# NEIGHBOURHOOD AND DEGREE CONDITIONS FOR THE EXISTENCE OF REGULAR FACTORS

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**Abstract**

We present sufficient conditions for the existence of a $k$-factor in a simple graph depending on $\sigma_2(G)$ and the neighbourhood of independent sets in our first theorem and on $\sigma_2(G)$ and $\alpha(G)$ in the second one.

## 1. INTRODUCTION

In this paper we consider only finite undirected graphs without loops or multiple edges. Terms not defined here can be found in [2].
Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v$ of $G$, we denote the degree of $v$ in $G$ by $d_G(v)$, or just $d(v)$. For $X \subseteq V(G)$, we define $d(X) := \sum_{v \in X} d(v)$. The neighbourhood $N_G(v)$ of $v$ is the set of all vertices in $V(G)$ adjacent to $v$ and for $X \subseteq V(G)$ we define $N(X) := \bigcup_{v \in X} N(v)$. For disjoint subsets $X, Y$ of $V(G)$, we denote the number of edges from $X$ to $Y$ by $e(X,Y)$. Instead of $e(\{x\},Y)$, we just write $e(x,Y)$. We use $\delta$ for the minimum degree, $\sigma_2$ for the minimum of $d(v_1) + d(v_2)$ over all pairs of independent vertices $v_1$ and $v_2$ of $G$ and $\alpha$ for the independence number of $G$. A spanning subgraph $F$ of $G$ is called $k$-factor, if $d_F(v) = k$ for all $v \in V(F)$. If $G_1$ and $G_2$ are disjoint graphs, the union is denoted by $G_1 \cup G_2$ and the join by $G_1 + G_2$.

Egawa/Enomoto [3] and Katerinis [5] proved the following sufficient condition for the existence of a $k$-factor.

**Theorem 1 (Egawa, Enomoto [3], Katerinis [5])** *Let $k \geq 1$ be an integer and $G$ a graph of order $n$ with $kn$ even and $n \geq 4k - 5$. If*

$$\delta \geq \frac{n}{2},$$

*then $G$ has a $k$-factor.*

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The following two theorems, which are different generalizations of Theorem 1, are also known. Tokushige proved Theorem 2 for the slightly weaker $n > 4k + 1 - 4\sqrt{k+2}$ instead of $n \geq 4k - 6$.

**Theorem 2 (Tokushige [9], Woodall [11])** *Let $k \geq 2$ be an integer and $G$ a graph of order $n$ with $n \geq 4k - 6$. If $k$ is odd, then $n$ is even and $G$ is connected. Let $G$ satisfy*

$$|N(X)| \geq \frac{1}{2k-1}(|X| + (k-1)n - 1)$$

*for every non-empty independent subset $X$ of $V(G)$, and*

$$\delta \geq \frac{k-1}{2k-1}(n+2) \ .$$

*Then $G$ has a $k$-factor.*

**Theorem 3 (Iida, Nishimura [4])** *Let $k$ be a positive integer and $G$ a graph of order $n$ with $kn$ even, $n \geq 4k - 5$ and $\delta \geq k$. Let $G$ satisfy*

$$\sigma_2 \geq n \ . \tag{1}$$

*Then $G$ has a $k$-factor.*

For connected graphs satisfying $n \geq 4k - 3$ and $k \geq 3$, Nishimura [8] could recently extend Theorem 3 again, replacing (1) by a condition on the maximum degree of any pair of independent vertices.

**Theorem 4 (Nishimura [8])** *Let $k \geq 3$ be an integer and $G$ a connected graph of order $n$ such that $kn$ is even, $n \geq 4k - 3$ and $\delta \geq k$. Suppose that*

$$\max\{d(u), d(v)\} \geq \frac{n}{2}$$

*for each pair of non-adjacent vertices $u, v \in V(G)$. Then $G$ has a $k$-factor.*

Niessen [7] proved the following sufficient condition for a $k$-factor depending on $\delta$ and $\alpha$.

**Theorem 5 (Niessen [7])** *Let $k \geq 2$ be an integer and $G$ a graph with $n$ vertices. If $k$ is odd, then suppose that $n$ is even and $G$ is connected. Let $G$ satisfy*

$$n > 4k + 1 - 4\sqrt{k+2} \ , \tag{2}$$

$$\delta \geq \frac{k-1}{2k-1}(n+2) \quad \textit{and}$$

$$\delta > \frac{1}{2k-2}((k-2)n + 2\alpha - 2) \ .$$

*Then $G$ has a $k$-factor.*

For $k \geq 2$, the following result is a generalization of Theorems 2 and 3.

**Theorem 6** *Let $k \geq 2$ be an integer and $G$ a graph of order $n$ with $n \geq 4k-6$ and $\delta \geq k$. If $k$ is odd, then $n$ is even and $G$ is connected. Let $G$ satisfy*

$$|N(X)| \geq \frac{1}{2k-1}(|X| + (k-1)n - 1) \tag{3}$$

*for every independent subset $X$ of $V(G)$ with $|X| \geq 2k$, and*

$$\sigma_2 \geq \frac{2k-2}{2k-1}n + \frac{4k-5}{2k-1} . \tag{4}$$

*Then $G$ has a $k$-factor.*

The following result is an extension of Theorem 5 for $n \geq 4k-6$.

**Theorem 7** *Let $k \geq 2$ be an integer and $G$ a graph of order $n$ with $n \geq 4k-6$ and $\delta \geq k$. If $k$ is odd, then suppose that $n$ is even and $G$ is connected. Let $G$ satisfy*

$$\sigma_2 \geq \frac{2k-2}{2k-1}n + \frac{4k-5}{2k-1} \quad \textit{and}$$

$$\sigma_2 > \frac{2}{2k-2}((k-2)n + 2\alpha - 2) . \tag{5}$$

*Then $G$ has a $k$-factor.*

## 2. PROOFS

To prove Theorems 6 and 7 we need the following result.

**Theorem 8** *Let $k \geq 2$ be an integer and $G$ a graph of order $n$ with $n \geq 4k-6$ and $\delta \geq k$. If $k$ is odd, then $n$ is even and $G$ is connected. Suppose that $G$ satisfies*

$$\sigma_2 \geq \frac{2k-2}{2k-1}n + \frac{4k-5}{2k-1} . \tag{6}$$

*Then $G$ has a $k$-factor or there exist disjoint non-empty subsets $A, B$ of $V(G)$ such that*

$$|X| := |\{v \in B \mid d_{G\setminus A}(v) = 0\}| \geq 2k \tag{7}$$

*and*

$$\omega \geq k|A| - k|B| + d_{G\setminus A}(B) + 2 , \tag{8}$$

*where $\omega$ denotes the number of components of $C := G \setminus (A \cup B)$ with at least three vertices.*

**Proof.** The proof is by contradiction. Let $G$ be a graph satisfying the hypotheses of Theorem 8, and having no $k$-factor. Then by the $k$-factor theorem of Belck [1] and Tutte [10] there exist two disjoint sets $A, B \subseteq V(G)$ such that

$$q \geq k|A| - k|B| + d_{G\setminus A}(B) + 2 \ , \tag{9}$$

where $q$ denotes the number of components $C_i$ of $C$ satisfying

$$k|C_i| + e(C_i, B) \equiv 1 \pmod 2 \ . \tag{10}$$

We choose $A$ and $B$ so that $|A \cup B|$ is maximal with respect to (9). From a lemma by Katerinis and Woodall [6] we can conclude that then every component of $C$ has at least $\max\{k - |B| + 2, 3\}$ vertices. Note that we also have $\omega \geq q$.

Let $a := |A|$, $b := |B|$, $c := |V(C)|$ and $\tilde{c} := c/\omega$.
If $B \neq \emptyset$, we choose a vertex $x \in B$ with

$$h := \min\{d_{G\setminus A}(v) \mid v \in B\} = d_{G\setminus A}(x) \ .$$

Further, if $B \setminus (N_G(x) \cup \{x\}) \neq \emptyset$, we define

$$m := \min\{d_{G\setminus A}(v) \mid v \in B \setminus (N_G(x) \cup \{x\})\} \ ,$$

and let $y \in B \setminus (N_G(x) \cup \{x\})$ be a vertex satisfying $d_{G\setminus A}(y) = m$. Obviously

$$\sigma_2 \leq d(x) + d(y) \leq 2a + h + m \ . \tag{11}$$

Note that, by the choice of $A$ and $B$ and by (9), we have

$$\begin{aligned} n &\geq a + b + 3\omega \\ &\geq (3k+1)a - (3k-1)b + 3d_{G\setminus A}(B) + 6 \ . \end{aligned} \tag{12}$$

Furthermore, $\delta \geq k$ yields

$$a \geq k - h \ . \tag{13}$$

If $\omega \geq 2$, let $C_1$ and $C_2$ be the smallest components of $C$ and $y_1$ and $y_2$ be vertices of $C_1$ and $C_2$, respectively. Then we obtain

$$\sigma_2 \leq d(y_1) + d(y_2) \leq 2a + |C_1| + |C_2| - 2 + e(y_1, B) + e(y_2, B) \ . \tag{14}$$

If $e(x, C) < \max\{k - b + 2, 3\}$, then in every component of $C$ exists a vertex non-adjacent to $x$. Let $C_1$ be the smallest component of $C$ and $x_1$ be a vertex of $C_1$ non-adjacent to $x$. Then we have

$$\sigma_2 \leq d(x) + d(x_1) \leq 2a + h + |C_1| + b - 2 \ . \tag{15}$$

Note that, using (6), $n = a + b + \tilde{c}\omega$ and (9), we obtain

$$\sigma_2 \geq \frac{2k-2}{2k-1}\,(a + b + \tilde{c}\,(ka - kb + hb + 2)) + \frac{4k-5}{2k-1} \quad . \tag{16}$$

We shall investigate different cases now.

*Case* 1. $B = \emptyset$.
Inequality (9) yields $q \geq 2$. Then according to (10) $k$ is odd, so $G$ is connected by hypothesis and it follows $a \geq 1$. By (14) and (16), we get

$$2\tilde{c} + 2a - 2 \geq \frac{2k-2}{2k-1}\,(a + \tilde{c}\,(ka + 2)) \; .$$

From $\tilde{c}, k \geq 3$, we obtain the following contradiction.

$$\frac{1}{2} \geq \frac{2\tilde{c} + 2a - 2}{a + \tilde{c}\,(ka + 2)} \geq \frac{2k-2}{2k-1} \; .$$

Thus we may assume $B \neq \emptyset$.

*Case* 2. $h \geq k + 1$.
Here (9) provides $\omega \geq 3$. Then (14) and (16) yield

$$2b + 2\tilde{c} + 2a - 2 \geq \frac{2k-2}{2k-1}\,(a + b + \tilde{c}\,(ka + b + 2)) + \frac{4k-5}{2k-1} \quad \text{, i.e.}$$

$$2ka\,(1 - k\tilde{c} + \tilde{c}) + 2b\,(k - k\tilde{c} + \tilde{c}) + 2\tilde{c} - 8k + 7 \geq 0 \; .$$

But $k \geq 2$ and $\tilde{c} \geq 3$ give the contradiction

$$\begin{aligned} &2ka\,(1 - k\tilde{c} + \tilde{c}) + 2b\,(k - k\tilde{c} + \tilde{c}) + 2\tilde{c} - 8k + 7 \\ &\qquad \leq \; 4a\,(1 - \tilde{c}) + 2b\,(2 - \tilde{c}) + 2\tilde{c} - 9 \\ &\qquad \leq \; -8a - 2b - 3 \\ &\qquad < \; 0 \; . \end{aligned}$$

*Case* 3. $h = k$.
Applying (9), we have $\omega \geq 2$.

*Case* 3.1. $b \geq k + 2$.
From (12) we deduce

$$n \geq (3k+1)\,a - (3k-1)\,b + 3k(k+1) + 3m(b - k - 1) + 6 \quad \text{, i.e.}$$

$$m \leq k + \frac{n - a - b - 3ka - 6}{3(b - k - 1)} \quad . \tag{17}$$

Because of $m \geq k$ we get

$$a \leq \frac{n-b-6}{3k+1} \quad . \tag{18}$$

Combining (11) and (17), we find

$$\sigma_2 \leq 2a + 2k + \frac{n-a-b-3ka-6}{3(b-k-1)} =: f_1(a) \tag{19}$$

with $f_1{}'(a) = 0$ if and only if $b = \frac{3}{2}k + \frac{7}{6}$.

*Case* 3.1.1. $b \geq \frac{3}{2}k + \frac{7}{6}$.
Using (19) and (18), we conclude

$$\sigma_2 \leq \frac{2}{3k+1}\, n + 2k - 1 - \frac{40}{9k+3} < \frac{2k-2}{2k-1}\, n + \frac{4k-5}{2k-1} \ ,$$

which is impossible according to (6).

*Case* 3.1.2. $b < \frac{3}{2}k + \frac{7}{6}$.
Applying (19) with $a = 0$, we have

$$\sigma_2 \leq 2k + \frac{n-b-6}{3\,(b-k-1)} =: f_2(b) \ . \tag{20}$$

Since $n \geq \max\{4k-6, b+6\}$ and $b \geq k+2$, we obtain $f_2{}'(b) \leq 0$, and hence

$$\sigma_2 \leq \frac{n}{3} + \frac{5k-8}{3} < \frac{2k-2}{2k-1}\, n + \frac{4k-5}{2k-1} \ ,$$

which again contradicts (6).

*Case* 3.2. $b \leq k+1$.

*Case* 3.2.1. $B \setminus (N_G(x) \cup \{x\}) \neq \emptyset$.
In this case (12) yields

$$n \geq (3k+1)\,a - (3k-1)\,b + 3k\,(b-1) + 3m + 6 \ , \text{ i.e.}$$

$$m \leq k - ka - 2 + \frac{n-a-b}{3} \ . \tag{21}$$

Then, using (11), (21), $a \geq 0$ and $b \geq 1$, we obtain the contradiction

$$\sigma_2 \leq \frac{n}{3} + \frac{6k-7}{3} < \frac{2k-2}{2k-1}\, n + \frac{4k-5}{2k-1} \ .$$

*Case* 3.2.2. $B \setminus (N_G(x) \cup \{x\}) = \emptyset$.

*Case* 3.2.2.1. $a \geq 1$.
Combining (14) and (16), we find

$$2\tilde{c} + 2b + 2a - 2 \geq \frac{2k-2}{2k-1}(a + b + \tilde{c}(ka+2)) + \frac{4k-5}{2k-1} \quad \text{, i.e.}$$

$$2ka(1 - k\tilde{c} + \tilde{c}) + 2kb + 2\tilde{c} - 8k + 7 \geq 0 \ .$$

But $b \leq k+1$ and $k \geq 2$ yield

$$\begin{aligned} 2ka\,(1 - k\tilde{c} + \tilde{c}) + 2kb + 2\tilde{c} - 8k + 7 & \\ \leq & \ 2ka\,(1 - k\tilde{c} + \tilde{c}) + 2k^2 - 6k + 2\tilde{c} + 7 \\ \leq & \ 4a\,(1 - \tilde{c}) + 2\tilde{c} + 3 \\ \leq & \ 7 - 2\tilde{c} \ . \end{aligned}$$

Hence, we get a contradiction except when $\tilde{c} = 3$, $a = 1$, $k = 2$ and $b = 3$. In this case, by (9), we have $\omega \geq 4$, this means $n \geq 16$. Then (15) and (16) give $8 \geq 35/3$, which is impossible.

*Case* 3.2.2.2. $a = 0$.
Note that $c \geq \max\{3k-7, 6\}$, since $n \geq 4k-6$. Using $e(x, C) = k-b+1$ and $|C_i| \geq k - b + 2$ for all components $C_i$ of $C$, by (15) we get

$$k + b + \frac{c}{2} - 2 \geq \frac{2k-2}{2k-1}(b + c) + \frac{4k-5}{2k-1} \quad \text{, i.e.}$$

$$2k^2 - 9k - kc + b + \frac{3}{2}c + 7 \geq 0 \ .$$

But because of $b \leq k+1$ we get the contradiction

$$2k^2 - 9k - kc + b + \frac{3}{2}c + 7 \leq 2k^2 - 8k + 8 + c\,(\frac{3}{2} - k) < 0 \ .$$

*Case* 4. $1 \leq h \leq k-1$

*Case* 4.1. $b \geq h+2$.
From (12) we deduce

$$n \geq (3k+1)\,a - (3k-1)\,b + 3h\,(h+1) + 3m\,(b-h-1) + 6 \quad \text{, i.e.}$$

$$m \leq \frac{n - a - b - 3ka + 3kb - 3h^2 - 3h - 6}{3(b-h-1)} \ . \tag{22}$$

Then $m \geq h$ yields

$$a \leq \frac{n - 3hb + 3kb - b - 6}{3k+1} . \qquad (23)$$

Combining (11) and (23), we see

$$\sigma_2 \leq 2a + h + \frac{n - a - b - 3ka + 3kb - 3h^2 - 3h - 6}{3(b-h-1)} =: f_3(a) \qquad (24)$$

with $f_3{}'(a) = 0$ if and only if $b = h + \frac{k}{2} + \frac{7}{6}$.

*Case* 4.1.1. $b \geq h + \frac{k}{2} + \frac{7}{6}$.

*Case* 4.1.1.1. $b > (kn+2)/(2k-h)$.
Using (24), $a \leq n - b$ and $h \geq 1$, we obtain

$$\begin{aligned}
\sigma_2 &\leq h + 2n - 2b + \frac{2kb - kn - h^2 - h - 2}{b - h - 1} \\
&< h + 2n - 2\,\frac{kn+2}{2k-h} + \frac{2k\,\frac{kn+2}{2k-h} - kn - h^2 - h - 2}{\frac{kn+2}{2k-h} - h - 1} \\
&\leq \frac{2k-2}{2k-1}\,n + \frac{4k-6}{2k-1} ,
\end{aligned}$$

which is impossible by (6).

*Case* 4.1.1.2. $b \leq (kn+2)/(2k-h)$.
By (24), (23) and $h \geq 1$, again we have the contradiction

$$\begin{aligned}
\sigma_2 &\leq h + \frac{2}{3k+1}\,(n - 3hb + 3kb - b - 6) + h \\
&\leq \frac{2k-2h}{2k-h}\,n + 2h - \frac{4}{2k-h} \\
&\leq \frac{2k-2}{2k-1}\,n + \frac{4k-6}{2k-1} .
\end{aligned}$$

*Case* 4.1.2. $b < h + \frac{k}{2} + \frac{7}{6}$.

*Case* 4.1.2.1. $n \geq 3(k-h)^2 - 2k + 3h + 7$.
In this case (24), $a \geq k - h$, $b \geq h + 2$ and $h \leq k - 1$ give

$$\sigma_2 \leq \frac{n}{3} + \frac{5k-5}{3} < \frac{2k-2}{2k-1}\,n + \frac{4k-5}{2k-1} ,$$

contradicting (6).

*Case* 4.1.2.2. $n < 3(k-h)^2 - 2k + 3h + 7$.
By (11), (22), $a \geq k-h$, and $b < h + \frac{k}{2} + \frac{7}{6}$, we have

$$\sigma_2 \leq k + 3h - \frac{1}{3} + \frac{2n + 6k - 10h - 6h^2 - 14}{3k+1} \quad ,$$

the right-hand side taking its maximum value for $h = \frac{3}{4}k - \frac{7}{12}$, i.e.

$$\begin{aligned} \sigma_2 &\leq \frac{2}{3k+1}\, n + \frac{153k^2 + 18k - 295}{72k + 24} \\ &< \frac{2k-2}{2k-1}\, n + \frac{4k-5}{2k-1} \quad , \end{aligned}$$

contradicting (6) again.

*Case* 4.2. $b \leq h + 1$.

*Case* 4.2.1. $B \setminus (N_G(x) \cup \{x\}) \neq \emptyset$.
Applying (12), we get

$$n \geq (3k+1)\,a - (3k-1)\,b + 3h\,(b-1) + 3m + 6 \quad \text{, i.e.}$$

$$m \leq kb - ka - hb + h - 2 + \frac{n-a-b}{3} \quad .$$

Then, using (11), $a \geq k-h$, and $h \leq k-1$, it follows

$$\sigma_2 \leq \frac{n}{3} + \frac{5k-7}{3} < \frac{2k-2}{2k-1}\, n + \frac{4k-5}{2k-1} \quad ,$$

contradicting (6).

*Case* 4.2.2. $B \setminus (N_G(x) \cup \{x\}) = \emptyset$.
By (9), $a \geq k-h$, $b \leq h+1$ and $h \leq k-1$, we have $\omega \geq 2$.

*Case* 4.2.2.1. $a \geq k - h + 1$.
Applying (14) and (16), we obtain

$$2\tilde{c} + 2b + 2a - 2 \geq \frac{2k-2}{2k-1}\,(a + b + \tilde{c}(ka - kb + hb + 2)) + \frac{4k-5}{2k-1} \quad \text{, i.e.}$$

$$2ka\,(1 - k\tilde{c} + \tilde{c}) + 2kb\,(1 + k\tilde{c} - \tilde{c}) + 2hb\tilde{c}\,(1-k) + 2\tilde{c} - 8k + 7 \geq 0 \quad .$$

But $\tilde{c} \geq 3$, $b \leq h+1$ and $h \leq k-1$ yield

$$\begin{aligned} & 2ka(1 - k\tilde{c} + \tilde{c}) + 2kb(1 + k\tilde{c} - \tilde{c}) + 2hb\tilde{c}(1-k) + 2\tilde{c} - 8k + 7 \\ &\leq 2ka\,(4 - 3k) + 2kb\,(3k-2) + 6hb\,(1-k) - 8k + 13 \\ &\leq 2k^2 - 6k^3 - 8hk + 6hk^2 + b\,(6k^2 - 4k - 6hk + 6h) + 13 \\ &\leq 8k^2 - 6k^3 - 4k + 13 + h^2\,(6 - 6k) + h\,(12k^2 - 18k + 6) \\ &\leq -4k^2 + 2k + 13 \quad , \end{aligned}$$

getting a contradiction except when $k = 2$, $h = 1$, $b = a = 2$ and $\tilde{c} = 3$. But in this case, applying (9), we obtain $\omega \geq 4$, which implies $n \geq 16$. Then (15) yields $8 \geq 35/3$, which is impossible.

*Case* 4.2.2.2. $a = k - h$.

Note that again, by $n \geq 4k - 6$, we have $c \geq \max\{3k - 7, 6\}$. Using $e(x, C) = h - b + 1$ and $|C_i| \geq k - b + 2 > h - b + 1$ for all components $C_i$ of $C$, by (15) and (16) we get

$$2(k-h) + h + b + \frac{c}{2} - 2 \geq \frac{2k-2}{2k-1}(k - h + b + c) + \frac{4k-5}{2k-1} ,$$

which analogously to case 3.2.2.2 yields a contradiction.

*Case* 5. $h = 0$.

Let $X := \{v \in B | d_{G\setminus A}(v) = 0\}$. By $n - a - b \geq \omega \geq ka - kb + b - |X| + 2$ and $b \leq n - a$, we have

$$a \leq \frac{1}{2k-1}(|X| + (k-1)n - 2) . \tag{25}$$

According to (7), we may assume $|X| \leq 2k - 1$.

*Case* 5.1. $|X| \geq 2$.

Combinig (11), (25) and $|X| \leq 2k - 1$, we find

$$\begin{aligned} \sigma_2 &\leq 2a \\ &\leq \frac{2k-2}{2k-1} n + \frac{2|X|-4}{2k-1} \\ &\leq \frac{2k-2}{2k-1} n + \frac{4k-6}{2k-1} , \end{aligned}$$

which contradicts (6).

*Case* 5.2. $|X| = 1$.

*Case* 5.2.1. $b \geq 2$.

Applying (12), we get

$$n \geq (3k+1)a - (3k-1)b + 3m(b-1) + 6 \text{ , i.e.}$$

$$m \leq \frac{n - a - b - 3ka + 3kb - 6}{3(b-1)} . \tag{26}$$

$|X| = 1$ implies $m \geq 1$, which yields

$$a \leq \frac{n + 3kb - 4b - 3}{3k + 1} \quad . \tag{27}$$

By (11) and (26), we have

$$\sigma_2 \leq 2a + \frac{n - a - b - 3ka + 3kb - 6}{3(b-1)} =: f_4(a) \; , \tag{28}$$

yielding $f_4{}'(a) = 0$ if and only if $b = \frac{k}{2} + \frac{7}{6}$.

*Case* 5.2.1.1. $b \geq \frac{k}{2} + \frac{7}{6}$.

*Case* 5.2.1.1.1. $b > (kn+1)/(2k-1)$.
In this case (28) and $a \leq n - b$ give

$$\begin{aligned} \sigma_2 &\leq 2n - 2b + \frac{2kb - kn - 2}{b - 1} \\ &< 2n - 2\frac{kn+1}{2k-1} + \frac{2k\,\frac{kn+1}{2k-1} - kn - 2}{\frac{kn+1}{2k-1} - 1} \\ &= \frac{2k-2}{2k-1}\,n + \frac{2k-3}{2k-1} \; , \end{aligned}$$

which is impossible according to (6).

*Case* 5.2.1.1.2. $b \leq (kn+1)/(2k-1)$.
By (28), (27) and $b \leq (kn+1)/(2k-1)$, here again we have

$$\begin{aligned} \sigma_2 &\leq 1 + \frac{2}{3k+1}\,(n + (3k-4)\,b - 3) \\ &\leq 1 + \frac{2}{3k+1}\,(n + (3k-4)\,\frac{kn+1}{2k-1} - 3) \\ &= \frac{2k-2}{2k-1}\,n + \frac{2k-3}{2k-1} \; , \end{aligned}$$

contradicting (6).

*Case* 5.2.1.2. $b < \frac{k}{2} + \frac{7}{6}$.
Using (9), we have $\omega \geq 1$, and hence (15) implies

$$2a + \tilde{c} + b - 2 \geq \frac{2k-2}{2k-1}\,(a + b + \tilde{c}\,(ka - kb + b + 1)) + \frac{4k-5}{2k-1} \; , \text{ i.e.}$$

$$2ka - 2k\tilde{c}\,(ka - kb + b) + 2\tilde{c}\,(ka - kb + b + 1) - 8k + b + 7 - \tilde{c} \geq 0 \; .$$

On the other hand, $a \geq k$, $b < \frac{k}{2} + \frac{7}{6}$ and $\tilde{c} \geq 3$ give the contradiction

$$\begin{aligned}
&2ka - 2k\tilde{c}(ka - kb + b) + 2\tilde{c}(ka - kb + b + 1) - 8k + b + 7 - \tilde{c} \\
\leq\ & 2k^2 - 2k\tilde{c}(k^2 - kb + b) + 2\tilde{c}(k^2 - kb + b + 1) - 8k + b - \tilde{c} + 7 \\
\leq\ & 2k^2 - 2k\tilde{c}(\frac{k^2}{2} - \frac{2}{3}k + \frac{7}{6}) + 2\tilde{c}(\frac{k^2}{2} - \frac{2}{3}k + \frac{13}{6}) - \frac{15}{2}k + \frac{49}{6} - \tilde{c} \\
\leq\ & -3k^3 + 9k^2 - \frac{37}{2}k + \frac{109}{6} \\
<\ & 0\ .
\end{aligned}$$

*Case* 5.2.2. $b = 1$.

Condition (9) yields $\omega \geq 1$. Then (15) together with (16) provides

$$2a + \tilde{c} - 1 \geq \frac{2k-2}{2k-1}\left(a + 1 + \tilde{c}(ka - k + 2)\right)\ .$$

But from $a \geq k$, $\tilde{c} \geq 3$ and $k \geq 2$ we get the following contradiction, which completes the proof.

$$\frac{1}{2} \geq \frac{2a + \tilde{c} - 1}{a + 1 + \tilde{c}(ka - k + 2)} \geq \frac{2k-2}{2k-1}\ .$$

With the help of Theorem 8 we will now proof Theorems 6 and 7.

**Proof of Theorem 6.** The proof is by contradiction. Let $G$ be a graph satisfying the hypotheses of Theorem 6, which has no $k$-factor. Then, by Theorem 8 there exist two disjoint non-empty sets $A, B \subseteq V(G)$ such that (7) and (8) hold.

Analogously to (25), we have

$$a \leq \frac{1}{2k-1}\left(|X| + (k-1)\,n - 2\right)\ .$$

Since $X$ is an independent set with $|X| \geq 2k$ and $N(X) \subseteq A$, we obtain

$$|N(X)| \leq \frac{1}{2k-1}\left(|X| + (k-1)\,n - 2\right)\ ,$$

contradicting (3), so the proof is complete.

The proof of Theorem 7 runs analogously to that of Niessen in [7]. For reasons of completeness we shall give it here again.

**Proof of Theorem 7.** This proof is by contradiction. Let $G$ be a graph without $k$-factor satisfying the conditions of Theorem 7. Again, by

Theorem 8 there exist two disjoint non-empty sets $A, B \subseteq V(G)$ such that (7) and (8) hold.
Using (7), we observe together with (11) that

$$|A| \geq \frac{\sigma_2}{2} . \tag{29}$$

Let $Y := \{v \in B \mid d_{G\setminus A}(v) = 1\}$.
Furthermore let $C_1, ..., C_{\omega_1}$ be the components of $G \setminus (A \cup B)$ with at least three vertices, having a vertex without neighbour in $Y$. Let $S_1$ be a set containing one such vertex from every $C_i$, $i = 1, ..., \omega_1$. The remaining components of $G \setminus (A \cup B)$ with at least three vertices are denoted by $D_1, ..., D_{\omega_2}$.
Let $Y_1 := \{v \in Y \mid N_{G\setminus A}(v) \subseteq B\}$ and $Y_2 := Y \setminus Y_1$. Then the graph induced by $Y_1$ in $G$ has maximum degree at most 1. Let $S_2$ be a maximum independent set of this graph. Then it is obvious that $|S_2| \geq \frac{1}{2}\,|Y_1|$.
Since every vertex of every $D_i$ has a neighbour in $Y_2$ and since these neighbours are distinct, we have

$$|Y_2| \geq \sum_{i=1}^{\omega_2} |V(D_i)| \geq 3\omega_2 \geq 2\omega_2 . \tag{30}$$

According to our definitions, $X \cup S_1 \cup S_2 \cup Y_2$ is an independent set of $G$. By (30), we have

$$\alpha \geq |X| + \omega_1 + \frac{1}{2}\,|Y_1| + \frac{1}{2}\,|Y_2| + \omega_2 = |X| + \omega + \frac{1}{2}\,|Y| . \tag{31}$$

Now, applying (31), (8) and (29), we get

$$\begin{aligned}
\alpha &\geq \omega + |X| + \frac{1}{2}\,|Y| \\
&\geq k|A| - k|B| + d_{G\setminus A}(B) + 2 + |X| + \frac{1}{2}\,|Y| \\
&\geq k\,\frac{\sigma_2}{2} - k|B| + d_{G\setminus A}(B \setminus (X \cup Y)) + 2 + |X| + \frac{3}{2}\,|Y| \\
&\geq k\,\frac{\sigma_2}{2} - k|B| + 2\,|B \setminus (X \cup Y)| + 2 + |X| + \frac{3}{2}\,|Y| \\
&\geq k\,\frac{\sigma_2}{2} - (k-2)\,|B| + 2 - \alpha ,
\end{aligned}$$

which means

$$(k-2)\,|B| \geq k\,\frac{\sigma_2}{2} - 2\alpha + 2 . \tag{32}$$

If $k = 2$, then (32) is equivalent to $\sigma_2 \leq 2\alpha - 2$, contradicting (5). If $k \geq 3$, then (29), (32) and (5) yield the contradiction

$$0 \leq n - |A| - |B| \leq n - \frac{\sigma_2}{2} - \frac{1}{k-2}\,(k\frac{\sigma_2}{2} - 2\alpha + 2) < 0 .$$

## 3. REMARKS

Because of $\sigma_2/2 \geq \delta$ it is clear that Theorem 6 implies Theorem 2 and for $n \geq 4k-6$ Theorem 7 implies Theorem 5. We now show that for $k \geq 2$ Theorem 6 also implies Theorem 3.

**Theorem 6 implies Theorem 3.** Let $G$ satisfying the hypotheses of Theorem 3 with $k \geq 2$. Obviously, $G$ is connected. (1) implies that one of every two independent vertices has degree at least $n/2$, which means that for every independent set $X \subseteq V(G)$ with $|X| \geq 2k$, we have

$$|X| \leq \frac{1}{2}n$$

and

$$\begin{aligned} |N(X)| &\geq \frac{1}{2}n \\ &= \frac{1}{2k-1}\left(\frac{1}{2}n + (k-1)n\right) \\ &\geq \frac{1}{2k-1}\left(|X| + (k-1)n - 1\right) . \end{aligned}$$

Furthermore, by $n \geq 4k-5$, we get

$$\begin{aligned} \sigma_2 &\geq n \\ &= \frac{2(k-1)(n+2) + (n-4k+4)}{2k-1} \\ &\geq \frac{2k-2}{2k-1}n + \frac{4k-5}{2k-1} . \end{aligned}$$

So $G$ satisfies the conditions of Theorem 6.

Examples showing that Theorems 6 and 7 do not imply each other can be found in [7].

Last we will show that the conditions of Theorems 6 and 7 are best possible. The connectedness of $G$, when $k$ is odd is needed to avoid that $G$ is the disjoint union of two odd complete graphs of roughly equal order.

To see that conditions (3), (4) and (5) of Theorems 6 and 7 are best possible, we consider graphs of the form

$$G := K_{r+2(sk-s-1)} + (rK_1 \cup (sk-1)K_2)$$

with $r \geq 0$ and $s \geq 2$, which do not contain a $k$-factor.

Choosing $r = 2k$, we see that condition (3) in Theorem 6 and condition (5) in Theorem 7 are best possible. For $r = 0$, we see that condition (4) in both theorems is needed.

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