Locally P_n^k Graphs

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ABSTRACT. We completely classify the graphs all of whose neighbourhoods of vertices are isomorphic to P_n^k $(2 \leq k < n)$, where P_n^k is the k-th power of the path P_n of length $n-1$.

1 Introduction

All graphs considered in this paper are undirected, without loops or multiple edges. K_n denotes the complete graph on n vertices, C_n the cycle of length n, P_n the path of length $n-1$, $P_{\aleph_0}$ the two-way infinite path of denumerable length and $\sim$ the adjacency relation. We shall say that a vertex v of degree d in a graph G is a *d-vertex* of G and we shall denote by $G(v)$ the *neighbourhood* of v, that is the subgraph induced by G on the set of vertices adjacent to v in G. Note that if v_i and v_j are two adjacent vertices of G, v_i is a d-vertex of $G(v_j)$ if and only if v_j is a d-vertex of $G(v_i)$. If $v_i \sim v_j$, $N_{i,j}$ will be the set of all neighbours of v_i in $G(v_j)$, and so $N_{i,j} = N_{j,i}$.

Given a positive integer k and a graph G, we denote by G^k the k-th power of G, that is the graph whose vertices are those of G, two vertices being adjacent in G^k iff their distance in G is at most k. Obviously $G^1 \cong G$.

Given a graph G', a connected graph G is said to be *locally* G' if, for every vertex v of G, the subgraph $G(v)$ is isomorphic to G'. There is an extensive literature on the determination of all graphs which are locally a given graph (see for example the bibliography at the end). The purpose of this paper is to answer a question raised by Topp and Volkmann [13]: which graphs are locally P_n^2? More generally, we will classify the graphs which are locally P_n^k for $2 \leq k < n$. When $k = 1$, it is already known (Brown and Connelly [4] [5], Hell [11]) that for any given $n > 3$, there are infinitely many non-isomorphic graphs which are locally P_n (it is easy to check that there is no locally P_3 graph and that K_3 is the only locally P_2 graph).

Our main result is the following:

Theorem. *Let k and n be integers such that $2 \leq k < n$ and let G be a locally P_n^k graph.*

(i) If $n = k+1$, then $G \cong K_{k+2}$.

(ii) If $n = 2k+2$, then G has at least $3k+4$ vertices. For every integer $m \geq 3k+4$, there is a unique locally P_{2k+2}^k graph on m vertices, namely C_m^{k+1}; the only infinite locally P_{2k+2}^k graph is $P_{\aleph_0}^{k+1}$.

(iii) If $n \neq k+1$ and $n \neq 2k+2$, there is no locally P_n^k graph.

2 Lemmas

The following properties of the graphs P_n^k will be used to establish our theorem. The proofs are omitted since they are very easy.

Lemma 1. *P_n^2 has two adjacent 3-vertices iff $n = 4$ or 5.*

Lemma 2. *If $n \geq k+2$, P_n^k has exactly two k-vertices and they are non-adjacent.*

Lemma 3. *If $n \geq 2k+2$, the subgraph induced by P_n^k on the set of neighbours of any vertex of degree $k+1$ is a complete graph on $k+1$ vertices with one missing edge (whose end vertices are respectively of degree k and $2k$ in P_n^k).*

If $a_1, \ldots, a_n$ are the vertices and $[a_i, a_{i+1}]$ $(i = 1, \ldots, n-1)$ the edges of P_n, we shall say that $a_1 \sim a_2 \sim \cdots \sim a_n$ is a *basic path* of P_n^k.

Lemma 4. *If $n \geq 2k+3$, then a_j and a_{n-j+1} $(j = 1, \ldots, k)$ are two $(k+j-1)$-vertices of P_n^k, all the other vertices being $2k$-vertices.*

Lemma 5. *If $n \geq 2k+3$ and if v and v' are two adjacent vertices of P_n^k such that v is a $(k+r)$-vertex $(r = 1$ or $2)$ and v' is a $(k+s)$-vertex of P_n^k with $r < s \leq k-1$, then v' is adjacent to every neighbour of v distinct from v'.*

3 Proof Of The Theorem

Let v_0 be any vertex of a graph G which is locally P_n^k with $2 \leq k < n$ and let $v_1, v_2, \ldots, v_n$ be the vertices of $G(v_0)$, the edges of $G(v_0)$ being those of a graph P_n^k constructed over the basic path $v_1 \sim \cdots \sim v_i \sim v_{i+1} \sim \cdots \sim v_n$.

1) If $n = k+1$, then $P_n^k \cong K_{k+1}$, and so obviously $G \cong K_{k+2}$.

2) If $k+2 \leq n \leq 2k+1$, then v_{k+1} is adjacent not only to v_0 but also to the $n-1$ vertices of $G(v_0)$ distinct from v_{k+1}. Since v_{k+1} must be of

degree n in G, it follows that $N_{k+1,1} = \{v_0, v_2, \ldots, v_k\}$. On the other hand, $N_{0,1} = \{v_2, \ldots, v_{k+1}\}$. Thus v_0 and v_{k+1} are two adjacent k-vertices of $G(v_1)$, contradicting Lemma 2.

3) If $n \geq 2k+2$, then $G(v_1)$ contains $v_0, v_2, \ldots, v_{k+1}$ and no other vertex of $G(v_0)$, and so v_1 must be adjacent to $n-k-1 \geq k+1 \geq 3$ new vertices $v_{n+1}, v_{n+2}, \ldots, v_{2n-k-1}$. Since v_0 is a k-vertex of $G(v_1) \cong P_n^k$, the set $N_{0,1} = \{v_2, \ldots, v_{k+1}\}$ contains exactly one $(k+i)$-vertex of $G(v_1)$ for every $i = 1, \ldots, k$. It is no restriction of generality to assume that v_{n+j} ($j = 1, \ldots, k$) is the unique vertex of $G(v_1)$ which has an index $> n$ and which is adjacent to exactly $k - j + 1$ vertices of $N_{0,1}$ (thus for example v_{n+1} is adjacent to all vertices of $N_{0,1}$).

Note that the $2k$ vertices $v_0, v_1, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}, v_{n+1}$ are all adjacent to v_k and v_{k+1}. Since $2k$ is the maximal degree of a vertex in P_n^k, it follows that $\{v_0, v_1, \ldots, v_{k-1}, v_{k+2}, \ldots, v_{2k}, v_{n+1}\} = N_{k,k+1}$.

We claim that v_{k+1} is a $(k+1)$-vertex of $G(v_1)$. This is clear if $n = 2k+2$ because the n neighbours of v_{k+1} in G are then exactly $v_0, v_1, \ldots, v_k, v_{k+2}, \ldots, v_{2k+1}, v_{n+1}$. If $n \geq 2k+3$ and if v_{k+1} is a $(k+i)$-vertex of $G(v_1)$ for some $i \geq 2$, then $v_{k+1} \sim v_{n+2}$ and so $v_k \not\sim v_{n+2}$ (because we have just proved that v_{n+2} cannot be a common neighbour of v_k and v_{k+1}). Therefore v_k is a $(k+1)$-vertex of $G(v_1)$, and so v_1 is a $(k+1)$-vertex of $G(v_k)$. But v_0 is clearly a $(2k-1)$- vertex of $G(v_k)$ and $v_1 \sim v_0$. If $k \geq 3$, Lemma 5 implies that v_0 is adjacent to every neighbour of v_1 distinct from v_0 in $G(v_k)$; in particular $v_0 \sim v_{n+1}$, a contradiction. If $k = 2$, v_0 and v_1 are two adjacent 3-vertices of $G(v_2) \cong P_n^2$ with $n \geq 7$, contradicting Lemma 1.

Thus we have proved that for every $n \geq 2k+2$, v_{k+1} is a $(k+1)$-vertex of $G(v_1)$; more precisely, $N_{k+1,1} = \{v_0, v_2, \ldots, v_k, v_{n+1}\}$. It follows that all vertices of $N_{0,1} - \{v_{k+1}\}$ are adjacent to v_{n+1} and v_{n+2}.

Since $v_2 \neq v_{k+1}$, v_2 is a $(k+j)$-vertex of $G(v_1)$ for some $j \in \{2, \ldots, k\}$, or equivalently v_1 is a $(k+j)$-vertex of $G(v_2)$ for some $j \in \{2, \ldots, k\}$. Since $N_{0,2} = \{v_1, v_3, \ldots, v_{k+2}\}$, v_0 is a $(k+1)$-vertex of $G(v_2) \cong P_n^k$ and so, by Lemma 3, the subgraph induced by $G(v_2)$ on $N_{0,2}$ is a complete graph with one missing edge whose end vertices are respectively of degree k and $2k$ in $G(v_2)$. But this missing edge is clearly $[v_1, v_{k+2}]$. We conclude that v_1 is a $2k$-vertex of $G(v_2)$, or equivalently that v_2 is a $2k$-vertex of $G(v_1)$.

Case I: $n \geq 2k+3$
If $k = 2$, v_2 is a 4-vertex of $G(v_1)$, thus $G(v_2)$ contains $v_0, v_1, v_3, v_4, v_{n+1}, v_{n+2}$ and no other vertex of $G(v_0) \cup G(v_1)$. In $G(v_2)$, v_0 is a 3-vertex and v_1, v_3 are 4-vertices with $v_1 \sim v_0 \sim v_3$. Since $v_4 \sim v_0$, v_4 must be a 2-vertex of $G(v_2)$ and so $N_{4,2} = \{v_0, v_3\} = N_{2,4}$. Since v_0 is a 4-vertex of $G(v_4)$, it follows that v_3 is a 3-vertex of $G(v_4)$. On the other hand, $N_{0,3} = \{v_1, v_2, v_4, v_5\}$, $N_{1,3} = N_{3,1} = \{v_0, v_2, v_{n+1}\}$ and $N_{2,3} = \{v_0, v_1, v_4, v_{n+1}\}$, thus v_1 is a 3-vertex and v_0, v_2 are 4-vertices of $G(v_3)$ with $v_0 \sim v_1 \sim v_2$.

Since v_4 is adjacent to v_0 and v_2 and since $n \geq 7$, v_4 must be a 4-vertex of $G(v_3) \cong P_n^2$, contradicting the fact that v_3 is a 3-vertex of $G(v_4)$.

If $k \geq 3$, v_3 is distinct from v_{k+1} and v_2, and so v_3 is a $(k+s)$-vertex of $G(v_1)$ for some $s \in \{2, \ldots, k-1\}$ (remember that v_{k+1} and v_2 are already known to be vertices of degree $k+1$ and $2k$ respectively in $G(v_1)$). Therefore v_1 is a $(k+s)$-vertex of $G(v_3)$ for some $s \in \{2, \ldots, k-1\}$. On the other hand, v_0 is clearly a $(k+2)$-vertex of $G(v_3)$. If $k = 3$, v_0 and v_1 are two adjacent 5-vertices of $G(v_3) \cong P_n^3$, a contradiction because $n \geq 9$. If $k \geq 4$, v_0 and v_1 are two adjacent vertices of degree $k+2$ and $k+s$ respectively in $G(v_3)$, with $2 < s \leq k-1$ ($s = 2$ is impossible because P_n^k does not contain two adjacent $(k+2)$-vertices when $n \geq 2k+3$). By Lemma 5, v_1 is adjacent to every vertex of $N_{0,3} - \{v_1\}$; in particular, $v_1 \sim v_{k+2}$, a contradiction.

Case II: $n = 2k+2$

Clearly G has at least $3k+4$ vertices (namely $v_0, v_1, \ldots, v_{3k+3}$) and $G(v_1)$ consists of the following $2k+2$ vertices: $v_0, v_2, \ldots, v_{k+1}, v_{2k+3}, \ldots, v_{3k+3}$ where v_0 and v_{3k+3} are the only two k-vertices of $G(v_1)$.

For any $i \in \{2, \ldots, k+1\}$, $G(v_i)$ contains at least the $k+i$ vertices $v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+i}$ (among which exactly k are adjacent to v_1) and so, since v_i is of degree $n = 2k+2$ in G, v_i cannot be adjacent to more than $k-i+2$ vertices in the set $\{v_{2k+3}, \ldots, v_{3k+3}\} \subseteq G(v_1)$. Thus v_i is of degree $\leq k + (k-i+2) = 2k-i+2$ in $G(v_1)$. But we know that $N_{0,1} = \{v_2, \ldots, v_{k+1}\}$ contains exactly one $(k+j)$-vertex of $G(v_1)$ for every $j = 1, \ldots, k$. Therefore, for any $i \in \{2, \ldots, k+1\}$, v_i is a $(2k-i+2)$-vertex of $G(v_1)$ and, since v_i is of degree $n = 2k+2$ in G, it follows that $G(v_i) = \{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+i}, v_{2k+3}, \ldots, v_{3k+4-i}\}$.

From all the adjacencies known at present, we easily deduce that the subgraph $G(v_2) \cong P_{2k+2}^k$ is necessarily constructed over the basic path $v_{k+2} \sim v_0 \sim v_{k+1} \sim \cdots \sim v_3 \sim v_1 \sim v_{2k+3} \sim \cdots \sim v_{3k+2}$. It follows that $v_{k+2} \not\sim v_{2k+3}, \ldots, v_{3k+2}$. More generally, the examination of each subgraph $G(v_i)$ for $i \in \{2, \ldots, k+1\}$ shows that $v_{k+i} \not\sim v_{2k+3}, \ldots, v_{3k+4-i}$ for these values of i.

Observe now that G contains a subgraph isomorphic to P_{3k+4}^{k+1}, constructed over the basic path $v_{3k+3} \sim v_{3k+2} \sim \cdots \sim v_{2k+3} \sim v_1 \sim v_2 \sim \cdots \sim v_{k+1} \sim v_0 \sim v_{k+2} \sim \cdots \sim v_{2k+2}$ and that there is exactly one missing vertex in the subgraph $G(v_{2k+3})$. Let w_1 be this missing vertex. The known adjacencies and non-adjacencies imply immediately that there are only two possibilities for w_1: either $w_1 = v_{2k+2}$ or w_1 is a new vertex v_{3k+4}.

If $w_1 = v_{2k+2}$, then in the subgraph $G(v_{2k+3})$ (which must be isomorphic to P_{2k+2}^k) there is exactly one missing edge through each of the k vertices $v_{2k+4}, \ldots, v_{3k+3}$ and there are exactly k missing edges through the vertex v_{2k+2}. Therefore v_{2k+2} must be adjacent to $v_{2k+4}, \ldots, v_{3k+3}$.

Note that each vertex v_{k+i} ($i = 2, \ldots, k+1$) is already adjacent to

$2k+3-i$ vertices in G, and so v_{k+i} has at most $i-1$ new neighbours in G. On the other hand, in the subgraph $G(v_{2k+2}) \cong P^k_{2k+2}$, the two vertices of degree k are necessarily v_0 and v_{2k+3} (because all their neighbours in G are already known) and each vertex $v_{k+i} \in N_{0,2k+2}$ $(i = 2, \ldots, k+1)$ must have $i-1$ new neighbours in the set $\{v_{2k+4}, \ldots, v_{3k+3}\} = N_{2k+3,2k+2}$. Therefore v_{k+i} $(i = 2 \ldots, k+1)$ must have exactly $i-1$ new neighbours in G and, since we already know that $v_{k+i} \not\sim v_{2k+3}, \ldots, v_{3k+4-i}$, these $i-1$ new neighbours are uniquely determined in the set $\{v_{2k+4}, \ldots, v_{3k+3}\}$. It follows that G itself is now completely determined and it is easy to check that $G \cong C^{k+1}_{3k+4}$.

If $w_1 = v_{3k+4}$, then $v_{2k+2} \not\sim v_{2k+3}$ and, reasoning as before in the subgraph $G(v_{2k+3})$, we see that v_{3k+4} must be adjacent to $v_{2k+4}, \ldots, v_{3k+3}$. Thus G contains a subgraph isomorphic to P^{k+1}_{3k+5}, constructed over the basic path $v_{3k+4} \sim v_{3k+3} \sim \cdots \sim v_{2k+3} \sim v_1 \sim v_2 \sim \cdots \sim v_{k+1} \sim v_0 \sim v_{k+2} \sim \cdots \sim v_{2k+2}$.

Note that $v_{2k+1} \not\sim v_{2k+4}$ because, if we assume that v_{2k+1} is adjacent to v_{2k+4}, then v_{2k+1} is necessarily a k-vertex of $G(v_{2k+4})$ and so v_{2k+1} is adjacent to $v_{2k+5}, \ldots, v_{3k+4}$, which implies that the degree of v_{2k+1} in G is at least $2k+3 > n$, a contradiction.

From this and the other known adjacencies and non-adjacencies, we deduce that there are only two possibilities for the missing vertex w_2 of the subgraph $G(v_{2k+4})$: either $w_2 = v_{2k+2}$ or w_2 is a new vertex v_{3k+5}.

If $w_2 = v_{2k+2}$, then in the subgraph $G(v_{2k+4}) \cong P^k_{2k+2}$ there is exactly one missing edge through each of the k vertices $v_{2k+5}, \ldots, v_{3k+4}$ and there are exactly k missing edges through the vertex v_{2k+2}. Therefore v_{2k+2} must be adjacent to $v_{2k+5}, \ldots, v_{3k+4}$.

Note that each vertex v_{k+i} $(i = 2, \ldots, k+1)$ is already adjacent to $2k+3-i$ vertices in G, and so v_{k+i} has at most $i-1$ new neighbours in G. Similarly, v_{2k+3-i} $(i = 2, \ldots, k+1)$ is already adjacent to $2k+3-i$ vertices in G, and so v_{2k+3-i} has at most $i-1$ new neighbours in G. On the other hand, in the subgraph $G(v_{2k+2}) \cong P^k_{2k+2}$, the two vertices of degree k are necessarily v_0 and v_{2k+4} (because all their neighbours in G are already known); moreover, $N_{0,2k+2} = \{v_{k+2}, \ldots, v_{2k+1}\}$ and $N_{2k+4,2k+2} = \{v_{2k+5}, \ldots, v_{3k+4}\}$. Using the preceding two remarks, we deduce that each vertex $v_{k+i} \in N_{0,2k+2}$ must have exactly $i-1$ new neighbours in the set $N_{2k+4,2k+2}$ and that these new neighbours are necessarily $v_{3k+6-i}, \ldots, v_{3k+4}$. It follows that G is now completely determined and that $G \cong C^{k+1}_{3k+5}$.

If $w_2 = v_{3k+5}$, reasoning as before, we are led to only two possibilities for the missing vertex w_3 in $G(v_{2k+5})$: either $w_3 = v_{2k+2}$ or w_3 is a new vertex v_{3k+6}. An easy induction argument finishes the proof.

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