General Upper Bounds for Covering Numbers

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Abstract. A $t-(n,k,\lambda)$ *covering design* consists of a collection of k-element subsets (blocks) of an n-element set $\mathcal{X}$ such that each t-element subset of $\mathcal{X}$ occurs in at least λ blocks. We use probabilistic techniques to obtain a general upper bound for the minimum size of such designs, extending a result of Erdős and Spencer [4].

1. Introduction

Let $n \geq k \geq t$ and consider a finite set $\mathcal{X}$ with cardinality n. Let $\mathcal{B}$ be a collection of k-element subsets of $\mathcal{X}$. Elements of $\mathcal{B}$ are called *blocks*. The pair $(\mathcal{X}, \mathcal{B})$ is said to form a $t-(n,k,\lambda)$ *covering design* if each t-element subset of $\mathcal{X}$ is contained in least λ blocks. The covering number $C_\lambda(n,k,t)$ is defined to be the number of blocks in a minimum $t-(n,k,\lambda)$ covering design. $C_1(n,k,t)$ will be denoted, for brevity, by $C(n,k,t)$. *Packing designs* are defined in an analogous fashion, and will not be discussed here. There is an extensive literature on covering and packing designs; for a survey of important results, see the recent papers by Mills and Mullin [6] and Sidorenko [8].

Our goal in this paper will be to derive general upper bounds for the covering numbers $C_\lambda(n,k,t)$; Erdős and Spencer [4] showed almost twenty

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years ago that for each choice of n, k and t,

$$C(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}}(1+\log\binom{k}{t}), \tag{1.1}$$

while Rödl [7] proved using probabilistic methods that for each fixed k and t

$$\lim_{n\to\infty} C(n,k,t)\frac{\binom{k}{t}}{\binom{n}{t}} = 1; \tag{1.2}$$

see Spencer [9] for an alternative proof of (1.2), and Alon and Spencer [1] for a discussion of the general technique (the "Rödl nibble") employed to prove this remarkable result first conjectured by Erdős and Hanani [3]. It is quite another matter, however, to use the proof of (1.2) to derive general upper bounds for $C(n, k, t)$ that are valid for each choice of the parameters; consequently (1.1) may yield a better bound than (1.2) when n is relatively small. For larger values of n, however, it is evident that Rödl's bound is far superior. For completeness, we outline the proof of (1.1) using the "method of alterations"; see [1] for other examples of this method.

Proof of (1.1): We start by selecting an unspecified number X of blocks at random and "with replacement". The probability that any specific t-set is uncovered is then equal to

$$(1-\frac{\binom{n-t}{k-t}}{\binom{n}{k}})^X = (1-\frac{\binom{k}{t}}{\binom{n}{t}})^X \leq \exp(-X\frac{\binom{k}{t}}{\binom{n}{t}}) \tag{1.3}$$

Now the number of uncovered t-sets can be written as

$$W = \sum_{j=1}^{\binom{n}{t}} I_j,$$

where I_j equals 1 if the jth t-set is uncovered and is zero otherwise. It follows from (1.3) that

$$\mathbf{E}(W) = \sum_{j=1}^{\binom{n}{t}} \mathbf{P}(I_j = 1) \leq \binom{n}{t}\exp(-X\frac{\binom{k}{t}}{\binom{n}{t}}) \tag{1.4}$$

To complete the covering, we next choose, in any *ad hoc* fashion, one k-set to cover each uncovered t-set. It follows that the expected number Z_X of sets in this random covering of the t-sets is at most

$$X + \binom{n}{t} \exp(-X \frac{\binom{k}{t}}{\binom{n}{t}}) \tag{1.5}$$

for any initial choice X. We next minimize (1.5) over X to obtain

$$\frac{\binom{n}{t}}{\binom{k}{t}} \log \binom{k}{t}$$

as the choice for X that minimizes Z_X. Since $\mathbf{P}(Y \leq \mathbf{E}(Y)) > 0$ for any random variable Y, it follows that there actually exists an initial selection as outlined above for which the covering number is at most $\frac{\binom{n}{t}}{\binom{k}{t}}(1 + \log\binom{k}{t})$. The result follows. □

Several remarks are in order at this point: We note firstly that it is conceivable that the non-constructive cover produced by the above process involves replicated blocks. It is immediate, however, that the desired property is preserved on elimination of the replicated blocks, so that the bound (1.1) holds without replication as well. In Section 2, we shall use probabilistic methods (notably exponential probability inequalities for binomial tails) to provide a generalization of (1.1) for arbitrary values of λ. The nature of our proof once again does not preclude replication, but we shall see how the method yields insight into the question of obtaining bounds on $C_\lambda(n,k,t)$ when replication of blocks is not allowed. Our result will reduce to (1.1) on setting $\lambda = 1$ and yields a bound that is far better than the trivial estimate $\lambda\frac{\binom{n}{t}}{\binom{k}{t}}(1 + \log\binom{k}{t})$ that can be immediately deduced from (1.1). Furthermore, the upper bound in Theorem 2.1 below exhibits the nature of the non-linearity in the growth rate (as a function of λ) of the bound for the covering numbers $C_\lambda(n,k,t)$.

2. Results

Theorem 2.1. *If $\log\binom{k}{t} \geq (\lambda - \frac{1}{2})\log(\lambda - 1) + 2$, then the minimum size $C_\lambda(n,k,t)$ of a $t-(n,k,\lambda)$ covering design with replicated blocks satisfies*

$$C_\lambda(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}}\{A + (\frac{A}{q\log\binom{k}{t}})^{\lambda-1}\}, \tag{2.1}$$

where $q = 1 - \frac{\binom{k}{t}}{\binom{n}{t}}$ and $A = \log\binom{k}{t} + \log\lambda - \log(\lambda-1)! + (\lambda-1)\log\log\binom{k}{t}$; if k and t are large and $\log\binom{k}{t} \gg \lambda$, the estimate in (2.1) is thus of order of magnitude $\frac{\binom{n}{t}}{\binom{k}{t}}(1 + \log\binom{k}{t} + (\lambda-1)\log\log\binom{k}{t})$.

Proof: We use the method of alterations. Assume that X k-sets have been selected randomly and with replacement. Then the probability that the ith t-set has been covered just j times ($0 \le j \le \lambda - 1$) equals

$$\binom{X}{j}\left[\frac{\binom{k}{t}}{\binom{n}{t}}\right]^j\left[1-\frac{\binom{k}{t}}{\binom{n}{t}}\right]^{X-j}, \tag{2.2}$$

so that the expected number of t-sets that are covered only j times (and thus undercovered) equals

$$\binom{n}{t}\binom{X}{j}\left[\frac{\binom{k}{t}}{\binom{n}{t}}\right]^j\left[1-\frac{\binom{k}{t}}{\binom{n}{t}}\right]^{X-j} \tag{2.3}$$

Now for each such undercovered set we choose, in any *ad hoc* fashion, an additional $\lambda - j$ blocks to complete the covering; it follows that if X blocks are initially drawn, we expect to need a total of at most

$$X+\binom{n}{t}\sum_{j=0}^{\lambda-1}(\lambda-j)\binom{X}{j}p^j q^{X-j} \tag{2.4}$$

blocks to form the λ-cover, where $p = \frac{\binom{k}{t}}{\binom{n}{t}}$ and $q = 1 - p$. As in the proof of (1.1), we observe that this process is potentially inefficient, but we will see that this inefficiency is not excessive [recall, for example, that the contribution to (1.5) of the initial drawing X was "almost" sufficient to form the entire cover, especially for large values of k and t]. We need, next, to minimize (2.4) over X. Towards this end, we bound (2.4) above by the more tractable quantity

$$X+\binom{n}{t}\exp\{-Xp\}\sum_{j=0}^{\lambda-1}\frac{(\lambda-j)}{j!}\left(\frac{Xp}{q}\right)^j \tag{2.5}$$

and set $\phi(X) = X + \binom{n}{t}\exp\{-Xp\}\sum_{j=0}^{\lambda-1}\frac{(\lambda-j)}{j!}(Xp)^j$, $\alpha = Xp$. It is easy to show that the following result holds:

Lemma 2.2. *$\phi'(\alpha) = 0$ if*

$$\frac{\sum_{j=0}^{\lambda-1}\alpha^j/j!}{e^\alpha} = \frac{1}{\binom{k}{t}} \qquad \square \tag{2.6}$$

(2.6) does not have an easily obtainable solution in general; before continuing with the proof of Theorem 2.1, we consider some special cases using approximate solutions:

(a) For $\lambda = 2$, the extremely suboptimal choice $\alpha = \log\binom{k}{t}$ [i.e. $X = \frac{\binom{n}{t}}{\binom{k}{t}} \log\binom{k}{t}$] yields, when substituted into (2.5), the result

$$C_2(n,k,t) \le \frac{\binom{n}{t}}{\binom{k}{t}}(\log\binom{k}{t}(1+\frac{1}{q})+2) \tag{2.7}$$

which is slightly more than twice the Erdős-Spencer bound (1.1). We clearly need to do better.

(b) For $\lambda = 2$ again, the choice $X = \frac{\binom{n}{t}}{\binom{k}{t}}\log(\binom{k}{t}\beta)$ $(\beta > 0)$ yields, when substituted into (2.5), the bound

$$C_2(n,k,t) \le \frac{\binom{n}{t}}{\binom{k}{t}}\{(1+\frac{1}{q\beta})\log\binom{k}{t} + (1+\frac{1}{q\beta})\log\beta + \frac{2}{\beta}\}$$

with a typical "good" choice for β being $\log(\binom{k}{t}\log[e\binom{k}{t}\log\{e\binom{k}{t}\}])$. This yields $C_2(n,k,t)$ of the order of magnitude of $\frac{\binom{n}{t}}{\binom{k}{t}}(1+\log\binom{k}{t}+\log\log\binom{k}{t})$.

(c) For $\lambda \ge 3$, the minimization problem (2.6) does not get any easier. We prove, however, that the bound obtained in (a) above can be extended as follows:

Proposition 2.3. *For each t, n, k and λ satisfying $\frac{\binom{k}{t}}{\binom{n}{t}} \le \frac{1}{2}$,*

$$C_\lambda(n,k,t) \le \frac{\binom{n}{t}}{\binom{k}{t}}(2\log\binom{k}{t} + 2\log\lambda + 2\lambda - \log 4); \tag{2.8}$$

note that (2.8) "almost" reduces to (2.7) on setting $\lambda = 2$.

Proof: We bound (2.4) above by

$$X + \binom{n}{t}\sum_{j=0}^{\lambda-1}\lambda\binom{X}{j}p^j q^{X-j} \tag{2.9}$$

and use the fact that the cumulative binomial probabilities $B(X,p,r) := \sum_{j=0}^{r}\binom{X}{j}p^j(1-p)^{X-j}$ satisfy

$$B(X,p,r) \le \exp\{-(Xp-r)^2/2Xp(1-p)\} \quad (r \le Xp \le X/2) \tag{2.10}$$

(see, e.g. Barbour et al [2]) to conclude that for $\lambda - 1 \leq Xp$,

$$C_\lambda(n,k,t) \leq \inf_X \{X + \lambda \binom{n}{t} \exp\{-(Xp - \lambda + 1)^2 / 2Xp(1-p)\}\}$$
$$\leq \inf_X \{X + \lambda \binom{n}{t} \exp\{\lambda - 1 - \frac{Xp}{2}\}\}, \qquad (2.11)$$

which yields, (with $X = 2\frac{\binom{n}{t}}{\binom{k}{t}} \log(\frac{\lambda e^{\lambda-1}\binom{k}{t}}{2})$), the required result; note that the condition $Xp \geq \lambda - 1$ is satisfied with this choice of X. □

We now continue with the proof of Theorem 2.1: (2.4) can be rewritten, on simplification, as

$$X + \binom{n}{t} \lambda B(X, p, \lambda - 1) - \binom{n}{t} XpB(X - 1, p, \lambda - 2); \qquad (2.12)$$

if $Xp > \lambda$, (2.12) reveals that an upper bound on $C_\lambda(n,k,t)$ is given by

$$C_\lambda(n,k,t)$$
$$\leq X + \binom{n}{t} \lambda b(X, p, \lambda - 1) + \binom{n}{t} \lambda \sum_{j=0}^{\lambda-2} \binom{X-1}{j} p^j q^{X-1-j} (\frac{Xq}{X-j} - 1)$$
$$\leq X + \binom{n}{t} \lambda b(X, p, \lambda - 1)$$
$$\leq X + \frac{\binom{n}{t} \lambda (\frac{Xp}{q})^{\lambda-1} e^{-pX}}{(\lambda - 1)!}, \qquad (2.13)$$

where $b(X, p, r)$ denotes the point binomial probability $\binom{X}{r} p^r q^{X-r}$. There are several choices for X that may be used to suboptimize over X in (2.13); we use

$$X = \frac{\binom{n}{t}}{\binom{k}{t}} \log[\frac{\binom{k}{t} \lambda \log^{\lambda-1}\binom{k}{t}}{(\lambda - 1)!}]$$

to conclude that

$$C_\lambda(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}} \{ \log \binom{k}{t} + \log \lambda - \log(\lambda-1)! + (\lambda - 1) \log\log \binom{k}{t}$$
$$+ (\frac{\log \binom{k}{t} + \log \lambda - \log(\lambda - 1)! + (\lambda - 1) \log\log \binom{k}{t}}{q \log \binom{k}{t}})^{\lambda-1}\},$$

establishing the required bound. Note that the condition $Xp > \lambda$ is satisfied if

$$\log\binom{k}{t} + \log\lambda - \log(\lambda-1)! + (\lambda-1)\log\log\binom{k}{t} > \lambda; \qquad (2.14)$$

(2.14) may be simplified somewhat on ignoring the contribution of the second and fourth terms on the left hand side and using the estimate $(\lambda-1)! \leq (\lambda-1)^{\lambda-\frac{1}{2}}e^{2-\lambda}$ (see, e.g., [2]). This leads to the condition $\log\binom{k}{t} \geq (\lambda-\frac{1}{2})\log(\lambda-1)+2$ that is quoted in the statement of the theorem. □

Example. For $n = 50, k = 30, t = 20$ and $\lambda = 5$, (2.1) yields the bound $C_5(50,30,20) \leq 33.37(1568620) = 33.37\frac{\binom{50}{20}}{\binom{30}{20}}$; the Erdős-Spencer bound for $\lambda = 1$ works out to be 18.22(156 8620).

Remarks.

(a) It is quite possible that for certain values of the parameters, our sub-optimal choice of X can be bettered: see [10] for bounds on $C_\lambda(n,k,t)$ obtained by using other X's including $X = \frac{\binom{n}{t}}{\binom{k}{t}}\log[\frac{\binom{k}{t}\lambda\log^{\lambda-1}\{\binom{k}{t}\log^{\lambda-1}\binom{k}{t}\}}{(\lambda-1)!}]$ and $X = \frac{\binom{n}{t}}{\binom{k}{t}}\log[\binom{k}{t}\log^{\lambda-1}\binom{k}{t}]$.
(b) It may be verified that the incorporation, into our calculations, of the third term on the right hand side of (2.13) vastly complicates the situation without yielding an appreciable improvement in the final result. The basic reason for this is that the (lower-tail) cumulative binomial probabilities are dominated by the last included term (see [2] for more details).
(c) Our bound (2.1) reduces to the Erdős-Spencer bound (1.1) on setting $\lambda = 1$, and, moreover, can be seen to be far smaller than λ times the latter.

We now turn to the question of estimating the number of blocks in a λ-cover that are not replicated; the non-constructive process used by us does not guarantee that the blocks are distinct. We shall argue, however, that since the original choice of X sets (which could equally well have been made *without* replication) constitutes a vast majority of the cover, the *support* of the latter almost equals its size. More specifically, we have

Theorem 2.4. *If* $\log\binom{k}{t} \geq (\lambda-\frac{1}{2})\log(\lambda-1)+2$, *there exists a* $t-(n,k,\lambda)$ *covering design with at most*

$$\frac{\binom{n}{t}}{\binom{k}{t}}\{D + \lambda(\frac{D}{q\log\binom{k}{t}})^{\lambda-1}\}$$

blocks and in which at least

$$\frac{\binom{n}{t}}{\binom{k}{t}}D$$

blocks are distinct, where $D = \log\binom{k}{t} + \log\lambda - \log(\lambda-1)! + (\lambda-1)\log\log\binom{k}{t}$ *and* Λ *is a constant of order of magnitude* $1 + \frac{\log^2\binom{k}{t}}{2\binom{n-t}{k-t}}$

Proof: We start by choosing X blocks randomly *but without replacement* from the $\binom{n}{k}$ k-subsets of $\mathcal{X}$. Set $A = \binom{n-t}{k-t}$ and $B = \binom{n}{k} - \binom{n-t}{k-t}$, and denote the cumulative and point hypergeometric probabilies $\sum_{j=0}^{r} \frac{\binom{A}{j}\binom{B}{X-j}}{\binom{A+B}{X}}$ and $\frac{\binom{A}{j}\binom{B}{X-j}}{\binom{A+B}{X}}$ by $H(A,B,X,r)$ and $h(A,B,X,j)$ respectively. It is clear then that the probability that the ith t-set is covered just j times $(0 \leq j \leq \lambda - 1)$ equals $h(A,B,X,j)$ for each i, so that the expected number of t-sets that are covered j times equals $\binom{n}{t}h(A,B,X,j)$. For each such set, we now select $(\lambda - j)$ blocks to complete the cover (in any arbitrary fashion). *These additional sets may be replicates of previously chosen ones, but the original X blocks are all distinct.* As in the proof of Theorem 2.1, the expected number of sets in the cover formed by using this process equals, for $\frac{AX}{A+B} > \lambda$,

$$\begin{aligned}
&X + \binom{n}{t}\sum_{j=0}^{\lambda-1}(\lambda - j)h(A,B,X,j)\\
=&X + \binom{n}{t}\lambda H(A,B,X,\lambda-1) - \binom{n}{t}\sum_{j=1}^{\lambda-1} jh(A,B,X,j)\\
=&X + \binom{n}{t}\lambda H(A,B,X,\lambda-1) - \frac{\binom{n}{t}AX}{A+B}H(A-1,B,X-1,\lambda-2)\\
\leq&X + \binom{n}{t}\lambda H(A,B,X,\lambda-1) - \binom{n}{t}\lambda H(A-1,B,X-1,\lambda-2)\\
=&X + \binom{n}{t}\lambda h(A,B,X,\lambda-1)\\
+&\binom{n}{t}\lambda\{H(A,B,X,\lambda-2) - H(A-1,B,X-1,\lambda-2)\}\\
=&X + \binom{n}{t}\lambda h(A,B,X,\lambda-1)
\end{aligned}$$

$$+\binom{n}{t}\lambda\sum_{j=0}^{\lambda-2}h(A-1,B,X-1.j)\{\frac{AX(B-X+j+1)}{(A-j)(X-j)(A+B)}-1\}; \quad (2.15)$$

in addition, it may easily be verified that the last term on the right hand side of (2.15) is non-positive if $Xp > \lambda$, where, as before, $p = \frac{\binom{k}{t}}{\binom{n}{t}} = \frac{A}{A+B}$. This is true since $\frac{A(B-X+j+1)}{A-j} \leq B$ if $Xp > \lambda$, so that

$$\{\frac{AX(B-X+j+1)}{(A-j)(X-j)(A+B)}-1\} \leq \frac{XB}{(X-j)(A+B)}-1=\frac{Xq}{X-j}-1\leq 0$$

if $Xp > \lambda$. It follows that for any such X,

$$C_\lambda(n,k,t) \leq X + \binom{n}{t}\lambda h(A,B,X,\lambda-1) \quad (2.16)$$

Several strategies may be used to further simplify (2.16); we use a result of Burr (see, e.g., [5], eq. (6.76),) to bound the hypergeometric point probability in (2.16) by a multiple of a binomial probability:

$$\begin{aligned} &h(A,B,X,\lambda-1) \\ \leq& b(X,p,\lambda-1)\{1+\frac{\lambda-1-(\lambda-1-Xp)^2}{2(A+B)p}+O(\frac{1}{(A+B)^2p^2})\}, \quad (2.17)\end{aligned}$$

which simplifies to

$$\begin{aligned} &h(A,B,X,\lambda-1) \\ \leq& b(X,p,\lambda-1)\{1+\frac{(\lambda-1-(\lambda-1-Xp)^2)\binom{n}{t}}{2\binom{n}{k}\binom{k}{t}}+O(\frac{\binom{n}{t}^2}{\binom{k}{t}^2\binom{n}{k}^2})\} \\ \leq& b(X,p,\lambda-1)\{1+\frac{X^2p^2\binom{n}{t}}{2\binom{n}{k}\binom{k}{t}}+O(\frac{1}{\binom{n-t}{k-t}^2})\} \\ =& b(X,p,\lambda-1)\{1+\frac{X^2\binom{k}{t}}{2\binom{n}{k}\binom{n}{t}}+O(\frac{1}{\binom{n-t}{k-t}^2})\}, \quad (2.18)\end{aligned}$$

where the second inequality above follows since $\lambda-1 \leq (\lambda-1)^2$ and $\lambda-1 \leq Xp$, so that by (2.13) and (2.16),

$$C_\lambda(n,k,t) \le X + \frac{\binom{n}{t}\lambda\{1+\frac{X^2\binom{k}{t}}{2\binom{n}{k}\binom{n}{t}} + O(\frac{1}{\binom{n-t}{k-t}^2})\}(\frac{Xp}{q})^{\lambda-1}e^{-pX}}{(\lambda-1)!} \tag{2.19}$$

If we now choose

$$X = \frac{\binom{n}{t}}{\binom{k}{t}}\log[\frac{\binom{k}{t}\lambda\log^{\lambda-1}\binom{k}{t}}{(\lambda-1)!}]$$

and set

$$\{1+\frac{X^2\binom{k}{t}}{2\binom{n}{k}\binom{n}{t}} + O(\frac{1}{\binom{n-t}{k-t}^2})\} = \Lambda,$$

it follows that

$$C_\lambda(n,k,t) \le \frac{\binom{n}{t}}{\binom{k}{t}}\{D + \Lambda(\frac{D}{q\log\binom{k}{t}})^{\lambda-1}\},$$

where D is as in the statement of Theorem 2.1. Note that $\Lambda \approx 1 + \frac{\log^2\binom{k}{t}}{2\binom{n-t}{k-t}}$. This completes the proof of Theorem 2.4. □

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