The Fifth Jump of the Point-Distinguishing Chromatic Index of $K_{n,n}$

Mirko Horňák and Roman Soták
Department of Geometry and Algebra
P.J. Šafárik University
Jesenná 5, 041 54 Košice, Slovakia
email: hornak@turing.upjs.sk, sotak@turing.upjs.sk

ABSTRACT. The point-distinguishing chromatic index $\chi_0(G)$ of a graph G represents the minimum number of colours in an edge colouring of G such that each vertex of G is distinguished by the set of colours of its incident edges. It is known that $\chi_0(K_{n,n})$ is a non-decreasing function of n with jumps of value 1. We prove that $\chi_0(K_{46,46}) = 7$ and $\chi_0(K_{47,47}) = 8$.

Harary and Plantholt [1] introduced the point-distinguishing chromatic index $\chi_0(G)$ of a graph G (with at most one component K_1 and without components K_2) as the minimum integer k admitting a k-colouring of edges of G such that for each pair (x, y) of different vertices of G the colour set of x - the set of colours of edges incident with x - is different from the colour set of y. They determined values of this invariant for several classes of graphs with simple structure (complete graphs, paths, cycles, cubes) and proved that for any integer $n \geq 2$

$$\lceil \log_2 n \rceil + 1 \leq \chi_0(K_{n,n}) \leq \lceil \log_2 n \rceil + 2.$$

Using results of Zagaglia Salvi [3] it is easy to see that $\chi_0(K_{n,n})$ is a non-decreasing integer function of n with jumps of value 1. By n_k will be denoted the maximum integer n such that $\chi_0(K_{n,n}) = k$. Zagaglia Salvi [3] found first values of n_k: $n_3 = 2$, $n_4 = 5$, $n_5 = 11$ and $n_6 = 22$. The same author in [4] claims to have determined all values of n_k, namely by recurrent relations $n_{k+1} = 2n_k$ for odd $k \geq 5$ and $n_{k+1} = 2n_k + 1$ for even $k \geq 6$. However, Horňák and Soták [2] proved an assertion contradicting implicitly results of [4]. The aim of the present paper is to show that $n_7 = 46$. This is the first contradicting statement as regards the recurrence above: n_7 is not equal to 45.

For integers p, q set

$$[p,q] := \cup_{i=p}^{q}\{i\}, \ [p,\infty) := \cup_{i=p}^{\infty}\{i\}$$

and for $k \in [3,\infty)$, $n \in [2,\infty)$ let $\mathcal{M}_{k,n}$ be the set of all square matrices M of order n with entries from $[1,k]$ such that sets of elements occurring in lines (rows or columns) of M are pairwise disjoint. As a straightforward consequence of the definition we get

Proposition 1. *If $n \in [2,\infty)$, then $\chi_0(K_{n,n}) = \min\{k \in [3,\infty): \mathcal{M}_{k,n} \neq \emptyset\}$.*

For a matrix M let $\mathcal{L}_i^1(M)$ be the set of all entries of the i-th row of M and $\mathcal{L}^1(M)$ the set of sets $\mathcal{L}_i^1(M)$ for i running over all row indices of M; $\mathcal{L}_i^2(M)$ and $\mathcal{L}^2(M)$ will be analogous sets concerning columns of M.

If $M \in \mathcal{M}_{k,n}$, then clearly

$$|\mathcal{L}^1(M)| = |\mathcal{L}^2(M)| = n, \quad \mathcal{L}^1(M) \cap \mathcal{L}^2(M) = \emptyset,$$
$$\mathcal{L}_i^1(M) \cap \mathcal{L}_j^2(M) \neq \emptyset \quad \text{for every } i,j \in [1,n].$$

Thus, provided $\mathcal{P}X$ denotes the set of all subsets of a set X, $\mathcal{P}^2X = \mathcal{P}(\mathcal{P}X)$ and $\mathcal{S}_{k,n}$ is the set

$$\{(\mathcal{S}^1,\mathcal{S}^2) \in (\mathcal{P}^2[1,k])^2 : |\mathcal{S}^1| = |\mathcal{S}^2| = n, \mathcal{S}^1 \cap \mathcal{S}^2 = \emptyset,$$
$$\forall S^1 \in \mathcal{S}^1 \ \forall S^2 \in \mathcal{S}^2 \ S^1 \cap S^2 \neq \emptyset\}$$

with $k \in [3,\infty)$, $n \in [2,\infty)$, non-emptiness of $\mathcal{M}_{k,n}$ implies non-emptiness of $\mathcal{S}_{k,n}$. The inverse implication in general does not hold, but according to [3, Theorem 4.5] it does for n great enough with respect to k:

Theorem 1. *If $k \in [3,\infty)$ and $n \in [\lceil \frac{2^k}{3} \rceil, 2^{k-1}]$, then $\mathcal{M}_{k,n} \neq \emptyset$ if and only if $\mathcal{S}_{k,n} \neq \emptyset$.*

Since by [3, Corollary 3.5] $\chi_0(K_{n,n}) = k$ for $n \in [2^{k-2}, \lfloor \frac{2^k}{3} \rfloor]$, an advantage following from Proposition 1 and Theorem 1 consists in the fact that deciding whether $\chi_0(K_{n,n}) = k$ or not for $n \in [\lceil \frac{2^k}{3} \rceil, 2^{k-1}]$ we need not analyze $\mathcal{M}_{k,n}$, but (more comfortably) $\mathcal{S}_{k,n}$.

Theorem 2. $n_7 = 46$.

Proof: As $[46,47] \subseteq [\lceil \frac{2^7}{3} \rceil, 2^6]$, to show that $\chi_0(K_{46,46}) = 7$ and $\chi_0(K_{47,47}) = 8$ it suffices to find a pair $(\mathcal{S}^1,\mathcal{S}^2)$ in $\mathcal{S}_{7,46}$ and to derive a contradiction from the assumption $\mathcal{S}_{7,47} \neq \emptyset$.

(a) Put

$$\begin{aligned}
\mathcal{S}_2^1 &= \{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\},\\
\mathcal{S}_3^1 &= \{\{2,6,7\},\{4,6,7\},\{5,6,7\}\} \cup \{A\cup\{i\}\colon A\in\mathcal{S}_2^1,\ i\in[1,7]\backslash A\},\\
\mathcal{S}_i^1 &= \{[1,7]\backslash A\colon A\in\mathcal{S}_{7-i}^1\}, && i=4,5,\\
\mathcal{S}_3^2 &= \{\{1,3,6\},\{1,3,7\},\{2,4,5\},\{2,4,6\},\{2,4,7\}\},\\
\mathcal{S}_i^2 &= \{A\subseteq[1,7]\colon |A|=i, A\notin\mathcal{S}_i^1\}, && i=4,5,\\
\mathcal{S}_j^2 &= \{A\subseteq[1,7]\colon |A|=j\}, && j=6,7,\\
\mathcal{S}^1 &= \cup_{i=2}^5\mathcal{S}_i^1,\\
\mathcal{S}^2 &= \cup_{i=3}^7\mathcal{S}_i^2.
\end{aligned}$$

One can easily check that $(\mathcal{S}^1,\mathcal{S}^2)\in\mathcal{S}_{7,46}$. ($\mathcal{S}_i^j$ is the intersection of $\mathcal{S}^j$ with $\mathcal{P}_i[1,7]$, the set of all i-element subsets of $[1,7]$.)

(b) Suppose $\chi_0(K_{47,47})=7$ and take $(\mathcal{S}^1,\mathcal{S}^2)\in\mathcal{S}_{7,47}$. If $\mathcal{T}_{ij}$ is the set of cardinality t_{ij} of all those $A\subseteq[1,7]$ for which $A\in\mathcal{S}^1\cup\mathcal{S}^2$ has truth-value i and $[1,7]\backslash A\in\mathcal{S}^1\cup\mathcal{S}^2$ has truth-value j, i, $j=0,1$, then $\{\mathcal{T}_{00},\mathcal{T}_{01},\mathcal{T}_{10},\mathcal{T}_{11}\}$ is a decomposition of $\mathcal{P}[1,7]$,

$$\begin{aligned}
t_{00}+t_{01}+t_{10}+t_{11} &= 128,\\
t_{10}+t_{11} = |\mathcal{S}^1\cup\mathcal{S}^2| &= 94,\\
t_{00}+t_{01} &= 34.
\end{aligned}$$

As evidently $t_{01}=t_{10}$, we have

$$t_{11}=94-(34-t_{00})\geq 60.$$

Setting for $j\in[1,2]$

$$\mathcal{T}_{11}^j=\mathcal{T}_{11}\cap\mathcal{S}^j,\quad t_{11}^j=|\mathcal{T}_{11}^j|,$$

we get $t_{11}^1+t_{11}^2=t_{11}$ and $t_{11}^j\equiv 0 \pmod 2$ (consider that $A\in\mathcal{T}_{11}^j$ implies $[1,7]\backslash A\in\mathcal{T}_{11}^j$), hence $|\mathcal{S}^j|=47$ and $t_{11}\geq 60$ yields

$$14\leq t_{11}^j\leq 46.$$

For

$$\begin{aligned}
\mathcal{S}_i^j &= \mathcal{S}^j\cap\mathcal{P}_i[1,7],\ s_i^j=|\mathcal{S}_i^j|,\\
\mathcal{S}_i &= \mathcal{S}_i^1\cup\mathcal{S}_i^2, \qquad s_i=|\mathcal{S}_i|,\ i\in[0,7],\ j\in[1,2],
\end{aligned}$$

we obtain immediately $s_0 = 0$, according to [3, Theorem 3.3] $s_1 = 0$ and as a consequence

$$\sum_{i=2}^{7} s_i = 94.$$

Let G_2^j be the graph $([1,7], \mathcal{S}_2^j)$, $j = 1, 2$. With respect to the obvious symmetry of the set $\mathcal{S}_{7,47}$ we can suppose without loss of generality $\Delta(G_2^1) \le \Delta(G_2^2)$.

(1) $\Delta(G_2^2) \ge 4$ If $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\}\} \subseteq \mathcal{S}_2^2$ and $A \in \mathcal{T}_{11}^1$, then $A \cap [1,5]$ must be either $\{1\}$ or $[2,5]$; since for $A \cap [6,7]$ there are 4 possibilities, $t_{11}^1 \le 2 \cdot 4 = 8 < 14$ - a contradiction.

(2) $\Delta(G_2^2) = 3$: As in the case (1), for $\{\{1,2\},\{1,3\},\{1,4\}\} \subseteq \mathcal{S}_2^2$ the assumption $A \in \mathcal{T}_{11}^1$ implies $A \cap [1,4] \in \{\{1\},[2,4]\}$ and the number of sets fulfilling the last condition is $2{\cdot}8 = 16$. However, according to $s_1 = 0$ A must be different from $\{1\}$ and $[2,7]$. Furthermore, $\{\{1,5\},\{1,6\},\{1,7\}\} \subseteq \mathcal{T}_{11}^1$ would analogously mean $t_{11}^2 \le 16 - 2 = 14$ and $t_{11} = t_{11}^1 + t_{11}^2 \le 28$ in contradiction with $t_{11} \ge 60$. Thus $t_{11}^1 \le 16 - 3 = 13$, which is impossible.

(3) $\Delta(G_2^2) \le 2$

(31) $s_2^1 s_2^2 > 0$: Let $\{1,2\} \in \mathcal{S}_2^1$, $\{1,3\} \in \mathcal{S}_2^2$. If $A \in \mathcal{S}_3$ and $A \cap [1,3] = \{i\} \in \{\{2\},\{3\}\}$, then necessarily $A \in \mathcal{S}_3^{4-i}$; since $[2,7]\backslash A$ is a 3-element subset of $[1,7]$ and its intersection with $[1,3]$ is $\{5-i\} \in \{\{2\},\{3\}\}$, it does not belong to $\mathcal{S}_3$ (it would be obliged to be in $\mathcal{S}_3^{4-(5-i)} = \mathcal{S}_3^{i-1}$ but then it would be disjoint with $A \in \mathcal{S}_3^{4-i} \neq \mathcal{S}_3^{i-1}$). Thus, as there are twelve 3-element sets $A \subseteq [1,7]$ fulfilling $A \cap [1,3] \in \{\{2\},\{3\}\}$, at least six of them do not belong to $\mathcal{S}_3$. Moreover, $[4,7]$ as well as each of its subsets (four of them are of cardinality 3) is missing in $\mathcal{S}^1 \cup \mathcal{S}^2$. These two observations lead to

$$s_3 \le \binom{7}{3} - (6+4) = 25.$$

In general, if $A \in \mathcal{S}_2^1$ and $B \in \mathcal{S}_2^2$, then $|A \cap B| = 1$, $|A \cup B| = 3$ and the 4-element set $[1,7]\backslash(A \cup B)$ is not in $\mathcal{S}_4$. At most one pair $(A', B') \in \mathcal{S}_2^1 \times \mathcal{S}_2^2$ different from (A, B) determines the same 4-element set $[1,7]\backslash(A' \cup B') = [1,7]\backslash(A \cup B)$ - there are only three 2-element subsets of $A \cup B$ and at most two ordered pairs of them are in $\mathcal{S}^1 \times \mathcal{S}^2$. That is why at least $\left\lceil \frac{s_2^1 s_2^2}{2} \right\rceil$ 4-element subsets of $[1,7]$ are not present in $\mathcal{S}_4$ and

$$s_4 \le \binom{7}{4} - \left\lceil \frac{s_2^1 s_2^2}{2} \right\rceil.$$

From the obtained inequalities with respect to $s_i \le \binom{7}{i} = 5, 6, 7$, we have

$$94 \le s_2 + 25 + 35 - \left\lceil \frac{s_2^1 s_2^2}{2} \right\rceil + 21 + 7 + 1,$$

$$5 \le s_2^1 + s_2^2 - \left\lceil \frac{s_2^1 s_2^2}{2} \right\rceil,$$

and, since $(s_2^1 - 1)(s_2^2 - 1) \ge 0$ implies $s_2^1 s_2^2 \ge s_2^1 + s_2^2 - 1$,

$$5 \le s_2^1 + s_2^2 - \left\lceil \frac{s_2^1 + s_2^2 - 1}{2} \right\rceil = \left\lfloor \frac{s_2^1 + s_2^2 + 1}{2} \right\rfloor = \left\lfloor \frac{s_2 + 1}{2} \right\rfloor,$$

so that finally $s_2 \ge 9$.

There exists $i \in \{1, 2\}$ with $s_2^i \ge 5$, hence at least three sets from $\mathcal{S}_2^i$ have the same intersection with $\{1, 4 - i\} \in \mathcal{S}_2^{3-i}$ which means that $\Delta(G_2^i) \ge 3$ - a contradiction.

(32) $s_2^1 = 0$

(321) G_2^2 contains a path P_4 on 4 vertices with non-adjacent endvertices: If $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\} \subseteq \mathcal{S}_2^2$, then $A \in \mathcal{T}_{11}^1$ fulfills $A \cap [1, 5] \in \{\{2, 4\}, \{1, 3, 5\}\}$ and $t_{11}^1 \le 2 \cdot 4 = 8 < 14$.

(322) The length of any path in G_2^2 with non-adjacent endvertices is at most 2.

(3221) G_2^2 contains a cycle C_3: The inclusion $\{\{1, 2\}, \{2, 3\}, \{3, 1\}\} \subseteq \mathcal{S}_2^2$ would imply $t_{11}^1 = 0 < 14$.

(3222) G_2^2 contains C_4: From $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\} \subseteq \mathcal{S}_2^2$ and $A \in \mathcal{T}_{11}^1$ we obtain $A \cap [1, 4] \in \{\{1, 3\}, \{2, 4\}\}$ and $3 \le |A| \le 4$ ($A \in \mathcal{S}_i^1$ for $i \in [5, 7]$ leads to $[1, 7] \backslash A \in \mathcal{T}_{11}^1 \cap \mathcal{S}_{7-i}^1$ in contradiction with $s_0 = s_1 = s_2^1 = 0$), hence $1 \le |A \cap [5, 7]| \le 2$ and $t_{11}^1 \le 2 \cdot (3 + 3) = 12 < 14$.

(3223) G_2^2 is acyclic (C_l has a subgraph P_{l-1} with non-adjacent endvertices) and its components are paths of lengths at most 2.

(32231) $s_2^2 \ge 4$: G_2^2 has c components, $2 \le c \le 3$. If P_l is a connected component of G_2^2 with $E(P_l) = \{\{i, i+1\} : i \in [m, m+l-2]\}$, then for any $A \in \mathcal{T}_{11}^1$ we have only two possibilities for $A \cap [m, m + l - 1]$, namely sets consisting of all elements of $[m, m + l - 1]$ of the same parity. Thus we can claim $t_{11}^1 \le 2^c \le 8 < 14$.

(32232) $s_2^2 \le 3$: In the remaining part of our analysis it is unimportant that $s_2^1 s_2^2 = 0$. We also release the assumption $\Delta(G_2^1) \le \Delta(G_2^2)$ - we suppose only $s_2 \le 3$. Then $t_{11}^i \ge 14$ together with $s_0^i = s_1^i = 0$ implies $s_2^i + s_3^i \ge \frac{14}{2}$ and $s_3^i \ge 4$, $i = 1, 2$.

Consider the Kneser graph $K(7, 3)$ with vertex set $\mathcal{P}_3[1, 7]$ and with edges joining just disjoint triples depicted in Fig.1. It is convenient for the study of the structure of $\mathcal{S}_3^1$ and $\mathcal{S}_3^2$ due to the fact that the subgraphs of $K(7, 3)$

induced by $\mathcal{S}_3^1$ and $\mathcal{S}_3^2$ are vertex-disjoint. However, before using it we point to some of its properties relevant for our proof.

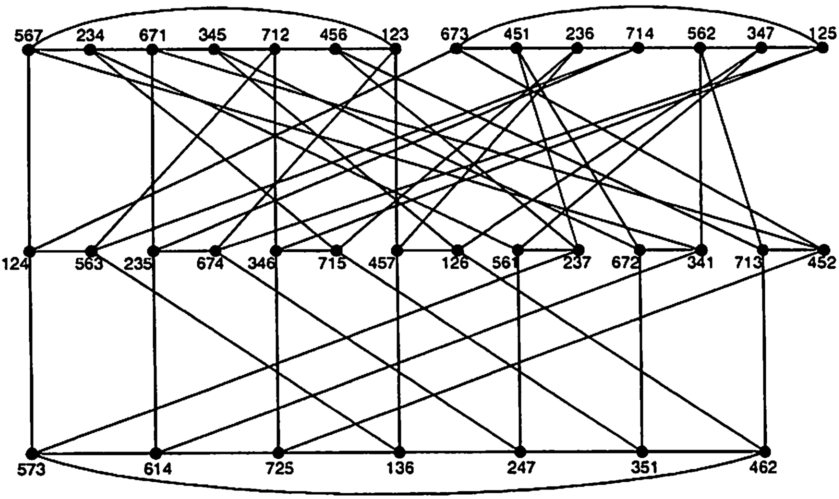

Figure 1. The Kneser Graph $K(7,3)$

Define the *ordered difference modulo* 7 of a set $\{a_1, a_2, a_3\} \subseteq [1,7]$ with $a_1 < a_2 < a_3$ as the minimum ordered triple from among $(a_2-a_1, a_3-a_2, 7+a_1-a_3)$, $(a_3-a_2, 7+a_1-a_3, a_2-a_1)$ and $(7+a_1-a_3, a_2-a_1, a_3-a_2)$ with respect to lexicographic ordering. Each of five possible differences, i.e. $(1,1,5)$, $(2,2,3)$, $(1,3,3)$, $(1,2,4)$ and $(1,4,2)$, corresponds to seven members of $\mathcal{P}_3[1,7]$.

If $\mathcal{C}_i$ is the set of all members of $\mathcal{P}_3[1,7]$ with difference containing two i's, then the subgraph of $K(7,3)$ induced on $\mathcal{C}_i$ is C_7 and that induced on $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ is $3C_7$. The set $\mathcal{K} = \mathcal{P}_3[1,7] \backslash (\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3)$ induces $7K_2$. (From now on we will use $3C_7$ and $7K_2$ exclusively for subgraphs of $K(7,3)$ induced on $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ or $\mathcal{K}$, respectively.) For $i \in [1,7]$ let $\mathcal{K}_i$ be the set of vertices of the i-th component of $7K_2$. One part of the bipartite graph $7K_2$ corresponds to the difference $(1,2,4)$, the other one to $(1,4,2)$. Every vertex of $\mathcal{K}$ has exactly one neighbour in $\mathcal{C}_i$, $i = 1,2,3$, and every vertex of $\mathcal{C}_j$ has exactly one neighbour in both parts of $7K_2$, $j = 1,2,3$.

$K(7,3)$ is a 4-regular vertex-transitive graph — for $\{a_1, a_2, a_3\}, \{a'_1, a'_2, a'_3\} \subseteq [1,7]$ any permutation $\pi\colon [1,7] \to [1,7]$ with $\pi(a_i) = a'_i$, $i = 1,2,3$, induces an automorphism of $K(7,3)$ mapping $\{a_1, a_2, a_3\}$ to $\{a'_1, a'_2, a'_3\}$. It is easy to see that a shortest cycle of $K(7,3)$ is of length 6.

The sets $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ are equivalent with respect to the structure of $K(7,3)$: The permutation $i \to (2i)_7$ of $[1,7]$, where $(2i)_7 \equiv 2i \pmod 7$, induces a permutation ϱ of $\mathcal{P}[1,7]$ which restricted on $\mathcal{P}_3[1,7]$ is an automorphism of $K(7,3)$ interchanging components of $3C_7$ by the following scheme of changes of differences:

$$(1,1,5) \to (2,2,3) \to (1,3,3) \to (1,1,5);$$

if it is applied twice, the corresponding scheme is

$$(1,1,5) \to (1,3,3) \to (2,2,3) \to (1,1,5).$$

Moreover, the i-th power of ϱ evidently transforms a pair $(\mathcal{S}^1, \mathcal{S}^2) \in \mathcal{S}_{7,47}$ to the pair $(\varrho^i(\mathcal{S}^1), \varrho^i(\mathcal{S}^2)) \in \mathcal{S}_{7,47}$, $i = 1,2$.

Put

$$\begin{aligned}
\mathcal{C}_i^j &= \mathcal{C}_i \cap \mathcal{S}^j, && i = 1,2,3, \quad j = 1,2,\\
\mathcal{C}_i^- &= \mathcal{C}_i \backslash (\mathcal{C}_i^1 \cup \mathcal{C}_i^2), && i = 1,2,3,\\
\mathcal{K}^j &= \mathcal{K} \cap \mathcal{S}^j, && j = 1,2,\\
\mathcal{K}^- &= \mathcal{K} \backslash (\mathcal{K}^1 \cup \mathcal{K}^2), &&\\
\mathcal{S}_3^- &= (\mathcal{P}_3[1,7]) \backslash (\mathcal{S}_3^1 \cup \mathcal{S}_3^2); &&
\end{aligned}$$

corresponding cardinalities will be c_i^j, c_i^-, k^j, k^- and s_3^-. From $s_2 \le 3$ we have

$$s_3 \ge 94 - 3 - \sum_{i=4}^{7} \binom{7}{i} = 27.$$

The contradiction will be reached by showing that $s_3^- \ge 9$, since then

$$35 = \binom{7}{3} = s_3^- + s_3 \ge 27 + 9 = 36.$$

Our analysis is based mainly on some simple observations.

(i) First of all, as sets $A \in \mathcal{S}^1$ and $B \in \mathcal{S}^2$ are not disjoint, for $i \in \{1,2\}$ in $K(7,3)$ any neighbour of a set from $\mathcal{S}_3^i$ must belong to $\mathcal{S}_3^i \cup \mathcal{S}_3^-$.

(ii) For $\mathcal{A} \subseteq \mathcal{S}_3^1$ and $\mathcal{B} \subseteq \mathcal{S}_3^2$ let $K(\mathcal{A},\mathcal{B})$ be the set of all $l \in [1,7]$ such that the distance $d(\mathcal{A},\mathcal{K}_l)$ in $K(7,3)$ between $\mathcal{A}$ and $\mathcal{K}_l$ is at most 1 (either $\mathcal{A} \cap \mathcal{K}_l \neq \emptyset$ or $\mathcal{A}$ has a neighbour in $\mathcal{K}_l$) simultaneously with $d(\mathcal{B},\mathcal{K}_l) \le 1$. Each $l \in K(\mathcal{A},\mathcal{B})$ corresponds to a path joining a vertex of $\mathcal{A} \subseteq \mathcal{S}_3^1$ to a vertex of $\mathcal{B} \subseteq \mathcal{S}_3^2$ with all interior vertices in $\mathcal{K}_l$; evidently, $\mathcal{K}_l \cap \mathcal{S}_3^- \neq \emptyset$, since at least one interior vertex of such a path must serve according to (i) as a "transition" between $\mathcal{S}_3^1$ and $\mathcal{S}_3^2$ and is therefore in $\mathcal{S}_3^-$. Thus $|K(\mathcal{A},\mathcal{B})|$ is a lower bound for k^-.

(iii) One can easily check that two different vertices of $3C_7$ have neighbours in 3 or 4 components of $7K_2$. For $i \in [1,3]$ any component of $7K_2$ has two neighbours in C_i whose distance (it is realized in C_i) is 3 - remember that the girth of $K(7,3)$ is 6. That is why a subpath of C_i on j vertices has neighbours in $2j$ components of $7K_2$ for $j \in [1,3]$ and in all seven components of $7K_2$ for $j \in [4,7]$.

(iv) For each $A \in \mathcal{P}_2[1,7]$ there exists a permutation σ of the set $[1,3]$ such that A is disjoint with exactly i members of $\mathcal{C}_{\sigma(i)}$, $i = 1,2,3$.

(v) On the other hand for every $i \in [1,3]$ and every 6-element subset $\mathcal{C}_i'$ of $\mathcal{C}_i$ there is exactly one set $B \in \mathcal{P}_2[1,7]$ such that B has a non-empty intersection with each member of $\mathcal{C}_i'$.

[1] $c_1^1 = 7$, $c_1^2 = c_1^- = 0$: The set of neighbours of $\mathcal{C}_1^1 = \mathcal{C}_1$ is $\mathcal{K}$ which implies $k^2 = 0$. By (iv) we have also $s_2^2 = 0$, hence the inequality $t_{11}^2 \geq 14$ implies $s_3^2 \geq 7$, in other words $c_2^2 + c_3^2 \geq 7$. Then $c_i^2 = \max\{c_2^2, c_3^2\} \geq 4$.

[11] $c_i^2 \geq 5$: Each of C_i^2 neighbours of $\mathcal{C}_i^2$ in $\mathcal{K}$ is in $\mathcal{K}^2 \cup \mathcal{K}^-$ so that $k^2 = 0$ supplies $s_3^- \geq k^- \geq 2c_i^2 \geq 10$ which is sufficient for a contradiction.

[12] $c_i^2 = 4$: As above, $k^- \geq 8$. Moreover, for $j \in [2,3]$ the set $\mathcal{C}_j^2$ (with at most four vertices) has at least two neighbours in $\mathcal{C}_j$; of course, these neighbours are in $\mathcal{C}_j^-$ and we have $s_3^- \geq 8 + 2 \cdot 2 = 12$.

[2] With respect to the interchangeability of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ and the symmetry of $C_{7,47}$ we can suppose $c_i^1 \leq 6$, $c_i^2 \leq 6$ and consequently $c_i^- \geq 1$, $i = 1,2,3$.

[21] $c_1^1 = 6$, $c_1^2 = 0$, $c_1^- = 1$: From among vertices of $\mathcal{K}$ only neighbours of $\mathcal{C}_1^-$ can be in $\mathcal{S}_3^2$ so that $k^2 \leq 2$. With respect to (v) $s_2^2 \leq 1$, $s_3^2 \geq 6$ and using (iii) the set $\mathcal{C}_1^1$ has neighbours in all components of $7K_2$.

[211] $c_i^2 = \max\{c_2^2, c_3^2\} = 6$, $c_i^1 = 0$, $c_i^- = 1$: Twelve neighbours of $\mathcal{C}_i^2$ in $\mathcal{K}$ are from $\mathcal{S}_3^2 \cup \mathcal{S}_3^-$ and twelve neighbours of $\mathcal{C}_1^1$ in $\mathcal{K}$ are from $\mathcal{S}_3^1 \cup \mathcal{S}_3^-$, hence $|\mathcal{K}| = 14$ implies $s_3^- \geq 10$.

[212] $c_i^2 \leq 5$, $i = 2,3$

[2121] $c_j^1 = \max\{c_2^1, c_3^1\} = 6$, $c_j^2 = 0$, $c_j^- = 1$: From (iv) it follows $s_2^2 = 0$, hence $7 \leq s_3^2 = c_1^2 + c_j^2 + c_{5-j}^2 + k^2 = c_{5-j}^2 + k^2$ and $k^2 \geq 7 - c_{5-j}^2 \geq 2$ in contradiction with the fact that only common neighbours of 1-element sets $\mathcal{C}_1^-$ and $\mathcal{C}_j^-$ can be in $\mathcal{K}^2$ and $K(7,3)$ does not contain cycles of length 4.

[2122] $c_i^1 \leq 5$, $i = 2,3$: For any $i \in [2,3]$ from $\max\{c_i^1, c_i^2\} \leq 5$ we can conclude that $c_i^- \geq 2$, since from $c_i^j \in [1,5]$ it follows that $\mathcal{C}_i^j$ has at least two neighbours in $\mathcal{C}_i$, each of them in $\mathcal{C}_i^-$, $j = 1,2$. From $k^2 \leq 2$ and $s_3^2 \geq 6$ we obtain $c_j^2 = \max\{c_2^2, c_3^2\} \geq 2$.

[21221] If the subgraph of $K(7,3)$ induced on $\mathcal{C}_l^2$ for some $l \in [1,3]$ is connected, $\mathcal{C}_l^2$ has neighbours in at least $\min\{2c_l^2, 7\} \geq 4$ components of $7K_2$, (iv) yields the inequality $k^- \geq |K(\mathcal{C}_1^1, \mathcal{C}_l^2)| \geq 4$ and $s_3^- = c_1^- + c_2^- + c_3^- + k^- \geq 9$.

[21222] If the subgraph above has at least two components, then C_l^2 has at least three neighbours in C_l and $c_1^- \geq 3$ together with $k^- \geq |K(C_1^1, C_l^2)| \geq 3$ is sufficient for a contradiction, too.

[22] Now we may suppose $c_i^1 \leq 5$, $c_i^2 \leq 5$ and $c_i^- \geq 2$, $i = 1,2,3$.

[221] $c_1^- = 2$: As $c_1^1 + c_1^2 = 5$, without loss of generality (the symmetry of $\mathcal{S}_{7,47}$) $c_1^1 \geq 3$ and the graph induced by C_1^1 is a path on at least three vertices. Then C_1^1 has neighbours in at least six components of $7K_2$.

[2211] $c_i^2 = \max\{c_1^2, c_2^2, c_3^2\} \geq 2$

[22111] There are two distinct vertices of C_i^2 whose distance (realized in C_i) is at most 2: The set C_i^2 has neighbours in four components of $7K_2$, hence $k^- \geq |K(C_1^1, C_i^2)| \geq 3$ and $s_3^- \geq 3 \cdot 2 + 3 = 9$.

[22112] C_i^2 has two vertices and their distance is 3: Now $k^- \geq |K(C_1^1, C_i^2)| \geq 2$, each of four neighbours of C_i^2 in C_i is in $\mathcal{S}_3^-$, $c_i^- \geq 4$ and $s_3^- \geq 10$.

[2212] $c_i^2 \leq 1$, $i = 1,2,3$

[22121] There exists $i \in [1,3]$ such that one of the components (paths) induced by C_i^1 has at least four vertices: The set C_i^1 has neighbours in all components of $7K_2$.

[221211] $c_1^2 + c_2^2 + c_3^2 \geq 2$: $C_1^2 \cup C_2^2 \cup C_3^2$ has neighbours in at least three components of $7K_2$, hence $k^- \geq |K(C_i^1, C_1^2 \cup C_2^2 \cup C_3^2)| \geq 3$ and $s_3^- \geq 3 \cdot 2 + 3 = 9$.

[221212] $c_1^2 + c_2^2 + c_3^2 \leq 1$: From $s_3^2 \geq 4$ we have $k^2 \geq 3$, $k^- \geq |K(C_i^1, \mathcal{K}^2)| \geq 3$ and again $s_3^- \geq 9$.

[22122] Each of the components induced by C_i^1 has at most three vertices, $i = 1,2,3$: In this case $c_1^2 = 0$ and C_1^1 induces two components, one on three vertices, the other one on two vertices ($c_1^2 = 1$ would force two neighbours of C_1^2 to be in C_1^- and, since remaining four vertices of C_1 are not all in C_1^1, c_1^- would be at least 3 contrary to the assumption [221]). Thus C_1^1 has neighbours in six components of $7K_2$.

[221221] $c_2^2 = c_3^2 = 1$: Proceeding as above we see that $c_2^- \geq 3$, $c_3^- \geq 3$. Moreover, $C_2^2 \cup C_3^2$ has neighbours in at least three components of $7K_2$, $k^- \geq |K(C_1^1, C_2^2 \cup C_3^2)| \geq 2$ and $s_3^- \geq 10$.

[221222] $c_2^2 + c_3^2 = 1$: As $c_2^- + c_3^- \geq 5$ and $k^2 \geq 3$, we get $k^- \geq |K(C_1^1, \mathcal{K}^2)| \geq 2$ and $s_3^- \geq 9$.

[221223] $c_2^2 = c_3^2 = 0$: From $k^2 \geq 4$ we obtain $k^- \geq |K(C_1^1, \mathcal{K}^2)| \geq 3$ and once more $s_3^- \geq 9$.

[222] Inequalities $c_i^- \geq 3$, $i = 1,2,3$, lead immediately to $s_3^- \geq 9$.

□

We have proved among others that $\min\{k \in [3,\infty)\colon n_{k+1} = 2n_k+2\} = 6$. Using results of [2] one can see that there are $k,\ l \in [7,\infty)$ such that $n_{k+1} \geq 2n_k+3$ and $n_{l+1} \leq 2n_l+2$. It could be interesting to decide whether there exists $p \in [3,\infty)$ fulfilling $n_{m+1} \leq 2n_m + p$ for every $m \in [3,\infty)$.

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