On the Construction of $m \times n$ Magic Rectangles Where m and n are Both Nonprime Integers

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In this paper we consider the problem of constructing magic rectangles of size $m \times n$ where m and n are nonprime integers. What seems to be two new methods of constructing such rectangles are given.

Introduction. An $m \times n$ magic rectangle is an $m \times n$ array Z containing the integers $1, 2, \cdots, mn$ such that if the (i, j)th entry of the array Z is denoted by $z_{ij}, i = 1, \cdots, m, j = 1, \cdots, n$, then

$$\sum_{j=1}^{n} z_{ij} = n(mn+1)/2 = R \text{ for } i = 1, \cdots, m$$

and (1)

$$\sum_{i=1}^{m} z_{ij} = m(mn+1)/2 = C \text{ for } j = 1, \cdots, n.$$

The construction of magic rectangles has largely been used as a source of mathematical recreation, e.g., see Andrews [2]. However, more recently, magic rectangles have proven useful for the construction of trend free factorial experimental designs, e.g., see Phillips [10]. A number of methods for constructing magic rectangles are known. In fact, Harmuth [4,5] proved the existence of all magic rectangles where m and n are both even or odd by giving some not very precise rules for their construction. Andrews [1] gives an excellent discussion of magic rectangles as well as a description of another method of constructing magic rectangles called the method of complementary differences which Andrews [2] attributes to Planck. Phillips [9] gives a simple method of construction when either m or n is a multiple of four and Jacroux [7] shows how to construct large numbers of magic rectangles when m and n are both multiples of two. In this paper we give what seems to be two new methods for constructing magic rectangles when both m and n are nonprime integers.

2. Construction Method One. In this section, we give a general method for constructing $m \times n$ magic rectangles where m and n are both nonprimes by using existing magic rectangles. To this end, let $\boldsymbol{W} = (w_{ij})$ and $\boldsymbol{X} = (x_{ij})$ denote $m_1 \times n_1$ and $m_2 \times n_2$ magic rectangles, respectively. Now, using W and X, construct the new array

$$\boldsymbol{Z} = \begin{pmatrix} (w_{11} - 1)m_2n_2\boldsymbol{J} + \boldsymbol{X} \cdots, (w_{1n_1} - 1)m_2n_2\boldsymbol{J} + \boldsymbol{X} \\ \vdots \\ (w_{m_11} - 1)m_2n_2\boldsymbol{J} + \boldsymbol{X} \cdots, (w_{m_1n_1} - 1)m_2n_2\boldsymbol{J} + \boldsymbol{X} \end{pmatrix} \tag{2}$$

where J is an $m_2 \times n_2$ matrix of ones.

Theorem 1. The array $\boldsymbol{Z}$ given in (2) above is an $m_1m_2 \times n_1n_2$ magic rectangle.

Proof. We shall show that the row and column sums of $\boldsymbol{Z}$ defined in (2) satisfy the conditions given in (1). Without loss of generality, consider the first row of $\boldsymbol{Z}$. Then

$$\begin{aligned} \sum_{i=1}^{n_1n_2} z_{1i} &= \sum_{p=1}^{n_1}\{\sum_{j=1}^{n_2}(w_{1p} - 1)m_2n_2 + x_{1j}\} = \sum_{p=1}^{n_1}\{(w_{1p} - 1)m_2n_2^2 + n_2 \\ &\quad (m_2n_2 + 1)/2\} \\ &= m_2n_2^2\sum_{p=1}^{n_1} w_{1p} - m_2n_1n_2^2 + n_1n_2(m_2n_2 + 1)/2 \\ &= m_2n_2^2(n_1(m_1n_1 + 1)/2) - m_2n_1n_2^2 + m_2n_1n_2^2/2 + n_1n_2/2 \\ &= (m_1m_2n_1^2n_2^2)/2 + n_1n_2/2 \end{aligned}$$

where the last expression is equal to the corresponding expression given in (1) with $m = m_1m_2$ and $n = n_1n_2$.

Now consider the elements in the first column of $\boldsymbol{Z}$. Then

$$\begin{aligned} \sum_{i=1}^{m_1m_2} z_{i1} &= \sum_{p=1}^{m_1}\{\sum_{j=1}^{m_2}(w_{p1} - 1)m_2n_2 + x_{j1}\} \\ &= \sum_{p=1}^{m_1}\{m_2^2n_2w_{p1} - m_2^2n_2 + m_2(m_2n_2 + 1)/2\} \\ &= m_2^2n_2m_1(m_1n_1 + 1)/2) - m_1m_2^2n_2 + m_1m_2(m_2n_2 + 1)/2 \end{aligned}$$

$$
\begin{aligned}
&= m_1^2 m_2^2 n_1 n_2/2 + m_1 m_2^2 n_2/2 - m_1 m_2^2 n_2 + m_1 m_2^2 n_2/2 + \\
&\quad m_1 m_2/2 \\
&= m_1^2 m_2^2 n_1 n_2/2 + m_1 m_2/2.
\end{aligned}
$$

where again this last expression is equal to the corresponding expression given in (1) with $m = m_1 m_2$ and $n = n_1 n_2$.

To illustrate the construction procedure described above, we now give two examples.

Example 2. In this example, we consider the construction of a 9×15 magic rectangle $\boldsymbol{Z}$ by using the 3×3 and 3×5 magic rectangles $\boldsymbol{W}$ and $\boldsymbol{X}$ given by

$$
\boldsymbol{W} = \begin{pmatrix} 1 & 6 & 8 \\ 9 & 2 & 4 \\ 5 & 7 & 3 \end{pmatrix} \quad \text{and } \boldsymbol{X} = \begin{pmatrix} 1 & 2 & 11 & 14 & 12 \\ 8 & 9 & 10 & 6 & 7 \\ 15 & 13 & 3 & 4 & 5 \end{pmatrix}.
$$

Now, using Theorem 1, we obtain the desired array

$$
\boldsymbol{Z} = \begin{pmatrix} 0\boldsymbol{J}+\boldsymbol{X} & 75\boldsymbol{J}+\boldsymbol{X} & 105\boldsymbol{J}+\boldsymbol{X} \\ 120\boldsymbol{J}+\boldsymbol{X} & 15\boldsymbol{J}+\boldsymbol{X} & 45\boldsymbol{J}+\boldsymbol{X} \\ 60\boldsymbol{J}+\boldsymbol{X} & 90\boldsymbol{J}+\boldsymbol{X} & 30\boldsymbol{J}+\boldsymbol{X} \end{pmatrix}
$$

$$
= \left(\begin{array}{rrrrrrrrrr}
1 & 2 & 11 & 14 & 12 & 76 & 75 & 86 & 89 & 87 \\
8 & 9 & 10 & 6 & 7 & 83 & 84 & 85 & 81 & 72 \\
15 & 13 & 3 & 4 & 5 & 90 & 88 & 78 & 79 & 80 \\
121 & 122 & 131 & 134 & 132 & 16 & 17 & 26 & 29 & 27 \\
128 & 129 & 130 & 126 & 127 & 23 & 24 & 25 & 21 & 22 \\
135 & 133 & 123 & 124 & 125 & 30 & 28 & 18 & 19 & 20 \\
61 & 62 & 71 & 74 & 72 & 91 & 92 & 101 & 104 & 102 \\
68 & 69 & 70 & 66 & 67 & 98 & 99 & 100 & 96 & 97 \\
75 & 73 & 63 & 64 & 65 & 105 & 103 & 93 & 94 & 95
\end{array} \right.
$$

$$
\left. \begin{array}{rrrrr}
106 & 109 & 116 & 119 & 117 \\
113 & 114 & 115 & 111 & 112 \\
120 & 118 & 108 & 109 & 110 \\
46 & 47 & 56 & 59 & 57 \\
53 & 54 & 55 & 51 & 52 \\
60 & 58 & 48 & 49 & 50 \\
31 & 32 & 41 & 44 & 42 \\
38 & 39 & 40 & 36 & 37 \\
45 & 43 & 33 & 34 & 35
\end{array} \right)
$$

having column sums equal to 612 and row sums equal to 945.

Example 3. In this example, we consider the construction of a 6×12 magic rectangle $\boldsymbol{Z}$ by using the 3×3 and 2×4 magic rectangles $\boldsymbol{W}$ and $\boldsymbol{X}$ given by

$$\boldsymbol{W} = \begin{pmatrix} 1 & 6 & 8 \\ 9 & 2 & 4 \\ 5 & 7 & 3 \end{pmatrix} \quad \text{and } \boldsymbol{X} = \begin{pmatrix} 1 & 7 & 6 & 4 \\ 8 & 2 & 3 & 5 \end{pmatrix}.$$

Now, using Theorem 1, we obtain the desired array

$$\boldsymbol{Z} = \begin{pmatrix} 0\boldsymbol{J} + \boldsymbol{X} & 40\boldsymbol{J} + \boldsymbol{X} & 56\boldsymbol{J} + \boldsymbol{X} \\ 64\boldsymbol{J} + \boldsymbol{X} & 8\boldsymbol{J} + \boldsymbol{X} & 24\boldsymbol{J} + \boldsymbol{X} \\ 32\boldsymbol{J} + \boldsymbol{X} & 48\boldsymbol{J} + \boldsymbol{X} & 16\boldsymbol{J} + \boldsymbol{X} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 7 & 6 & 4 & 41 & 47 & 46 & 44 & 57 & 63 & 62 & 60 \\ 8 & 2 & 3 & 5 & 48 & 42 & 43 & 48 & 64 & 58 & 59 & 61 \\ 65 & 71 & 70 & 68 & 9 & 15 & 14 & 12 & 25 & 21 & 30 & 28 \\ 72 & 66 & 67 & 69 & 16 & 10 & 11 & 13 & 32 & 26 & 27 & 29 \\ 33 & 39 & 38 & 36 & 49 & 55 & 54 & 52 & 17 & 23 & 22 & 20 \\ 40 & 34 & 35 & 37 & 56 & 50 & 51 & 53 & 24 & 18 & 19 & 21 \end{pmatrix}.$$

having column sums equal to 219 and row sums equal to 438.

3. Construction Method Two. In this section, we give our second method of construction. The method of construction given here is slightly more complicated than that given in Section 2 but is also slightly more general. We now give a method for constructing an $m_1 \times n_1 n_2$ magic rectangle from an existing $m_1 \times n_1$ magic rectangle. We note before beginning that m_1 and n_1 must both be even or odd and that m_1 and $n_1 n_2$ must both be even or odd for the construction process to work.

Construction Process:
Step 1;

Construct the $m_1 n_2 \times 1$ column vector $\boldsymbol{x}' = (\boldsymbol{x}'_0, \cdots, \boldsymbol{x}'_{m_1-1})$ where each $\boldsymbol{x}_i$ is an $n_2 \times 1$ subvector of $\boldsymbol{x}$ satisfying

(i) if m_1 is even, $\boldsymbol{x}'_i = (0, 1, \cdots, n_2 - 1)$ if i is even and $\boldsymbol{x}'_i = (n_2 - 1, n_2 - 2, \cdots, 0)$ if i is odd for $i = 0, \cdots, m_1 - 1$ or

(ii) if m_1 is odd,
$\boldsymbol{x}'_0 = (0, 1, \cdots, n_2 - 1)$,

$x'_1 = ((n_2-1)/2, (n_2+1)/2, \cdots, n_2-1, 0, 1, \cdots, (n_2-3)/2)$,
$x'_2 = (n_2-1, (n_2-3)/2, n_2-2, (n_2-5)/2, \cdots, 0, (n_2-1)/2)$,
and $x'_i = (0, 1, \cdots, n_2-1)$ if $i \geq 3$ is even and $x'_i = (n_2-1, n_2-2, \cdots, 0)$ if $i \geq 4$ is odd.

We shall denote the qth component of x_i defined above by $x_i(q), i = 0, \cdots, m_1-1, q = 1, \cdots, n_2$.

Comment: It is easy to verify that the position numbers corresponding to the coordinates of x occupied by each of the entries $0, 1, \cdots, n_2-1$ of x add up to $m_1(m_1n_2+1)/2$.

Step 2;

From the $m_1 \times n_1$ magic rectangle $W = (w_{ij}), i = 0, 1, \cdots, m_1 - 1, j = 0, 1, \cdots, n_1-1$, and the vector x obtained in step 1, construct the $m_1n_1n_2 \times 3$ matrix $Y' = (y'_1, \cdots, y'_{m_1n_1n_2})$ where if $p = n_2(w_{ij}-1)+q, p = 1, \cdots, m_1n_1n_2, 1 \leq q \leq n_2$, the 1×3 row vector y_p of Y is given by $y_p = (i, j, x_i(q))$.

Comment: Since the column entries of W all add up to the same value, it follows that the row numbers of Y corresponding to the entries $0, 1, \cdots, m_1-1$ in the first column of Y add up to $n_1n_2(m_1n_1n_2+1)/2$. Also, from the comment following step 1 and again using the fact that the column sums of W are all the same, it follows that the row numbers of Y corresponding to each ordered pair of entries $(j, x_i(q))$ in the last two columns of Y add up to $m_1(m_1n_1n_2+1)/2$.

Step 3;

Construct the desired $m_1 \times n_1n_2$ magic rectangle Z whose m_1 rows are indexed by $0, 1, \cdots, m_1-1$ and whose n_1n_2 columns are indexed by $(0,0), (0,1), \cdots, (0, n_2-1), (1,0), \cdots, (n_1-1, 0), \cdots, (n_1-1, n_2-1)$. Using the matrix Y obtained in step 2, if $y_p = (i, j, x_i(q))$, place p in the row of Z corresponding to i and in the column of Z corresponding to $(j, x_i(q))$.

Comment: The fact that the row entries of Z all add up to the same value and that the column entries of Z all add up to the same value follows from the comment following step 2.

To illustrate the construction process described above, we now give an example.

Example 4. Consider constructing a 3×15 magic rectangle Z using the above stepwise procedure and the 3×3 magic rectangle

$$W = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1 & 6 & 8 \\ 9 & 2 & 4 \\ 5 & 7 & 3 \end{pmatrix} \end{array}.$$

In this example, $m_1 = 3, n_1 = 3$ and $n_2 = 5$.

Step 1;

Construct the 15×1 column vector $x' = (x'_0, x'_1, x'_2)$ where $x'_0 = (0, 1, 2, 3, 4), x'_1 = (2, 3, 4, 0, 1)$ and $x'_2 = (4, 1, 3, 0, 2)$.

Comment: Note that the position numbers in x corresponding to the coordinate numbers occupied by each of $0, 1, 2, 3$ and 4 add up to 24.

Step 2;

Construct the 45×3 rectangle Y where

$$Y' = \begin{pmatrix} 0&0&0&0&0 & 1&1&1&1&1 & 2&2&2&2&2 \\ 0&0&0&0&0 & 1&1&1&1&1 & 2&2&2&2&2 & \cdots \\ 0&1&2&3&4 & 2&3&4&0&1 & 4&1&3&0&2 \\ \\ 1&1&1&1&1 & 2&2&2&2&2 & 0&0&0&0&0 \\ 2&2&2&2&2 & 0&0&0&0&0 & 1&1&1&1&1 & \cdots \\ 2&3&4&0&1 & 4&1&3&0&2 & 0&1&2&3&4 \\ \\ 2&2&2&2&2 & 0&0&0&0&0 & 1&1&1&1&1 \\ 1&1&1&1&1 & 2&2&2&2&2 & 0&0&0&0&0 \\ 4&1&3&0&2 & 0&1&2&3&4 & 2&3&4&0&1 \end{pmatrix}.$$

Comment. We note that the row numbers of Y corresponding to each of the entries 0, 1 and 2 in the first column of Y add up to 207 where as the row numbers of Y corresponding to each of the entries $(i, j), i = 0, 1, 2, j = 0, 1, 2, 3, 4$, in columns two and three of Y add up to 69.

Step 3;

Using Y from step 2, construct the desired magic rectangle Z whose rows are indexed by the entries in the first column of Y and whose columns are indexed by the entries in the last two columns of Y. Proceeding as described in the construction process, we obtain

$$
Z = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}
\begin{array}{cccccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (1,0) & (1,1) & (1,2) & (1,3) & (1,4) \\
1 & 2 & 3 & 4 & 5 & 26 & 27 & 28 & 29 & 30 \\
44 & 45 & 41 & 42 & 43 & 9 & 10 & 6 & 7 & 8 \\
24 & 22 & 25 & 23 & 21 & 34 & 32 & 35 & 33 & 31
\end{array}
$$

$$
\begin{array}{ccccc}
(2,0) & (2,1) & (2,2) & (2,3) & (2,4) \\
36 & 37 & 38 & 39 & 40 \\
19 & 20 & 16 & 17 & 18 \\
14 & 12 & 15 & 13 & 11
\end{array} .
$$

It is easy to verify that the column sums of Z are all equal to 69 and the row sums of Z are all equal to 345.

The construction process given in this section is more general than that given in Section 2 since it only requires the existence of a single $m_1 \times n_1$ magic rectangle whereas the construction process given in Section 2 requires the existence of an $m_1 \times n_1$ and an $m_2 \times n_2$ magic rectangle. Using the construction process given in this section, once an $m_1 \times n_1$ magic rectangle W is determined, any $m \times n$ magic rectangle Z can be constructed which has m_1 as a divisor of m and n_1 as a divisor of n by sequentially applying the process to the rows and columns of W.

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