|$. Clearly G has an odd-parity cover iff $BX = 1$ has a solution, and each solution corresponds to an odd-parity cover. Recall that by theorem 4, $BX = 1$ has no solution iff there is an odd number of row vectors of B whose sum is O in $GF(2)$.

Suppose such an OZF of row vectors exists. These rows correspond to vertices in G, say $v_1, v_2, \dots, v_k$. Consider the subgraph H of G induced by these vertices.

Since in B, $\Sigma_{i=1}^{k} b_{ij} = 0$ for $j = 1, \dots, |G|$, and $b_{ii} = 1$, it follows that in H all degrees are odd and $|H| = k \equiv 1 \pmod 2$ which is impossible. Hence B cannot contain an OZF of row vectors and $BX = 1$ is solvable over $GF(2)$, thus G has an odd-parity cover. □

Let us now consider theorem 3 in the light of binary matroids. For a basic information on matroids we refer to [7]. One of the many characterizations of binary matroids is: ([7, p. 162]) M is a binary matroid it for every two distinct circuits of M, say A and B, the symmetric difference $A \Delta B$ contains a circuit (and is also the union of disjoint circuits).

Lemma 1. *Let M be a binary matroid containing two intersecting circuits A and B. Then there are in M three circuits X, Y, Z forming an OZF.*

Proof: Choose from all the pairs of intersecting circuits in M a pair X and Y for which $|X \cup Y|$ is the minimum possible. Let Z be a circuit in $X \Delta Y$.

Then both $X \cap Z$ and $Y \cap Z$ are non-empty, for in the other case a circuit (Z) would be a proper subset of another circuit (X or Y), which is impossible. If $(X \Delta Y) - Z$ is not empty, then either $|X \cup Z| < |X \cup Y|$ or $|Y \cup Z| < |X \cup Y|$. In the first case the pair X, Z contradicts the choice of X, Y. In the second case the pair Y, Z contradicts the choice of X, Y. Hence $X \Delta Y = Z$ and X, Y, Z forms an OZF. □

Let us modify the notion of odd-cycle property to the context of binary matroids. A binary matroid M is said to have the odd-circuit property if there exists a subset Q of the elements of M such that for every circuit C

of M, $|C \cap Q| \equiv 1 \pmod 2$. Note that the cycles of a graph G are the circuits of the binary graphic matroid (see e.g. [7] chapter 10).

We can now extend theorem as follows:

Theorem 5.

(1) A binary matroid M has the odd-circuit property iff no two circuits intersect.

(2) A graph G has the odd-cycle property it every two cycles are edge-disjoint.

(3) If G has the odd-cycle property, and girth $g(G) \geq k$, then $|E(G)| \leq \lfloor k(n-1)/(k-1) \rfloor$.

Proof: Consider the hypergraph $H(V, E)$, with V the elements of M and E the circuits of M. Let B be the incidence matrix of E and consider $BX = 1$. Clearly M has odd-circuit property iff $BX = 1$ has a solution which holds (by proposition 1 and by lemma 1) iff M contains no intersecting circuits.

This establishes the proof of the first two parts of the theorem.

Lastly, let G be a graph with the odd-cycle property and girth $g(G) \geq k$. We proceed by induction on $n = |G|$.

For $n \leq k$ claim (3) holds trivially. Consider an end-block D of G. This block D is either an edge or a cycle C_t, $t \geq k$ by part (2) of the theorem.

Consider two cases.

Case 1: D is an edge with an end-vertex v. Consider $F = G \backslash \{v\}$, and apply induction on F to obtain $|E(G)| = |E(F)| + 1 \leq \lfloor k(n-2)/(k-1) \rfloor + 1 = \lfloor k(n-2)/(k-1) \rfloor + \lfloor k/(k-1) \rfloor \leq \lfloor k(n-1)/(k-1) \rfloor$.

Case 2: D is a cycle C_t, $t \geq k$. Consider $F = G \backslash \{C_t\}$. Clearly $|F| = n - t + 1$ and by induction $|E(G)| = |E(F)| + t \leq \lfloor k(n-t)/(k-1) \rfloor + t = \lfloor k(n-t)/(k-1) \rfloor + \lfloor (k-l)t/(k-1) \rfloor \leq \lfloor (k(n-t) + (k-l)t)/(k-1) \rfloor = \lfloor (kn-t)/(kl) \rfloor \leq \lfloor k(n-l)/(k-1) \rfloor$ completing the proof. □

Our last result relates Gallai's partitions and odd-parity covers thereby producing another simple indirect proof of Gallai's theorem.

Theorem 6. *The following two statements are equivalent.*

(1) Every graph G has a Gallai's partition.

(2) Every graph G has an odd parity cover.

Proof: (1) $\rightarrow$ (2). Consider G and let V_0 be the set of vertices of even degree in G. Form a graph H from G by adding a new vertex w which is adjacent to all vertices of V_0. Apply Gallai's partition on H to obtain two parts A and B inducing subgraphs with all degrees even.

Suppose $w \in A$ then B is an odd parity cover of G. Indeed all degrees in B are even hence for every vertex v in B $|N(v) \cap B| - 1 \pmod 2$, recall that $v \in N(v)$. For every vertex v in A, $v \neq w$, the degree of v in A is even but the degree of v in $(A \backslash w)$ is even iff $(w, v) \notin E(A)$ which holds it v has odd degree in G. Hence every vertex $v \in A$, $v \neq w$ is adjacent to an odd number of vertices in B, namely $|N|v| \cap B| - 1 \pmod 2$ and B is indeed an odd-parity cover.

(2) $\rightarrow$ (1). Consider G and to every vertex v of even degree attach a new vertex v' which is adjacent only to v. In the resulting graph H all degrees are odd. Let $D = V(H) \backslash V(G)$. Since it has odd-parity cover say A, put $B = V(H) \backslash A$. B is also an odd-parity cover because in H all degrees are odd and the degrees in $\langle A \rangle$ and $\langle B \rangle$ are all even. Hence for every vertex v of even degree in G, it follows that $v \in A$ iff $v' \in B$ and $v \in B$ iff $v' \in A$.

Hence in $\langle A \backslash D \rangle$ and $\langle B \backslash D \rangle$ all degrees are even producing the Galas partition of G. □

Remarks:

(1) Since and odd parity cover can be found solving $BX = 1$ (as in the proof of theorem 2), and since the confirmation of the graph H in the proof of the second part of theorem 6 can be implemented in time $O(n^3)$ respectively $O(n^2)$. It follows that we can find a Gallai's partition and an odd-parity cover in time $O(n^3)$.

(2) The upper bound $\lfloor k(n-1)/(k-1) \rfloor$ in part 3 of theorem 5 is essentially best possible. It is not hard to give a construction that realizes this bound. However, I refer to a theorem of Bollobas [2, p. 32] which states that the maximum number of edges in a graph G without a pair of vertices x, y having three vertex disjoint paths between them is $\lfloor 3(n-1)/2 \rfloor$ and this bound is sharp. Clearly this is exactly a graph in which every two cycles are edge-disjoint which shows that our bound is exact for $k = 3$.

References

[1] Alon, N., Caro, Y.: On three zero-sum Ramsey-type problems. Journal of Graph Theory. 17(1993) 177–192.

[2] Bollobas, B.: Extremal graph theory. Academic Press 1978.

[3] Harary, F.: Graph theory. Addison-Wesley Publishing Company. Third Printing 1972.

[4] Lovasz, L.: Combinatorial problems and exercises. North-Holland, Amsterdam. 1979.

[5] Manber, R., Shao, J.Y.: On digraphs with the odd cycle property. Journal of Graph Theory 10(1986) 155–165.

[6] Sutner, K.: Linear cellular automata and the Garden-of-Eden. The Mathematical Intelligencer, Vol. 11, No. 2 (1989) 49–53.

[7] Welsh, D.: Matroid Theory. Academic Press 1976.