On The Boundary Inequality For Bandwidth Of Graphs

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ABSTRACT. The quantity $B(G) = \min\max\{|f(u)-f(v)|: (u,v) \in E(G)\}$ is called the bandwidth of a graph $G = (V(G), E(G))$ where min is taken over all bijections $f: V(G) \to \{1, 2, \ldots, |V(G)|\}$ called *labelings*. L.H. Harper presented an important inequality related to the boundary ∂S of subsets $S \subseteq V(G)$. This paper gives a refinement of Harper's inequality which will be more powerful in determining bandwidths for several classes of graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$, and $|V(G)| = n$. A bijection (one-to-one mapping) $f: V \to \{1, 2, \ldots, n\}$ will be called a *labeling*. The *bandwidth of a labeling* f for G is defined by

$$B(G, f) = \max\{|f(u) - f(v)|: (u, v) \in E(G)\};$$

and the *bandwidth* of G is defined by

$$B(G) = \min\{B(G, f): f \text{ is a labeling of } G\}.$$

Here, a labeling that attains the minimum is called *an optimal labeling.*

The bandwidth problem for graphs arises from sparse matrix computation, code theory and the circuit layout of VLSI designs. The recognition version of determining the minimum bandwidth of a graph has been proved to be NP-hard [8], so people make efforts to find bandwidth representations or algorithms for as many special graphs as possible. In this direction, the following inequality due to L. H. Harper [4] is significant.

Theorem 1.

$$B(G) \geq \max_{1 \leq k \leq |V|} \min_{|S|=k} |\partial S|,$$

where $\partial S = \{x \in S\colon$ *there is a* $y \in V - S$ *such that* $(x, y) \in E(G)\}$ *denotes the* boundary *of* $S \subseteq V(G)$.

This lower bound is decisive in solving the bandwidth problem for n-cubes, $P_m \times P_n$, $C_m \times P_n$, $C_m \times C_n$, (m, n)-multipaths and others (See [1,4,7]). In the present paper, we try to strengthen this theoretic tool so as to solve different problems more efficiently. In Section 2, we present a generalization of Theorem 1. In Section 3, we illustrate how the generalized inequality works. Some of the applications are simplifications of known results, some are new.

2 The Boundary Inequality

For a labeling f, let $u_i = f^{-1}(i)$ $(1 \leq i \leq n)$ be the vertex of label i. Denote $S_k = \{u_1, u_2, \ldots, u_k\} = f^{-1}(\{1, 2, \ldots, k\})$ for $1 \leq k \leq n$. Theorem 1 can be written as

Theorem 1'. *For any labeling* f *of* G,

$$B(G, f) \geq \max_{1 \leq k \leq |V|} |\partial S_k|.$$

Here, the inner boundary ∂S can be replaced by the outer boundary, that is the neighbor set $N(S) = \{y \in V - S\colon$ there is an $x \in S$ such that $(x, y) \in E(G)\}$. In order to get a more accurate lower bound, we will extend the range of boundaries. The sets

$$D^-(S_k) = \{u_j \in S_k\colon \text{ there is a } u_i \text{ such that } i \leq j \leq k \text{ and } u_i \in \partial S_k\},$$
$$D^+(S_k) = \{u_i \in V - S_k\colon \text{ there is a } u_j \text{ such that } k < i \leq j \text{ and } u_j \in N(S_k)\}$$

will be called the *backward boundary* and the *forward boundary* of S_k respectively. Obviously $\partial S_k \subseteq D^-(S_k)$, $N(S_k) \subseteq D^+(S_k)$. Furthermore, if $X \subseteq D^-(S_k)$, $Y \subseteq D^+(S_k)$ satisfy

(i) $x \in X$, $y \in Y \Longrightarrow (x, y) \in E(G)$,

(ii) $|X \cup Y|$ is maximum,

then $X \cup Y$ is called an *intersection boundary* of S_k, denoted by $D^0(S_k)$. In other words, the subgraph of G induced by $D^0(S_k)$ represents a maximum complete bipartite subgraph between $D^-(S_k)$ and $D^+(S_k)$. By using this notation, we have a generalization of Theorem 1' as follows.

Theorem 2. *For any labeling f of G,*

$$B(G,f) \geq \max_{1\leq k\leq |V|} \max\{|D^-(S_k)|,\ |D^+(S_k)|, |D^0(S_k)|-1\}.$$

Proof: For any labeling f, recall that

$$u_i = f^{-1}(i), \quad i = 1,2,\ldots,n$$

and

$$S_k = \{u_1, u_2, \ldots, u_k\}.$$

Let $\alpha = \min\{i\colon u_i \in D^-(S_k)\}$. Then u_α is adjacent to a $u_\beta \in V - S_k$. Note that $\alpha \leq k - |D^-(S_k)| + 1$, $\beta \geq k+1$. So

$$\begin{aligned} B(G,f) &\geq |f(u_\alpha) - f(u_\beta)| \\ &= \beta - \alpha \geq |D^-(S_k)|. \end{aligned}$$

Similarly for $D^+(S_k)$. It remains to show $B(G,f) \geq |D^0(S_k)| - 1$. Suppose that $D^0(S_k) = X \cup Y$ with $X \subseteq D^-(S_k)$, $Y \subseteq D^+(S_k)$ and $X \times Y \subseteq E(G)$. Let $\alpha = \min\{i\colon u_i \in X\}$, $\beta = \max\{j\colon u_j \in Y\}$. Then $\beta \geq \alpha + |D^0(S_k)| - 1$. Note that u_α is adjacent to u_β. We have

$$\begin{aligned} B(G,f) &\geq |f(u_\alpha) - f(u_\beta)| \\ &= \beta - \alpha \geq |D^0(S_k)| - 1. \end{aligned}$$

By the arbitrariness of k, Theorem 2 follows. □

In next section, we shall explain how to use this generalized boundary inequality to get sharp lower bounds.

3 Applications

We first concentrate on the result of complete k-partite graphs which is a typical application of Theorem 2.

Proposition 1 (P.G. Eitner [3]). *If $n_1 \geq n_2 \geq \cdots \geq n_k$, then*

$$B(K_{n_1,n_2,\ldots,n_k}) = n - \lceil \frac{n_1+1}{2} \rceil,$$

where $n = \Sigma_{i=1}^{k} n_i$ *is the number of vertices.*

Proof: Let V_i be the vertex set of the i-th part of $G = K_{n_1,n_2,\ldots,n_k}$, $|V_i| = n_i (i = 1, 2, \ldots, k)$. For a given labeling f, suppose that $u_1 \in V_i$. Then take $k = \lceil (n_i + 1)/2 \rceil$. If S_k intersects more than one part, then

$$\begin{aligned} |N(S_k)| &= n - |S_k| \\ &= n - \lceil \frac{n_i + 1}{2} \rceil \\ &\geq n - \lceil \frac{n_1 + 1}{2} \rceil. \end{aligned}$$

Otherwise $S_k \subseteq V_i$, then $D^0(S_k)$ contains all vertices in G except the $\lfloor (n_i - 1)/2 \rfloor$ vertices of $V_i - S_k$. So

$$\begin{aligned} |D^0(S_k)| - 1 &= n - \lfloor \frac{n_i - 1}{2} \rfloor - 1 \\ &= n - \lfloor \frac{n_i + 1}{2} \rfloor \\ &\geq n - \lceil \frac{n_1 + 1}{2} \rceil. \end{aligned}$$

In any case, it follows from Theorem 2 that

$$\begin{aligned} B(G, f) &\geq \max\{|D^+(S_k)|,\ |D^0(S_k)| - 1\} \\ &\geq n - \lceil \frac{n_1 + 1}{2} \rceil. \end{aligned}$$

On the other hand, the following labeling f^* attains the above lower bound

$$f^*(v_{1j}) = \begin{cases} j, & 1 \leq j \leq \lfloor \frac{n_1}{2} \rfloor, \\ n - n_1 + j, & \lfloor \frac{n_1}{2} \rfloor + 1 \leq j \leq n_1 \end{cases},$$

$$f^*(v_{ij}) = \lfloor \frac{n_1}{2} \rfloor + \sum_{h=2}^{i-1} n_h + j, \text{ for } i \geq 2.$$

where $V_i = \{v_{ij} : 1 \leq j \leq n_i\}$. This completes the proof. □

The following applications are based on this result to some extent. The labeling f^* given above will be used again in the remainder of this paper. In particular, we label $\lfloor n_1/2 \rfloor$ vertices of V_1 by integers $1, 2, \ldots, \lfloor n_1/2 \rfloor$; then label V_i by successive n_i integers ($i = 2, 3, \ldots, k$); finally label the remaining $\lceil n_1/2 \rceil$ vertices of V_1 by the integers from $n - \lceil n_1/2 \rceil + 1$ to n.

The sum (join) of graphs $G_1, G_2, \ldots, G_k$, denoted by $G_1 + G_2 + \cdots + G_k$, is defined to be the graph with vertex set $V(G_1) \cup V(G_2) \cup \cdots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_k) \cup \{(u, v) : u \in V(G_i),\ v \in V(G_j)$ for $i \neq j\}$. As a consequence of Proposition 1, we have the following.

Proposition 2 (Lai, Liu and Williams [5]). *Let $G = G_1+G_2+\cdots+G_k$ with $n_i = |V(G_i)|$, $n = \Sigma_{i=1}^k n_i$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. If $B(G_1) < \lceil n_1/2 \rceil$, then*

$$B(G) = n - \lceil \frac{n_1+1}{2} \rceil.$$

Proof: Since $K_{n_1,n_2,\ldots,n_k} \subseteq G$, it follows from Proposition 1 that $B(G) \geq n - \lceil (n_1+1)/2 \rceil$. On the other hand, the labeling f^* defined in Proposition 1 (where the order of vertices in V_i is according to an optimal labeling of G_i) attains the lower bound. In fact, in addition to the edges of $K_{n_1,n_2,\ldots,n_k}$, we have

$$\begin{aligned} \max_{(u,v)\in E(G_1)} |f^*(u) - f^*(v)| &\leq B(G_1) + n - n_1 \\ &\leq \lceil \frac{n_1}{2} \rceil - 1 + n - n_1 \\ &= n - \lceil \frac{n_1+1}{2} \rceil, \\ \max_{(u,v)\in E(G_i)} |f^*(u) - f^*(v)| &\leq n_i - 1 \\ &\leq n - n_1 - 1 \\ &\leq n - \lceil \frac{n_1+1}{2} \rceil \text{ (for } i \geq 2). \end{aligned}$$

So the labelling f^* attains the bound. This is what we wanted to show. □

Proposition 2 is a partial result on the sum of k graphs. We can obtain the general result by using Theorem 2.

Proposition 3. *Let $G = G_1 + G_2 + \cdots + G_k$ with $n_i = |V(G_i)|$ and $n = \Sigma_{i=1}^k n_i$. Then*

$$B(G) = \min_{1\leq i\leq k} \max\{B(G_i) + n - n_i, n - \lceil \frac{n_i+1}{2} \rceil\}$$

Proof: For a given labeling f, suppose that $u_1 = f^{-1}(1) \in V(G_i)$. By the proof of Proposition 1 and $K_{n_1,n_2,\ldots,n_k} \subseteq G$, we have

$$B(G,f) \geq n - \lceil \frac{n_i+1}{2} \rceil. \tag{1}$$

Additionally, we will prove that

$$B(G,f) \geq B(G_i) + n - n_i. \tag{2}$$

If $B(G_i) < \lceil n_i/2 \rceil$, then (1) implies (2). Hence we may assume $B(G_i) \geq \lceil n_i/2 \rceil$. Let $g\colon V(G_i) \to \{1, 2, \ldots, n_i\}$ be a labeling of G_i induced from

f, i.e., $g(u) < g(v) \Leftrightarrow f(u) < f(v)$ for all $u, v \in V(G_i)$. Furthermore, we denote $x_j = g^{-1}(j)$, $1 \le j \le n_i$ (note that $x_1 = u_1$). Then there is a edge $(x_k, x_r) \in E(G_i)$ $(k < r)$ such that

$$\begin{aligned} B(G_i, g) &= |g(x_k) - g(x_r)| = r - k \\ &\ge B(G_i) \ge \lceil \frac{n_i}{2} \rceil. \end{aligned}$$

Since $r \le n_i$, it follows that $k \le \lceil n_i/2 \rceil$.

Case 1 $f(x_k) = k$. That is, $S_k = f^{-1}(\{1, 2, \ldots, k\}) = \{x_1, x_2, \ldots, x_k\} \subseteq V(G_i)$. Then

$$\{x_{k+1}, x_{k+2}, \ldots, x_r\} \cup (V(G) - V(G_i)) \subseteq D^+(S_k).$$

Thus

$$\begin{aligned} B(G, f) &\ge |D^+(S_k)| \ge r - k + n - n_i \\ &\ge B(G_i) + n - n_i. \end{aligned}$$

Case 2 $f(x_k) > k$. According to the order of f, let y be the first vertex which does not belong to $V(G_i)$, and $f(y) = h$. Then $S_k = \{x_1, x_2, \ldots, x_{h-1}, y\}$, and

$$\begin{aligned} B(G, f) &\ge |N(S_h)| = n - |S_h| \ge n - k \\ &= n - (r - B(G_i, f)) \\ &\ge B(G_i) + n - n_i, \end{aligned}$$

By combining the above two cases, we obtain (2). Further, by combining (1) and (2), it follows that

$$\begin{aligned} B(G, f) &\ge \max\{B(G_i) + n - n_i, n - \lceil \frac{n_i + 1}{2} \rceil\} \\ &\ge \min_i \max\{B(G_i) + n - n_i, n - \lceil \frac{n_i + 1}{2} \rceil\}. \end{aligned}$$

Without loss of generality, we may assume that G_1 attains the above minimum. By noting the last two representations of the proof of Proposition 2, we see that the labeling f^* attains the lower bound. The proof is complete. □

As a consequence of Proposition 3, we have the following.

Proposition 4 (J. Yuan [9]).

$$\begin{aligned} B(G + H) = \min\{&|V(H)| + \max\{B(G), \lfloor \frac{|V(G)| - 1}{2} \rfloor\}, \\ &|V(G)| + \max\{B(H), \lfloor \frac{|V(H)| - 1}{2} \rfloor\}\}. \end{aligned}$$

As a special case of Proposition 2, the paper [6] gave a partial result of $B(G+H)$.

A similar application is about the composition of graphs as follows. The composition of two graphs G and H, denoted by $G[H]$, is defined to be the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent to (u_2, v_2) if u_1 is adjacent to u_2 in G or if $u_1 = u_2$ and v_1 is adjacent to v_2 in H. Since $K_m[H] = H_1 + H_2 + \cdots + H_m$ where $H_1, H_2, \ldots, H_m$ are m copies of H, the next Proposition follows directly from Proposition 3.

Proposition 5.

$$B(K_m[H]) = \max\{B(H), \lfloor \frac{|V(H)|-1}{2} \rfloor\} + (m-1)|V(H)|.$$

The paper [2] gave results on $P_m^r[H]$ and $C_m^r[H]$ when r is so small that they are not isomorphic to $K_m[H]$. Here, Proposition 5 proposes a complement. This also provides a lower bound for $B(G[H])$ when G has a subgraph K_m (a clique of m vertices).

Proposition 6. *If G has the maximum degree $\Delta(G)$, then*

$$B(G[H]) \geq |V(H)| + \lfloor \frac{\Delta(G)|V(H)|-1}{2} \rfloor.$$

Proof: Since $G[H]$ has a complete bipartite subgraph whose one part is $V(H)$, another is $\Delta(G)$ copies of $V(H)$, the inequality follows directly from Proposition 1. □

This lower bound is attainable for $P_m[P_n]$, $P_m[C_n]$, etc.

4 Concluding Remarks

In determining the bandwidth of a graph, a common way is to derive a lower bound which can be attained by a constructed labeling. Therefore, looking for a sharp lower bound is usually critical. In the foregoing discussion, a type of lower bounds based on the concept of boundary has been established. In addition to the applications mentioned above, more examples for illustrating the efficiency of this methodology could be found in the literature. Further, it is possible to get more variants of this type of lower bounds, which would be suitable to different classes of graphs.

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