On the Number of Mutually Orthogonal Partial Latin Squares

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Abstract. An upper bound on the size of any collection of mutually orthogonal partial latin squares is derived as a function of the number of compatible cells that are occupied in all squares. It is shown that the bound is strict if the number of compatible cells is small.

1 Introduction

A *partial latin square* of order n is an $n \times n$ array such that each of the integers $1, 2, \ldots, n$ appears at most once in any row or column [4]. Two partial latin squares (not necessarily distinct) of the same order form an *orthogonal pair* if all ordered pairs of entries obtained by superimposing the two squares are distinct. A collection of partial latin squares with the property that each pair of squares in the collection is an orthogonal pair is called a collection of *mutually orthogonal partial latin squares* (MOPLS's). A collection of partial latin squares will be called *p-compatible* if every square has exactly p entries and the cells of these entries are the same for all squares. Euler, having failed to construct a pair of orthogonal latin squares of order six, constructed an orthogonal pair with 34 compatible cells as shown in Figure 1 (see e.g., [2]).

1	2	3	4	5	6
2	3	4	6	1	5
3	6	1	5	2	4
	5		3	6	1
5	1	6	2	4	3
6	4	5	1	3	2

1	2	3	4	5	6
6	1	5	3	4	2
4	5	2	6	3	1
	4		2	1	3
3	6	4	1	2	5
2	3	1	5	6	4

Figure 1.

In this paper, we are interested in the maximum number of p-compatible MOPLS's of order n. We denote this number by $M_n(p)$. For example, Figure 2 shows a collection of five 8-compatible MOPLS's of order four. On the other hand, there are no more than three 8-compatible MOPLS's of order four if four of the compatible cells are in a row or a column. In fact, it will be shown that there are no more than five 8-compatible MOPLS's of order four regardless of the choice of the compatible cells; and thus $M_4(8) = 5$. In general, we will determine $M_n(p)$ for $n+1 \leq p \leq 2n$. The problem of determining $M_n(p)$ is of interest only for $p \geq n+1$. Indeed, if $p \leq n$, we can always construct a partial latin square which is orthogonal to itself and thus we have a collection of MOPLS's containing an infinite number of copies of this square. We will assume in the following that $n \geq 2$ and $n+1 \leq p \leq n^2$. Clearly, $M_n(p)$ is a nonincreasing function of p and a nondecreasing function of n.

The number $M_n(p)$ plays a role in coding theory. It can be shown that any code of p codewords of length N over an alphabet of n letters has Hamming distance of at most $N - \lceil \log_n p \rceil + 1$ (see e.g., [1]). This number reduces to $N-1$ in the case $n+1 \leq p \leq n^2$. However, this bound is not always attainable. In fact, the maximum value of N for which this bound can be attained is $M_n(p)+2$ [1]. An application of this result to file distribution in database systems is explained in [1]. In this application, it is required to distribute the records of a large file on several disks in order to reduce the retrieval time of records in response to queries. Typical queries request records that are close together in terms of Hamming distance. If these records are located on a single disk, then the retrieval time may be large. To reduce the retrieval time, records which are close in terms of Hamming distance are distributed on several disks. This can be accomplished by allocating the records such that the records of each disk form a code with large Hamming distance. The coding-theoretic results based on $M_n(p)$ give bounds on the best possible retrieval time.

1	3		
	1	3	
		2	4
4			2

1	3		
	2	4	
		1	3
4			2

1	3		
	2	4	
		2	4
3			1

3	1		
	4	2	
		1	3
4			2

2	1		
	3	2	
		4	3
4			1

Figure 2.

2 Upper Bound

In this section, we derive an upper bound on $M_n(p)$. First, we give an upper bound on the number of MOPLS's with a given set $\mathcal{P}$ of p compatible cells. We then maximize this upper bound over all sets of p compatible cells.

Let $\mathcal{P}$ be a set of p cells. Let r_i and c_i, for $i = 1, 2, \ldots, n$, denote the number of cells in the i^{th} row and column, respectively. Thus,

$$r_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} r_i = p, \tag{1}$$

and

$$c_i \geq 0 \text{ for } i = 1, 2, \ldots, n, \quad \text{and} \quad \sum_{i=1}^{n} c_i = p. \tag{2}$$

Let $\nu(\mathcal{P})$ be the number of (unordered) pairs of distinct cells in $\mathcal{P}$ that lie in the same row or column. Then,

$$\nu(\mathcal{P}) = \sum_{i=1}^{n} \binom{r_i}{2} + \sum_{i=1}^{n} \binom{c_i}{2} = \frac{1}{2} \sum_{i=1}^{n} r_i(r_i - 1) + \frac{1}{2} \sum_{i=1}^{n} c_i(c_i - 1). \tag{3}$$

Consider a partial latin square with p occupied cells. Let μ_j, for $j = 1, 2, \ldots, n$, denote the number of occurrences of j as an entry in the square.

Thus,

$$\mu_j \geq 0 \text{ for } j = 1, 2, \ldots, n, \quad \text{and} \quad \sum_{j=1}^{n} \mu_j = p. \tag{4}$$

We define $\mathcal{F}$ to be the set of pairs of distinct cells such that each pair contains the same entry. Hence,

$$|\mathcal{F}| = \sum_{j=1}^{n} \binom{\mu_j}{2} = \frac{1}{2} \sum_{j=1}^{n} \mu_j(\mu_j - 1). \tag{5}$$

Assume that we have M p-compatible partial latin squares. Let $\mathcal{P}$ be the set of compatible cells. Let $\mu_{j,k}$, for $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, M$, be the number of occurrences of j as an entry in the k^{th} square. Furthermore, let $\mathcal{F}_k$, for $k = 1, 2, \ldots, M$, be the set of pairs of distinct cells such that each pair contains the same entry in the k^{th} square. A necessary and sufficient condition for the k^{th} and the l^{th} squares to be orthogonal is that $\mathcal{F}_k \cap \mathcal{F}_l = \phi$. Hence, the M squares are mutually orthogonal if and only if the $\mathcal{F}_k$'s are mutually disjoint, i.e.,

$$\sum_{k=1}^{M} |\mathcal{F}_k| = \left| \bigcup_{k=1}^{M} \mathcal{F}_k \right|. \tag{6}$$

From the definition of a partial latin square, it follows that $\mathcal{F}_k$ does not contain any pair of cells in the same row or column. Hence,

$$\left| \bigcup_{k=1}^{M} \mathcal{F}_k \right| \leq \binom{p}{2} - \nu(\mathcal{P}) = \frac{1}{2} p(p-1) - \nu(\mathcal{P}). \tag{7}$$

Since $p \geq n + 1$ implies that $|\mathcal{F}_k| > 0$ for all k, then (6) and (7) give

$$M \leq \frac{\frac{1}{2} p(p-1) - \nu(\mathcal{P})}{\min_{1 \leq k \leq M} |\mathcal{F}_k|}. \tag{8}$$

Recall that $|\mathcal{F}_k|$ is given in (5) with μ_j replaced by $\mu_{j,k}$ where the $\mu_{j,k}$'s satisfy (4). In the following lemma, we derive a lower bound on $|\mathcal{F}|$ as given in (5) for any integers μ_j, $j = 1, 2, \ldots, n$, satisfying (4).

Lemma 1 *Suppose that the integers μ_j, $j = 1, 2, \ldots, n$, satisfy (4). Then,*

$$\sum_{j=1}^{n} \mu_j(\mu_j - 1) \geq \lfloor p/n \rfloor (2p - n - n\lfloor p/n \rfloor),$$

where equality holds if and only if $n - p + n\lfloor p/n \rfloor$ of the μ_j's equal $\lfloor p/n \rfloor$ and the remaining $p - n\lfloor p/n \rfloor$ of them equal $\lfloor p/n \rfloor + 1$.

Proof. Suppose that $\max_{1\le j\le n}\mu_j - \min_{1\le j\le n}\mu_j \ge 2$. Then, without loss of generality, we may assume $\mu_1-\mu_2\ge 2$. Define $\mu'_1=\mu_1-1$, $\mu'_2=\mu_2+1$, and $\mu'_j=\mu_j$ for $3\le j\le n$. It is easy to check that $\mu'_1,\mu'_2,\ldots,\mu'_n$ satisfy (4) and

$$\sum_{j=1}^{n}\mu_j(\mu_j-1)-\sum_{j=1}^{n}\mu'_j(\mu'_j-1)=2(\mu_1-\mu_2-1)>0.$$

Hence, the minimum of $\sum_{j=1}^{n}\mu_j(\mu_j-1)$ is attained only if $\max\mu_j-\min\mu_j\le 1$. This holds if and only if $n-p+n\lfloor p/n\rfloor$ of the μ_j's equal $\lfloor p/n\rfloor$ and the remaining $p-n\lfloor p/n\rfloor$ of them equal $\lfloor p/n\rfloor+1$. In this case, the inequality stated in the lemma is strict. □

From (5), (8), and Lemma 1, we get

$$M\le\frac{p(p-1)-2\nu(\mathcal{P})}{\lfloor p/n\rfloor(2p-n-n\lfloor p/n\rfloor)}. \tag{9}$$

In order to obtain an upper bound that holds for any p-compatible MOPLS's, we maximize the upper bound given in (9) over all sets $\mathcal{P}$ of p cells. This is accomplished by minimizing $\nu(\mathcal{P})$ as given in (3), where the r_i's and the c_i's satisfy the conditions of (1) and (2), respectively. These conditions are the same as the conditions stated in (4) for the μ_j's. From Lemma 1, we get

$$\nu(\mathcal{P})\ge 2\lfloor p/n\rfloor(2p-n-n\lfloor p/n\rfloor),$$

where equality holds if and only if $n-p+n\lfloor p/n\rfloor$ of the r_i's (c_i's) equal $\lfloor p/n\rfloor$ and the remaining $p-n\lfloor p/n\rfloor$ of them equal $\lfloor p/n\rfloor+1$. Substituting this bound in (9), we get the following result.

Theorem 2 *For $n+1\le p\le n^2$, we have*

$$M_n(p)\le\left\lfloor\frac{p(p-1)}{\lfloor p/n\rfloor(2p-n-n\lfloor p/n\rfloor)}\right\rfloor-2.$$

For instance, Theorem 2 implies that $M_n(n^2)\le n-1$. Indeed, it is well known that there are at most $n-1$ mutually orthogonal latin squares of order n [4]. We will show that this bound not only applies if all squares are full, but also applies if the squares are slightly more than half full. This is stated in the following result which is a corollary to Theorem 2.

Corollary 3 *Suppose that*

$$p\ge\begin{cases}\dfrac{n(n+1)}{2}+1 & \textit{if } n \textit{ is odd},\\[2ex] \dfrac{n^2}{2}+\left\lfloor\dfrac{2n+1-\sqrt{2n^2+4n+1}}{2}\right\rfloor+1 & \textit{if } n \textit{ is even}.\end{cases}$$

Then $M_n(p) \le n-1$ and equality holds if n is a prime power.

Proof. Since $M_n(p)$ is a nonincreasing function of p, it suffices by Theorem 2 to show that

$$\left\lfloor \frac{p(p-1)}{\lfloor p/n\rfloor(2p-n-n\lfloor p/n\rfloor)} \right\rfloor - 2 \le n-1,$$

i.e.,

$$p(p-1) < (n+2)\lfloor p/n\rfloor(2p-n-n\lfloor p/n\rfloor), \tag{10}$$

if p equals the lower bound stated in the corollary. In the case p equals this lower bound, we have

$$\left\lfloor \frac{p}{n} \right\rfloor = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases} \tag{11}$$

Let $\alpha = p - n\lfloor p/n\rfloor$. Then, (10) is equivalent to

$$\alpha^2 - (4\lfloor p/n\rfloor + 1)\alpha - n\lfloor p/n\rfloor(2\lfloor p/n\rfloor - n - 1) < 0. \tag{12}$$

Substituting $\lfloor p/n\rfloor$ from (11) in (12), we find that the smallest nonnegative integer α that satisfies (12) is

$$\alpha = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \lfloor(2n+1-\sqrt{2n^2+4n+1})/2\rfloor + 1 & \text{if } n \text{ is even.} \end{cases}$$

This shows that (10) holds if and only if p satisfies the lower bound. If n is a prime power, it is known that there are $n-1$ mutually orthogonal latin squares of order n [4]. Thus, $M_n(n^2) = n-1$. Since $M_n(p)$ is a nonincreasing function of p, then in this case $M_n(p) = n-1$ for all p that satisfies the lower bound stated in the corollary. □

In general, if n is not a prime power, then $M_n(p)$ may be less than $n-1$ even if p satisfies the condition stated in Corollary 3. The following result may be useful in such case.

Theorem 4

$$M_n(n^2-1) = M_n(n^2).$$

Proof. Suppose that we are given a collection $\mathcal{L}$ of $M_n(n^2-1)$ MOPLS's with n^2-1 compatible cells. We will demonstrate that an element can be placed in each partial latin square to form a complete latin square and that the collection of the completed squares are pairwise orthogonal. This will imply that $M_n(n^2-1) \le M_n(n^2)$. Since $M_n(p)$ is a nonincreasing function of p, this gives the required result. We may assume, without loss of generality, that cell $(1,1)$ is empty in all squares and all other cells are occupied. Consider some latin square L in $\mathcal{L}$. The element that does not appear in the first row of this square appears exactly $n-1$ times in the square. Similarly, the element that does not appear in the first column appears exactly $n-1$ times. If these two elements are different, then the remaining $n-2$ elements appear n^2-2n+1 times in L. This implies that one of the remaining elements appears more than n times, which contradicts the assumption that L is a partial latin square. Thus, the element missing in the first row is the same as the element missing in the first column. By inserting this element in cell $(1,1)$, we obtain a complete latin square L'. Now, we prove that the collection $\mathcal{L}'$ of completed latin squares are pairwise orthogonal. Let L_1 and L_2 be two latin squares in $\mathcal{L}$. Suppose that k_1 and k_2 are the entries in cell $(1,1)$ in the completed squares L_1' and L_2', respectively. If L_1' and L_2' are not orthogonal, then k_1 and k_2 are the entries in some other cell in L_1' and L_2', respectively. This implies that for some l_1, where $1 \leq l_1 \leq n$ and $l_1 \neq k_1$, no cell has l_1 in L_1 and k_2 in L_2. Since l_1 appears n times in L_1, then there is an element l_2 such that in at least two cells, l_1 appears in L_1 and l_2 appears in L_2. This contradicts the assumption that L_1 and L_2 are orthogonal. □

The function $M_n(n^2)$, which denotes the maximum number of pairwise orthogonal latin squares of order n, has been studied extensively (see e.g., [3],[4]). In particular, it is known that a pair of orthogonal latin squares of order n exists, i.e., $M_n(n^2) \geq 2$, if and only if $n \notin \{2,6\}$. Combining this fact with Theorem 4 and the construction given in Figure 1, we obtain the following result.

Theorem 5 *$M_n(p) \geq 2$ except if $n=2$ or $n=6$ and $p \in \{35,36\}$.*

3 $M_n(p)$ for $n=3$ and $n=4$

In this section, we construct MOPLS's for all p such that $n+1 \leq p \leq n^2$, where $n=3$ and $n=4$. In all cases considered, we show that the bound stated in Theorem 2 is strict.

Theorem 6

$$M_3(p) = \begin{cases} 4 & \text{for } p=4, \\ 3 & \text{for } 5 \leq p \leq 6, \\ 2 & \text{for } p \geq 7. \end{cases}$$

In particular, the bound in Theorem 2 is strict for $n=3$.

Proof. Figure 3 shows four 4-compatible MOPLS's and Figure 4 shows three 6-compatible MOPLS's; all of order three. Thus, $M_3(4) \geq 4$, and $M_3(5) \geq M_3(6) \geq 3$. From Theorem 2, it follows that equalities hold in all three cases. Corollary 3 implies that $M_3(p) = 2$ for $p \geq 7$. □

2	3	
	1	
		1

1	2	
	1	
		3

1	2	
	3	
		1

2	1	
	3	
		1

Figure 3.

3	2	
	1	3
2		1

1	2	
	3	2
3		1

1	3	
	1	2
2		3

Figure 4.

Theorem 7

$$M_4(p) = \begin{cases} 8 & \text{for } p = 5, \\ 5 & \text{for } 6 \leq p \leq 8, \\ 4 & \text{for } p = 9, \\ 3 & \text{for } p \geq 10. \end{cases}$$

In particular, the bound in Theorem 2 is strict for $n = 4$.

Proof. Figures 5, 2, and 6 show eight 5-compatible, five 8-compatible, and four 9-compatible MOPLS's, respectively; all of order four. Thus, $M_4(5) \geq 8$, $M_4(6) \geq M_4(7) \geq M_4(8) \geq 5$, and $M_4(9) \geq 4$. From Theorem 2, it follows that equalities hold in all five cases. Corollary 3 implies that $M_4(p) = 3$ for $p \geq 10$. □

1	2		
	3		
		1	
			4

2	3		
	1		
		4	
			1

1	2		
	3		
		4	
			1

2	3		
	1		
		1	
			4

2	3		
	4		
		1	
			1

1	2		
	1		
		3	
			4

2	1		
	3		
		1	
			4

2	1		
	3		
		4	
			1

Figure 5.

1	2	3	
		4	3
	4		2
4			1

1	2	3	
		1	4
	4		1
3			2

1	2	3	
		4	2
	1		3
2			4

1	2	3	
		2	1
	3		4
4			3

Figure 6.

4 $M_n(p)$ for $n+1 \le p \le 2n$

In this section, we construct MOPLS's for all n and p such that $n+1 \le p \le 2n$. Let $\mathcal{P}$ be a set of p cells, where $n+1 \le p \le 2n$. Suppose that we are given a collection of M sets $\mathcal{A}_1, \ldots, \mathcal{A}_M$ of (unordered) pairs of distinct cells in $\mathcal{P}$ that satisfy the following conditions:

(i) $|\mathcal{A}_k| = p - n$ for $1 \le k \le M$.

(ii) If $\langle \pi_1, \pi_1' \rangle \in \mathcal{A}_k$ and $\langle \pi_2, \pi_2' \rangle \in \mathcal{A}_k$, where π_1, π_1', π_2, and π_2' are cells in $\mathcal{P}$, then π_1, π_1', π_2, and π_2' are distinct.

(iii) If $\langle \pi, \pi' \rangle \in \mathcal{A}_k$, where π and π' are cells in $\mathcal{P}$, then π and π' do not lie in the same row or the same column.

(iv) $\mathcal{A}_k \cap \mathcal{A}_l = \phi$ for $1 \le k < l \le M$.

Notice that we use $\langle\ ,\ \rangle$ to denote unordered pairs and $(\ ,\ )$ to denote ordered pairs. From condition (i), we can write $\mathcal{A}_k = \{\langle \pi_r, \pi'_r \rangle : 1 \le r \le p-n\}$. Based on $\mathcal{A}_k$, we construct a partial latin square Γ_k with entries occupying the p cells in the set $\mathcal{P}$. The element r, where $1 \le r \le p-n$, is the entry at cells π_r and π'_r. Condition (ii) implies that there are exactly $2(p-n)$ such cells. If $p < 2n$, then the remaining $2n-p$ cells in $\mathcal{P}$ are occupied by distinct elements from the set $\{p-n+1, p-n+2, \ldots, n\}$. Condition (iii) ensures that Γ_k is indeed a partial latin square. Furthermore, condition (iv) guarantees that $\Gamma_1, \Gamma_2, \ldots, \Gamma_M$ form a collection of p-compatible MOPLS's of order n. As an example, the five 8-compatible MOPLS's of order four shown in Figure 2 can be constructed from the sets

$$\begin{aligned}
\mathcal{A}_1 &= \{\langle(1,1),(2,2)\rangle, \langle(3,3),(4,4)\rangle, \langle(1,2),(2,3)\rangle, \langle(3,4),(4,1)\rangle\}, \\
\mathcal{A}_2 &= \{\langle(1,1),(3,3)\rangle, \langle(2,2),(4,4)\rangle, \langle(1,2),(3,4)\rangle, \langle(2,3),(4,1)\rangle\}, \\
\mathcal{A}_3 &= \{\langle(1,1),(4,4)\rangle, \langle(2,2),(3,3)\rangle, \langle(1,2),(4,1)\rangle, \langle(2,3),(3,4)\rangle\}, \\
\mathcal{A}_4 &= \{\langle(3,3),(1,2)\rangle, \langle(4,4),(2,3)\rangle, \langle(1,1),(3,4)\rangle, \langle(2,2),(4,1)\rangle\}, \\
\mathcal{A}_5 &= \{\langle(4,4),(1,2)\rangle, \langle(1,1),(2,3)\rangle, \langle(2,2),(3,4)\rangle, \langle(3,3),(4,1)\rangle\}.
\end{aligned} \tag{13}$$

Similarly, the four 4-compatible MOPLS's of order three shown in Figure 3 can be constructed from the sets

$$\begin{aligned}
\mathcal{A}_1 &= \{\langle(2,2),(3,3)\rangle\}, \\
\mathcal{A}_2 &= \{\langle(1,1),(2,2)\rangle\}, \\
\mathcal{A}_3 &= \{\langle(1,1),(3,3)\rangle\}, \\
\mathcal{A}_4 &= \{\langle(3,3),(1,2)\rangle\}.
\end{aligned} \tag{14}$$

In general, to specify the sets $\mathcal{A}_k$'s, we will construct certain arrays $S(q; \lambda_1, \ldots, \lambda_m)$ for any given positive integers q and m and any non-negative integers $\lambda_1, \ldots, \lambda_m$ less than or equal to $\lfloor q/2 \rfloor$, and satisfying $\lambda_1 + \cdots + \lambda_m \le q(q-1)/2$, for which the following three properties hold:

(I) $S(q; \lambda_1, \ldots, \lambda_m)$ is a $q \times q$ symmetric array and its main diagonal is empty.

(II) Each $k = 1, 2, \ldots, m$ appears exactly $2\lambda_k$ times in $S(q; \lambda_1, \ldots, \lambda_m)$.

(III) No entry appears twice in the same row or the same column of $S(q; \lambda_1, \ldots, \lambda_m)$.

In Lemmas 9, 11, and 12, we will construct the arrays $S(q; \lambda_1, \ldots, \lambda_m)$ for q odd, $q \equiv 2 \pmod 4$, and $q \equiv 0 \pmod 4$, respectively. First, we assume that q is odd and define the following sequences of cells off the main diagonal of a $q \times q$ array:

$$\begin{aligned}
\mathcal{U}_r &= (1, r+1), (2, r), \ldots, (\lceil r/2 \rceil, \lfloor r/2 \rfloor + 2), \\
\mathcal{V}_r &= (q-r, q), (q-r+1, q-1), \ldots, (q - \lfloor r/2 \rfloor - 1, q - \lceil r/2 \rceil + 1), \\
\mathcal{U}'_r &= (r+1, 1), (r, 2), \ldots, (\lfloor r/2 \rfloor + 2, \lceil r/2 \rceil), \\
\mathcal{V}'_r &= (q, q-r), (q-1, q-r+1), \ldots, (q - \lceil r/2 \rceil + 1, q - \lfloor r/2 \rfloor - 1),
\end{aligned} \tag{15}$$

where $r = 1, 2, \ldots, q-1$. Notice that the sequences $\mathcal{U}_r$ and $\mathcal{V}_r$ contain cells (i,j) such that $i < j$. Furthermore, every cell (i,j) with $i < j$ is in $\mathcal{U}_r$ or $\mathcal{V}_r$ for some r. Similar statements hold for $\mathcal{U}'_r$ and $\mathcal{V}'_r$ with regard to the cells (i,j) with $i > j$. In fact, $\mathcal{U}'_r$ and $\mathcal{V}'_r$ are obtained by interchanging i and j for every cell (i,j) in $\mathcal{U}_r$ and $\mathcal{V}_r$, respectively. The cells in $\mathcal{U}_r$ and $\mathcal{U}'_r$ lie in different rows and columns. A similar statement holds for the cells in $\mathcal{V}_r$ and $\mathcal{V}'_r$. Each of the four sequences $\mathcal{U}_r$, $\mathcal{V}_r$, $\mathcal{U}'_r$, and $\mathcal{V}'_r$ contains $\lceil r/2 \rceil$ cells. We also have $\mathcal{U}_{q-1} = \mathcal{V}_{q-1}$ and $\mathcal{U}'_{q-1} = \mathcal{V}'_{q-1}$.

We order the $q(q-1)/2$ cells (i,j), where $1 \le i < j \le q$, as specified by the list

$$\mathcal{V}_{q-2}, (\mathcal{V}_{q-3}, \mathcal{U}_1), (\mathcal{V}_{q-4}, \mathcal{U}_2), \ldots, (\mathcal{V}_1, \mathcal{U}_{q-3}), \mathcal{U}_{q-2}, \mathcal{U}_{q-1}, \tag{16}$$

which reduces to $\mathcal{V}_1, \mathcal{U}_1, \mathcal{U}_2$ in the case $q = 3$. Here, the parentheses are only used for clarification. Similarly, the $q(q-1)/2$ cells (i,j), where $1 \le j < i \le q$, are ordered according to the list

$$\mathcal{V}'_{q-2}, (\mathcal{V}'_{q-3}, \mathcal{U}'_1), (\mathcal{V}'_{q-4}, \mathcal{U}'_2), \ldots, (\mathcal{V}'_1, \mathcal{U}'_{q-3}), \mathcal{U}'_{q-2}, \mathcal{U}'_{q-1}, \tag{17}$$

which reduces to $\mathcal{V}'_1, \mathcal{U}'_1, \mathcal{U}'_2$ in the case $q = 3$. Let $o(i,j)$ be the order of cell (i,j) starting from the left in list (16) if $i < j$ or list (17) if $i > j$. Notice that $o(i,j) = o(j,i)$ for $i \ne j$. Figure 7 illustrates the orders in the cases $q = 3$, 5, and 9. In each array, the entry in cell (i,j), where $i \ne j$, represents its order $o(i,j)$.

For any integers l_1 and l_2, where $0 \le l_1 \le l_2 \le q(q-1)/2$, let

$$\Pi(l_1+1, l_2) = \{(i,j) : 1 \le i, j \le q, i \ne j, l_1 + 1 \le o(i,j) \le l_2\}, \tag{18}$$

and

$$\Pi^*(l_1+1, l_2) = \{(i,j) : 1 \le i < j \le q, l_1 + 1 \le o(i,j) \le l_2\}.$$

Clearly, $\Pi(l_1+1, l_2)$ and $\Pi^*(l_1+1, l_2)$ contain $2(l_2 - l_1)$ and $l_2 - l_1$ cells, respectively.

	2	3
2		1
3	1	

	4	6	7	9
4		8	10	1
6	8		2	3
7	10	2		5
9	1	3	5	

	8	12	15	19	22	26	29	33
8		16	20	23	27	30	34	1
12	16		24	28	31	35	2	5
15	20	24		32	36	3	6	9
19	23	28	32		4	7	10	13
22	27	31	36	4		11	14	17
26	30	35	3	7	11		18	21
29	34	2	6	10	14	18		25
33	1	5	9	13	17	21	25	

Figure 7.

Lemma 8 *If $q \geq 3$ is odd, then*

1. *for $k = 1, 2, \ldots, q$, no two cells among the $q-1$ cells in the set $\Pi((k-1)(q-1)/2+1, k(q-1)/2)$ lie in the same row or the same column, and*

2. *for $l = 0, 1, \ldots, (q^2-2q+3)/2$, no two cells among the $q-3$ cells in the set $\Pi(l+1, l+(q-3)/2)$ lie in the same row or the same column.*

Proof. From Figure 7, the lemma can be verified for $q = 3$. Therefore, we assume in the following that $q \geq 5$. According to (16) and (17), we deduce that the orders of the cells in the sequences $\mathcal{V}_{q-2}$ and $\mathcal{V}'_{q-2}$ are $1, 2, \ldots, (q-1)/2$ since each of these sequences contains $\lceil (q-2)/2 \rceil = (q-1)/2$ cells. Thus, $\Pi(1, (q-1)/2)$ contains the cells in $\mathcal{V}_{q-2}$ and $\mathcal{V}'_{q-2}$, which lie in different rows and columns. The sequences $\mathcal{V}_{q-k-1}$ and $\mathcal{U}_{k-1}$, where $2 \leq k \leq q-2$, contain in total

$$\left\lceil \frac{q-k-1}{2} \right\rceil + \left\lceil \frac{k-1}{2} \right\rceil = \frac{q-1}{2}$$

cells. Similarly, $\mathcal{V}'_{q-k-1}$ and $\mathcal{U}'_{k-1}$ contain in total $(q-1)/2$ cells. Therefore, by induction on k, it follows that $\Pi((k-1)(q-1)/2+1, k(q-1)/2)$, where $2 \leq k \leq q-2$, is the set of cells in $\mathcal{V}_{q-k-1}$, $\mathcal{U}_{k-1}$, $\mathcal{V}'_{q-k-1}$, and $\mathcal{U}'_{k-1}$. It can be checked from (15) that no two cells among the $q-1$ cells in these four sequences lie in the same row or the same column. From the above, we deduce that the last element in $\mathcal{U}_{q-3}$ has order $(q-1)(q-2)/2$. Hence, $\Pi((q-1)(q-2)/2+1, (q-1)^2/2)$ is the set of cells in $\mathcal{U}_{q-2}$ and $\mathcal{U}'_{q-2}$, which lie in different rows and columns. The cells of the set $\Pi((q-1)^2/2+1, q(q-1)/2)$ are those in $\mathcal{U}_{q-1}$ and $\mathcal{U}'_{q-1}$, and they also lie in different rows and columns. This concludes the proof of the first part of the lemma.

Now, we consider the second part of the lemma. Notice that if the $(q-3)/2$ cells in the set $\Pi^*(l+1, l+(q-3)/2)$ are confined to the sequences $\mathcal{V}_{q-2}$, $(\mathcal{V}_{q-k-1}, \mathcal{U}_{k-1})$ for some k where $2 \leq k \leq q-2$, $\mathcal{U}_{q-2}$, or $\mathcal{U}_{q-1}$, then $\Pi(l+1, l+(q-3)/2) \subset \Pi((r-1)(q-1)/2+1, r(q-1)/2)$ for some r. In this case, the first part of the lemma implies that the cells in $\Pi(l+1, l+(q-3)/2)$ lie in different rows and columns. Next, assume that the $(q-3)/2$ cells in $\Pi^*(l+1, l+(q-3)/2)$ are confined to $(\mathcal{V}_{q-2}, \mathcal{V}_{q-3})$ such that each sequence contains at least one cell in the set, i.e., these cells are the last t cells in $\mathcal{V}_{q-2}$ and the first $(q-3)/2-t$ cells in $\mathcal{V}_{q-3}$ for some t, where $1 \leq t \leq (q-5)/2$. From (15), it follows that the cells in $\Pi(l+1, l+(q-3)/2)$ lie in different rows and columns. A similar argument shows that the same conclusion holds if the cells in the set $\Pi^*(l+1, l+(q-3)/2)$ are confined to $(\mathcal{U}_{q-3}, \mathcal{U}_{q-2})$ or $(\mathcal{U}_{q-2}, \mathcal{U}_{q-1})$. Next, assume that the cells in the set $\Pi^*(l+1, l+(q-3)/2)$ are confined to $(\mathcal{U}_{k-1}, \mathcal{V}_{q-k-2})$, for some k where $2 \leq k \leq q-3$. From (15), it can be shown that the cells in the sequences $\mathcal{U}_{k-1}$, $\mathcal{V}_{q-k-2}$, $\mathcal{U}'_{k-1}$, and $\mathcal{V}'_{q-k-2}$, and therefore the cells in $\Pi(l+1, l+(q-3)/2)$, lie in different rows and columns. Thus, we conclude that if the cells in the set $\Pi^*(l+1, l+(q-3)/2)$ are confined to two consecutive sequences in list (16), then no two cells in this set lie in the same row or the same column. Therefore, we may assume in the following that the cells in $\Pi^*(l+1, l+(q-3)/2)$ are confined to three or more consecutive sequences in list (16) such that each sequence contains at least one cell from the set. Since each of $\mathcal{U}_r$ and $\mathcal{V}_r$ contains $\lceil r/2 \rceil$ cells, any two consecutive sequences in the list contain at least $(q-3)/2$ cells, and the $(q-3)/2$ cells in $\Pi^*(l+1, l+(q-3)/2)$ are confined to three consecutive sequences in list (16), which are $(\mathcal{V}_{q-k-1}, \mathcal{U}_{k-1}, \mathcal{V}_{q-k-2})$ or $(\mathcal{U}_{q-k-2}, \mathcal{V}_{k-1}, \mathcal{U}_{q-k-1})$ for some k, where $2 \leq k \leq q-6$. In the first case, the $(q-3)/2$ cells in $\Pi^*(l+1, l+(q-3)/2)$ are the last t cells in $\mathcal{V}_{q-k-1}$, all the $\lceil (k-1)/2 \rceil = \lfloor k/2 \rfloor$ cells in $\mathcal{U}_{k-1}$, and the first $(q-3)/2-t-\lfloor k/2 \rfloor$ cells in $\mathcal{V}_{q-k-2}$ for some t, where $1 \leq t \leq (q-5)/2 - \lfloor k/2 \rfloor$. We have already stated in the proof of the first part of the lemma that no two cells among the sequences $\mathcal{V}_{q-k-1}, \mathcal{U}_{k-1}, \mathcal{V}'_{q-k-1}$, and $\mathcal{U}'_{k-1}$ lie in the same row or the same column. A similar statement holds for the sequences $\mathcal{U}_{k-1}$, $\mathcal{V}_{q-k-2}, \mathcal{U}'_{k-1}$, and $\mathcal{V}'_{q-k-2}$. Moreover, it can be verified from (15) that the last t cells in $\mathcal{V}_{q-k-1}$ and $\mathcal{V}'_{q-k-1}$, and the first $(q-5)/2 - t - \lfloor k/2 \rfloor$ cells in $\mathcal{V}_{q-k-2}$ and $\mathcal{V}'_{q-k-2}$ lie in different rows and columns. Hence, no two cells in $\Pi(l+1, l+(q-3)/2)$ lie in the same row or the same column. A similar argument shows that this is true if the cells in $\Pi(l+1, l+(q-3)/2)$ are confined to $(\mathcal{U}_{q-k-2}, \mathcal{V}_{k-1}, \mathcal{U}_{q-k-1})$. This concludes the proof of the lemma. □

Lemma 9 *For any positive odd integer q and any nonnegative integers $\lambda_1, \ldots, \lambda_m$ less than or equal to $(q-1)/2$ such that $\lambda_1 + \cdots + \lambda_m \leq q(q-1)/2$, there exists an array $S(n; \lambda_1, \ldots, \lambda_m)$ satisfying properties (I), (II), and (III).*

Proof. If $q = 1$, then we take $S(q; \lambda_1, \ldots, \lambda_m)$ to be the 1×1 empty array. Therefore, let $q \geq 3$ in the following construction of $S(q; \lambda_1, \ldots, \lambda_m)$. By renaming the entries if necessary, we may assume, without loss of generality, that $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$. Let k, where $k = 1, 2, \ldots, m$, be the entry of the cells in the set $\Pi_k = \Pi((\sum_{t=1}^{k-1} \lambda_t) + 1, \sum_{t=1}^{k} \lambda_t)$. From (18) and the way $o(i,j)$ is defined according to lists (16) and (17), it follows that properties (I) and (II) are satisfied. If $\lambda_k \leq (q-3)/2$, then the second part of Lemma 8 implies that no two cells in the set Π_k lie in the same row or the same column. If $\lambda_k = (q-1)/2$, then $\lambda_t = (q-1)/2$ for $t \leq k$. Therefore, $\Pi_t = \Pi((t-1)(q-1)/2 + 1, t(q-1)/2)$ for $1 \leq t \leq k$, and the first part of Lemma 8 implies that no two cells in Π_k lie in the same row or the same column. This shows that property (III) holds. □

The construction of the array $S(q; \lambda_1, \ldots, \lambda_m)$ given in the proof of Lemma 9 can be explained as follows. The entries are renamed, if necessary, such that $\lambda_1 \geq \cdots \geq \lambda_m$. The first λ_1 cells in each list of (16) and (17) are assigned the entry 1, the following λ_2 cells in each list are assigned the entry 2, etc., until λ_m cells in each list are assigned the entry m. Figure 8 shows $S(3; 1, 1, 1)$, $S(5; 2^{(5)})$, and $S(9; 4^{(7)}, 2^{(3)})$ constructed according to this method, where the orders of the cells in lists (16) and (17) are given in Figure 7 for $q = 3$, 5, and 9. Here we use the notation $a^{(b)}$, where b is a positive integer, to denote the sequence $(a, a, \ldots, a)$ of length b, e.g., $S(9; 4^{(7)}, 2^{(3)}) = S(9; 4, 4, 4, 4, 4, 4, 4, 2, 2, 2)$. Figure 9 shows $S(9; 2^{(3)}, 4^{(7)})$ obtained from $S(9; 4^{(7)}, 2^{(3)})$ by renaming the entries $1, 2, \ldots, 10$ as $10, 9, \ldots, 1$, respectively.

	2	3
2		1
3	1	

$S(3;1,1,1)$

	2	3	4	5
2		4	5	1
3	4		1	2
4	5	1		3
5	1	2	3	

$S(5;2^{(5)})$

	2	3	4	5	6	7	8	10
2		4	5	6	7	8	10	1
3	4		6	7	9		1	2
4	5	6		9		1	2	3
5	6	7	9		1	2	3	4
6	7	9		1		3	4	5
7	8		1	2	3		5	6
8	10	1	2	3	4	5		7
10	1	2	3	4	5	6	7	

$S(9;4^{(7)},2^{(3)})$

Figure 8.

	9	8	7	6	5	4	3	1
9		7	6	5	4	3	1	10
8	7		5	4	2		10	9
7	6	5		2		10	9	8
6	5	4	2		10	9	8	7
5	4	2		10		8	7	6
4	3		10	9	8		6	5
3	1	10	9	8	7	6		4
1	10	9	8	7	6	5	4	

$S(9;2^{(3)},4^{(7)})$

Figure 9.

As we will see later, the array $S(q; (\frac{q-1}{2})^{(q)})$ is of special interest. In this array, each element $k = 1, 2, \ldots, q$ appears in exactly $q - 1$ cells. Therefore, there is a unique row in which k does not appear. Let this row be the $\theta(k)^{th}$ row. Since the array is symmetric, the element k does not appear in the $\theta(k)^{th}$ column also. Notice that $\theta(1), \theta(2), \ldots, \theta(q)$ are distinct since each row contains $q - 1$ distinct entries. As an example, for the array $S(5; 2^{(5)})$ shown in Figure 8, $\theta(1) = 1$, $\theta(2) = 4$, $\theta(3) = 2$, $\theta(4) = 5$, and $\theta(5) = 3$. In general, it can be shown that for the array $S(q; (\frac{q-1}{2})^{(q)})$ constructed in the proof of Lemma 9,

$$\theta(k) = \begin{cases} (k+1)/2 & \text{if } k \text{ is odd,} \\ (q+k+1)/2 & \text{if } k \text{ is even.} \end{cases} \tag{19}$$

For $\theta = 1, 2, \ldots, q$, let $k(\theta)$ be the unique element k, where $k = 1, 2, \ldots, q$, for which $\theta(k) = \theta$, i.e., $k(\theta)$ is the entry that does not appear in the θ^{th} row or column. For example, if $S(q; (\frac{q-1}{2})^{(q)})$ is constructed as shown in the proof of Lemma 9, then (19) implies that

$$k(\theta) = \begin{cases} 2\theta - 1 & \text{if } 1 \le \theta \le (q+1)/2, \\ 2\theta - q - 1 & \text{if } (q+1)/2 + 1 \le \theta \le q. \end{cases} \tag{20}$$

In particular, for the array $S(5; 2^{(5)})$ shown in Figure 8, $k(1) = 1$, $k(2) = 3$, $k(3) = 5$, $k(4) = 2$, and $k(5) = 4$.

In the following, we will show how to construct an array $S(q; \lambda_1, \ldots, \lambda_m)$ satisfying properties (I), (II), and (III), in the case q is even. The array $S(q; \lambda_1, \ldots, \lambda_m)$ will be specified in the form

$$S(q; \lambda_1, \ldots, \lambda_m) = \begin{array}{|c|c|} \hline B_1 & C \\ \hline C^T & B_2 \\ \hline \end{array}\ , \tag{21}$$

where B_1, B_2, and C are $q/2 \times q/2$ arrays described below and C^T is the transpose of C. By ordering the rows and columns of the array C as $1, 2, \ldots, q/2$, the set of cells (i, j), where $1 \le i, j \le q/2$, in C can be partitioned into the $q/2$ sequences $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{q/2}$, where

$$\begin{aligned} \mathcal{W}_r = {} & (1, r), (2, r+1), \ldots, (q/2 - r + 1, q/2), \\ & (q/2 - r + 2, 1), (q/2 - r + 3, 2), \ldots, (q/2, r - 1), \end{aligned}$$

which is composed of the $q/2 - r + 1$ cells $(i, i + r - 1)$ as i runs over $1, 2, \ldots, q/2 - r + 1$, followed by the $r - 1$ cells $(q/2 - r + i + 1, i)$ as i runs over $1, 2, \ldots, r-1$. Notice that if $q = 2$, then $\mathcal{W}_1 = (1, 1)$. The $q^2/4$ cells in the array C are ordered according to the list

$$\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_{q/2}. \tag{22}$$

The proof of the next result is straightforward.

Lemma 10 *If $q \geq 2$ is even, then*

1. *for $r = 1, 2, \ldots, q/2$, the $q/2$ cells in $\mathcal{W}_r$ lie in different rows and columns, and*
2. *any $q/2 - 1$ consecutive cells in list (22) lie in different rows and columns.*

Lemma 11 *For any $q \equiv 2 \pmod 4$, where $q \geq 2$, and any nonnegative integers $\lambda_1, \ldots, \lambda_m$ less than or equal to $q/2$ such that $\lambda_1 + \cdots + \lambda_m \leq q(q-1)/2$, there exists an array $S(q; \lambda_1, \ldots, \lambda_m)$ satisfying properties (I), (II), and (III).*

Proof. By renaming the entries if necessary, we may assume, without loss of generality, that $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$. First, suppose that $m \geq q/2$ and $\lambda_1 = \cdots = \lambda_{q/2} = q/2$. In this case, let $B_1 = B_2 = S(q/2; ((q-2)/4)^{(q/2)})$, which can be constructed as shown in the proof of Lemma 9. Each element $k = 1, 2, \ldots, q/2$ is assigned as the entry of cell $(\theta(k), \theta(k))$ on the main diagonal of C. Thus, the entries in the cells of the sequence $\mathcal{W}_1$ are specified. None of the elements $1, 2, \ldots, q/2$ appears elsewhere in the array given in (21). From Lemma 9 and the definition of $\theta(k)$, it follows that none of these elements appears more than once in the same row or the same column. If $m > q/2$, then the first $\lambda_{q/2+1}$ cells in C in the list $\mathcal{W}_2, \mathcal{W}_3, \ldots, \mathcal{W}_{q/2}$ are assigned the entry $q/2+1$, and the following $\lambda_{q/2+2}$ cells are assigned the entry $q/2+2$, etc., until λ_m cells are assigned the entry m. The construction of C is complete. Since $q/2 \geq \lambda_{q/2+1} \geq \cdots \geq \lambda_m$, it can be shown using Lemma 10 that $S(q; \lambda_1, \ldots, \lambda_m)$ satisfies properties (I), (II), and (III).

Next, suppose that $m < q/2$ or $\lambda_{q/2} < q/2$. Then, $\lambda_t < q/2$ for $q/2 \leq t \leq m$ since $\lambda_1 \geq \cdots \geq \lambda_m$. Let t^* be the least integer such that $\sum_{t=1}^{t^*} \lambda_t > q^2/4$ if $\sum_{t=1}^{m} \lambda_t > q^2/4$, otherwise define $t^* = m+1$. The first λ_1 cells in list (22) are assigned the entry 1, and the following λ_2 cells are assigned the entry 2, etc, until λ_{t^*-1} cells are assigned the entry t^*-1. In the case $t^* = m+1$, the construction of the array C is complete and we take B_1 and B_2 in (21) to be the $q/2 \times q/2$ empty arrays. If $t^* \leq m$, then the last $q^2/4 - \sum_{t=1}^{t^*-1} \lambda_t$ cells in C are assigned the entry t^*. Lemma 10 implies that no entry in C appears more than once in the same row or the same column. Thus, if $t^* = m+1$, then $S(q;\lambda_1,\ldots,\lambda_m)$ satisfies properties (I), (II), and (III). In the following, we will assume that $t^* \leq m$. Notice that $\lambda_{t^*} \leq q/2-1$, and therefore $\lambda_t \leq q/2-1$ for $t \geq t^*$, otherwise $q/2 \geq \lambda_1 \geq \cdots \geq \lambda_m$ and $\lambda_{q/2} < q/2$ imply that $t^* < q/2$ and $\lambda_t = q/2$ for $t \leq t^*$ which contradicts that $\sum_{t=1}^{t^*} \lambda_t > q^2/4$. It remains to construct symmetric arrays B_1 and B_2 with empty diagonals such that t^* appears in $2[\lambda_{t^*} - (q^2/4 - \sum_{t=1}^{t^*-1} \lambda_t)] = 2[(\sum_{t=1}^{t^*} \lambda_t) - q^2/4]$ cells, each k, where $t^*+1 \leq k \leq m$, appears in $2\lambda_k$ cells in the arrays B_1 and B_2, and no entry appears more than once in the same row or the same column. Furthermore, since the entry t^* appears in $q^2/4 - \sum_{t=1}^{t^*-1} \lambda_t$ cells in C, t^* should not appear in the same row in C and B_1 or the same column in C and B_2. For the moment, we will ignore this last requirement involving the array C.

Let $\{t_1, t_2, \ldots, t_r\}$ be the set of all t, where $t^*+1 \leq t \leq m$, such that λ_t is odd. For $1 \leq k \leq m$, define

$$\lambda'_k = \begin{cases} 0 & \text{if } 1 \leq k \leq t^*-1, \\ \left\lfloor ((\sum_{t=1}^{t^*} \lambda_t) - (q^2/4))/2 \right\rfloor & \text{if } k = t^*, \\ \lambda_k/2 & \text{if } k \geq t^*+1 \text{ and } k \notin \{t_1, t_2, \ldots, t_r\}, \\ \lceil \lambda_k/2 \rceil & \text{if } k = t_l \text{ for some odd } l, \\ \lfloor \lambda_k/2 \rfloor & \text{if } k = t_l \text{ for some even } l, \end{cases} \tag{23}$$

and

$$\lambda''_k = \begin{cases} 0 & \text{if } 1 \leq k \leq t^*-1, \\ (\sum_{t=1}^{t^*} \lambda_t) - (q^2/4) - \lambda'_{t^*} & \text{if } k = t^*, \\ \lambda_k - \lambda'_k & \text{if } t^*+1 \leq k \leq m. \end{cases} \tag{24}$$

Since $\lambda_k \leq q/2-1$ for $k \geq t^*$, $\sum_{t=1}^{t^*-1} \lambda_t \leq q^2/4$, and $\sum_{t=1}^{t^*} \lambda_t > q^2/4$, it follows that for $k = 1, 2, \ldots, m$,

$$0 \leq \lambda'_k, \lambda''_k \leq \lceil ((q/2-1)/2) \rceil = (q-2)/4. \tag{25}$$

From (23), (24), and the assumption that $\sum_{t=1}^{m} \lambda_t \leq q(q-1)/2$, we have

$$\sum_{t=1}^{m} \lambda'_t + \sum_{t=1}^{m} \lambda''_t = (\sum_{t=1}^{m} \lambda_t) - (q^2/4) \leq q(q-2)/4,$$

which is an even integer. Combining this with the fact that

$$-1 \leq \sum_{t=1}^{m} \lambda'_k - \sum_{t=1}^{m} \lambda''_k \leq 1,$$

which follows from (23) and (24), we get

$$\max\left\{\sum_{t=1}^{m}\lambda'_t, \sum_{t=1}^{m}\lambda''_t\right\} \leq q(q-2)/8.$$

Let the $q/2 \times q/2$ arrays B_1 and B_2 in (21) be $S(q/2; \lambda'_1, \lambda'_2, \ldots, \lambda'_m)$ and $S(q/2; \lambda''_1, \lambda''_2, \ldots, \lambda''_m)$, respectively, constructed as in Lemma 9, for which properties (I), (II), and (III) hold. This implies that each $k = 1, 2, \ldots, m$ appears in exactly $2\lambda_k$ cells in $S(q; \lambda_1, \ldots, \lambda_m)$ and these cells lie in different rows and columns, with the possible exception that t^* may appear in the same row in B_1 and C, and hence in the same column in B_1 and C^T, or in the same column in B_2 and C, and hence in the same row in B_2 and C^T. We will show that this possibility can be eliminated by a suitable permutation of the rows and columns of C, and the corresponding columns and rows of C^T. Such a permutation maintains the property that no element appears more than once in the same row or the same column of C. The element t^* appears in exactly $(q^2/4) - \sum_{t=1}^{t^*-1}\lambda_t$ cells in C, $2\lambda'_{t^*}$ cells in B_1, and $2\lambda''_{t^*}$ cells in B_2. Therefore, the permutation should avoid placing any of the $(q^2/4) - \sum_{t=1}^{t^*-1}\lambda_t$ cells in C containing t^* in any of the $2\lambda'_{t^*}$ rows in which t^* appears in B_1, or in any of the $2\lambda''_{t^*}$ columns in which t^* appears in B_2. From (23) and (24), $\lambda'_{t^*} \leq \lambda''_{t^*} \leq \lceil((\sum_{t=1}^{t^*}\lambda_t) - (q^2/4))/2\rceil$. Therefore,

$$\frac{q}{2} - 2\lambda'_{t^*} \geq \frac{q}{2} - 2\lambda''_{t^*} \geq \frac{q}{2} - 2\left\lceil\frac{(\sum_{t=1}^{t^*}\lambda_t) - (q^2/4)}{2}\right\rceil$$
$$\geq \frac{q}{2} - \left((\sum_{t=1}^{t^*}\lambda_t) - \frac{q^2}{4} + 1\right) \geq \frac{q^2}{4} - \sum_{t=1}^{t^*-1}\lambda_t,$$

where we used the fact that $\lambda_{t^*} \leq (q/2)-1$. Hence, it is possible to permute the rows and columns of C to avoid having any of the $q^2/4 - \sum_{t=1}^{t^*-1}\lambda_t$ cells in C containing t^* in any of the $2\lambda'_{t^*}$ rows containing t^* in B_1, or in any of the $2\lambda''_{t^*}$ columns containing t^* in B_2. With this permutation, t^* does not appear twice in the same row or the same column in $S(q; \lambda_1, \ldots, \lambda_m)$. This concludes the proof of the lemma. □

Figure 10 shows $S(6; 3^{(4)}, 2)$ and $S(6; 3, 3, 2, 2, 2, 1)$ constructed according to the proof of Lemma 11.

	2	3	1	4	5
2		1	5	3	4
3	1		4		2
1	5	4		2	3
4	3		2		1
5	4	2	3	1	

$S(6;3^{(4)},2)$

	6		2	1	3
6		5	1	3	2
	5		4	2	1
2	1	4		5	
1	3	2	5		4
3	2	1		4	

$S(6;3,3,2,2,2,1)$

Figure 10.

For $S(6;3^{(4)},2)$, we take the 3×3 arrays B_1 and B_2 to be $S(3;1,1,1)$, which is shown in Figure 8. The entry k, where $k = 1,2$, and 3, is assigned to cell $(\theta(k),\theta(k))$ in the array C as given in (19). The remaining cells in C are ordered according to the list

$$(1,2),(2,3),(3,1),(1,3),(2,1),(3,2).$$

The first three cells in this list are assigned the entry 4 and the following two cells are assigned the entry 5.

For $S(6;3,3,2,2,2,1)$, we order the cells of the array C according to the list

$$(1,1),(2,2),(3,3),(1,2),(2,3),(3,1),(1,3),(2,1),(3,2).$$

The first three cells in this list, $(1,1),(2,2),(3,3)$, are assigned the entry 1. The following three cells, $(1,2),(2,3),(3,1)$, are assigned the entry 2. The cells $(1,3),(2,1)$ are assigned the entry 3. Notice that $t^* = 4$ and the cell $(3,2)$ is assigned the entry 4. Thus,

$C =$

1	2	3
3	1	2
2	4	1

From (23) and (24), we have $\lambda'_1 = \lambda'_2 = \lambda'_3 = \lambda'_4 = 0$, $\lambda'_5 = \lambda'_6 = 1$, $\lambda''_1 = \lambda''_2 = \lambda''_3 = 0$, $\lambda''_4 = \lambda''_5 = 1$, and $\lambda''_6 = 0$. We take $B_1 = S(3;0,0,0,0,1,1)$ and $B_2 = (3;0,0,0,1,1,0)$ as constructed in the proof of Lemma 9, i.e.,

$B_1 =$

	6	
6		5
	5	

and $B_2 =$

	5	
5		4
	4	

Since the entry 4 appears in the second column in both of C and B_2, and also in the third column in B_2, we need to permute the columns in C in order to avoid having the entry 4 in the last two columns. This is done by interchanging the first two columns in C, which gives the array $S(6; 3, 3, 2, 2, 2, 1)$ shown in Figure 10.

Lemma 12 *For any $q \equiv 0 \pmod 4$, where $q \geq 4$, and any nonnegative integers $\lambda_1, \ldots, \lambda_m$ less than or equal to $q/2$ such that $\lambda_1 + \cdots + \lambda_m \leq q(q-1)/2$, there exists an array $S(q; \lambda_1, \ldots, \lambda_m)$ satisfying properties (I), (II), and (III).*

Proof. The array $S(q; \lambda_1, \ldots, \lambda_m)$ is specified in terms of $q/2 \times q/2$ arrays B_1, B_2, and C as shown in (21). By renaming the entries if necessary, we may assume, without loss of generality, that $\lambda_1 \geq \cdots \geq \lambda_m \geq 1$. First, suppose that $m \geq q/2$ and $\lambda_1 = \cdots = \lambda_{q/2} = q/2$. In this case, let $B_1 = B_2 = S(q/2; (q/4)^{(q/2-1)})$, which can be constructed using induction on q based on Lemma 11. The first $\lambda_{q/2}$ cells in C in list (22) are assigned the entry $q/2$, and the following $\lambda_{q/2+1}$ cells are assigned the entry $q/2+1$, etc., until λ_m cells are assigned the entry m. The construction of C is complete. Since $q/2 \geq \lambda_{q/2} \geq \cdots \geq \lambda_m$, it can be shown using Lemma 10 that $S(q; \lambda_1, \ldots, \lambda_m)$ satisfies properties (I), (II), and (III). Next, suppose that $m < q/2$ or $\lambda_{q/2} < q/2$. In this case, the construction of $S(q; \lambda_1, \ldots, \lambda_m)$ is similar to the corresponding construction explained in the proof of Lemma 11. Notice that the expression $(q-2)/4$ which appears in (25) should be replaced by $q/4$. Furthermore, the arrays $S(q/2; \lambda'_1, \ldots, \lambda'_m)$ and $S(q/2; \lambda''_1, \ldots, \lambda''_m)$ can be constructed using induction on q based on Lemma 11. □

Figure 11 shows $S(4; 2, 2, 2)$ and $S(4; 1^{(6)})$ constructed according to the proof of Lemma 12. For $S(4; 2, 2, 2)$, we take the 2×2 arrays B_1 and B_2 to be $S(2; 1)$, as constructed in the proof of Lemma 11, i.e.,

$$B_1 = B_2 = \begin{array}{|c|c|} \hline & 1 \\ \hline 1 & \\ \hline \end{array} \; .$$

The cells in C are ordered according to the list $(1,1), (2,2), (1,2), (2,1)$. The first two cells in this list are assigned the entry 2 and the following two cells are assigned the entry 3.

	1	2	3
1		3	2
2	3		1
3	2	1	

$S(4;2,2,2)$

	6	1	3
6		4	2
1	4		5
3	2	5	

$S(4;1^{(6)})$

Figure 11.

For $S(4;1^{(6)})$, we order the cells of the array C according to the list $(1,1)$, $(2,2)$, $(1,2)$, $(2,1)$. These cells are assigned the entries 1, 2, 3, and 4, respectively. Thus,

$$C = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline \end{array} \ .$$

Notice that $t^* = 5$. From (23) and (24), we have $\lambda'_1 = \cdots = \lambda'_5 = 0$, $\lambda'_6 = 1$, $\lambda''_1 = \cdots = \lambda''_4 = 0$, $\lambda''_5 = 1$, and $\lambda''_6 = 0$. We take $B_1 = S(2;0^{(5)},1)$ and $B_2 = (2;0^{(4)},1,0)$ as constructed in the proof of Lemma 9, i.e.,

$$B_1 = \begin{array}{|c|c|}\hline & 6 \\ \hline 6 & \\ \hline \end{array} \qquad \text{and} \qquad B_2 = \begin{array}{|c|c|}\hline & 5 \\ \hline 5 & \\ \hline \end{array} \ ,$$

which gives the array $S(4;1^{(6)})$ shown in Figure 11.

Lemmas 9, 11, and 12 can be combined in the following result.

Lemma 13 *For any positive integer q and any nonnegative integers $\lambda_1,\ldots,\lambda_m$ less than or equal to $\lfloor q/2 \rfloor$ such that $\lambda_1+\cdots+\lambda_m \le q(q-1)/2$, there exists an array $S(q;\lambda_1,\ldots,\lambda_m)$ satisfying properties (I), (II), and (III).*

Based on these arrays, we will construct in Theorem 15 a collection of p-compatible MOPLS's of order n. In order to accomplish this, $\lambda_1,\ldots,\lambda_m$ need to be specified properly depending on the values of n and p. This issue is addressed in the following lemma.

Lemma 14 *If $n+1 \le p \le 2n$, and $p \le (2n-p)^2$ in the case n is odd, then there exist integers $\lambda_1,\ldots,\lambda_m$ such that*

$$\lceil (p-n)/2 \rceil \le \lambda_1,\ldots,\lambda_m \le \min\{\lfloor n/2 \rfloor, p-n\}$$

and

$$(p-n)(2m-p+n+1)/2 \le \lambda_1+\cdots+\lambda_m \le n(n-1)/2,$$

where

$$m = \left\lfloor \frac{n(n-1)+(p-n)(p-n-1)}{2(p-n)} \right\rfloor.$$

Proof. First, we will show that

$$\lfloor n/2 \rfloor m \geq n(n-1)/2. \tag{26}$$

We have

$$\begin{aligned} m-(n-1) &= \left\lfloor \frac{n(n-1)+(p-n)(p-n-1)}{2(p-n)} \right\rfloor - (n-1) \\ &= \left\lfloor \frac{(2n-p)(2n-p-1)}{2(p-n)} \right\rfloor \geq 0, \end{aligned}$$

which proves (26) in the case n is even. Furthermore, if n is odd, then $p \leq (2n-p)^2$ and

$$\begin{aligned} m-(n-1) &= \left\lfloor \frac{(2n-p)(2n-p-1)}{2(p-n)} \right\rfloor = \left\lfloor \frac{(2n-p)^2-(2n-p)}{2(p-n)} \right\rfloor \\ &\geq \left\lfloor \frac{p-(2n-p)}{2(p-n)} \right\rfloor = 1, \end{aligned}$$

and (26) holds in the case n is odd. Next, we will show that

$$(p-n)m \geq n(n-1)/2. \tag{27}$$

If $p = n+1$, then $m = \lfloor n(n-1)/2 \rfloor = n(n-1)/2$ and (27) holds. If $p = n+2$, then $m = \lfloor (n(n-1)+2)/4 \rfloor \geq n(n-1)/4$ and (27) also holds. If $p \geq n+3$, then

$$\begin{aligned} (p-n)m &= (p-n)\left\lfloor \frac{n(n-1)+(p-n)(p-n-1)}{2(p-n)} \right\rfloor \\ &\geq (p-n)\left\lfloor \frac{n(n-1)+2(p-n)}{2(p-n)} \right\rfloor > \frac{n(n-1)}{2}. \end{aligned}$$

From (26) and (27), we obtain

$$\min\{\lfloor n/2 \rfloor, p-n\}m \geq n(n-1)/2. \tag{28}$$

Next, we will show that

$$\lceil (p-n)/2 \rceil m \leq (p-n)(2m-p+n+1)/2, \tag{29}$$

which is equivalent to

$$\lfloor (p-n)/2 \rfloor m \geq (p-n)(p-n-1)/2. \tag{30}$$

From (26) we have $m \geq n-1$. If $p \leq 2n-1$, then

$$\lfloor (p-n)/2 \rfloor m \geq (p-n-1)(n-1)/2 \geq (p-n)(p-n-1)/2,$$

and (30) holds. If $p = 2n$, then $p > (2n-p)^2$ and n is even. In this case, (30) also holds. Therefore, we have

$$\lceil (p-n)/2 \rceil m \leq (p-n)(2m-p+n+1)/2 \leq n(n-1)/2 \leq \min\{\lfloor n/2 \rfloor, p-n\} m, \tag{31}$$

where the first inequality follows from (29), the second inequality follows from the definition of m, and the third inequality follows from the (28). From (31), we conclude that there exist m integers $\lambda_1, \ldots, \lambda_m$ that satisfy the statement of the lemma. □

Theorem 15 *For* $n+1 \leq p \leq 2n$,

$$M_n(p) = \left\lfloor \frac{p(p-1)}{2(p-n)} \right\rfloor - 2.$$

In particular, the bound in Theorem 2 is strict for $n+1 \leq p \leq 2n$.

Proof. The result obviously holds for $n = 2$. Hence, assume that $n \geq 3$. Let

$$\mathcal{P} = \{(i,i) : i = 1, 2, \ldots, n\} \cup \{(i, (i+1)_n) : i = 1, 2, \ldots, n\},$$

where the notation $(l)_n$, for an integer l, denotes the integer satisfying $1 \leq (l)_n \leq n$ and $(l)_n \equiv l \pmod{n}$. Let

$$m = \left\lfloor \frac{n(n-1) + (p-n)(p-n-1)}{2(p-n)} \right\rfloor. \tag{32}$$

First, assume that $p \leq (2n-p)^2$ in the case n is odd. We will specify a collection of $m+n-2$ sets $\mathcal{A}_1, \ldots, \mathcal{A}_{m+n-2}$ of pairs of distinct cells in $\mathcal{P}$ that satisfies conditions (i)–(iv) stated at the beginning of this section. From Lemma 14, there exist integers $\lambda_1, \ldots, \lambda_m$ that satisfy the conditions stated in the lemma. In particular, $0 \leq \lambda_1, \ldots, \lambda_m \leq \lfloor n/2 \rfloor$ and $\lambda_1 + \cdots + \lambda_m \leq n(n-1)/2$. Hence, Lemma 13 ensures that there exists an array $S(n; \lambda_1, \ldots, \lambda_m)$ that satisfies properties (I), (II), and (III). Furthermore, the conditions satisfied by $\lambda_1, \ldots, \lambda_m$ in Lemma 14 imply that

$$0 \leq p-n-\lambda_1, \ldots, p-n-\lambda_m \leq \lfloor (p-n)/2 \rfloor$$

and

$$(p-n-\lambda_1) + \cdots + (p-n-\lambda_m) \leq (p-n)(p-n-1)/2.$$

Hence, Lemma 13 ensures that there exists an array $S(p-n; p-n-\lambda_1, \ldots, p-n-\lambda_m)$ that satisfies properties (I), (II), and (III). Let $S(i,j)$ and $S'(i,j)$ denote the entries of cell (i,j) in the arrays $S(n;\lambda_1,\ldots,\lambda_m)$ and $S(p-n;p-n-\lambda_1,\ldots,p-n-\lambda_m)$, respectively, if the cell is nonempty. We define for $k = 1, 2, \ldots, m$,

$$\begin{aligned}\mathcal{A}_k = \ & \{\langle (i,i),(j,j)\rangle : 1 \le i,j \le n, S(i,j) = k\} \cup \\ & \{\langle (i,(i+1)_n),(j,(j+1)_n)\rangle : 1 \le i,j \le p-n, S'(i,j) = k\},\end{aligned} \tag{33}$$

and for $k = m+1, m+2, \ldots, m+n-2$,

$$\mathcal{A}_k = \{\langle ((k-m+t+1)_n,(k-m+t+1)_n),(t,(t+1)_n)\rangle : 1 \le t \le p-n\}. \tag{34}$$

From properties (I) and (II), it follows that $\mathcal{A}_k$, for $1 \le k \le m$, contains $p-n$ pairs of distinct cells since $S(i,j) = k$ for $2\lambda_k$ cells (i,j) in $S(n;\lambda_1,\ldots,\lambda_m)$ and $S'(i,j) = k$ for $2(p-n-\lambda_k)$ cells (i,j) in $S(p-n;p-n-\lambda_1,\ldots,p-n-\lambda_m)$, and if $S(i,j) = k$ or $S'(i,j) = k$, then $S(j,i) = k$ or $S'(j,i) = k$, respectively, and $i \ne j$. Furthermore, property (III) implies that the cells in any pairs of cells in $\mathcal{A}_k$, for $1 \le k \le m$, are distinct. It also follows that the two cells in any pair of cells in $\mathcal{A}_k$, for $1 \le k \le m$, lie in different rows and columns. Therefore, $\mathcal{A}_k$, where $k = 1, 2, \ldots, m$, satisfies conditions (i), (ii), and (iii). From (34), it follows that $\mathcal{A}_k$, where $k = m+1, m+2, \ldots, m+n-2$, satisfies conditions (i) and (ii). Condition (iii) is also satisfied since if this is not the case, then $(k-m+t+1)_n = t$ or $(t+1)_n$, for some k and t, where $m+1 \le k \le m+n-2$ and $1 \le t \le p-n$. This implies that n divides $k-m+1$ or $k-m$, which is impossible. It can be also checked that condition (iv) also holds for $\mathcal{A}_1, \ldots, \mathcal{A}_{m+n-2}$, i.e., $\mathcal{A}_k \cap \mathcal{A}_l = \phi$, where $1 \le k < l \le m+n-2$. Therefore, $\mathcal{A}_1, \ldots, \mathcal{A}_{m+n-2}$ satisfy conditions (i)–(iv). The discussion following the statement of these conditions shows how to construct a collection of $m+n-2$ MOPLS's of order n that are p-compatible. Therefore, $M_n(p) \ge m+n-2$ if $n+1 \le p \le 2n$, and $p \le (2n-p)^2$ in the case n is odd.

Next, we consider the case n is odd and $p = 2n$. We will specify a collection of $2n-3$ sets $\mathcal{A}_1, \ldots, \mathcal{A}_{2n-3}$ of pairs of distinct cells that satisfies conditions (i)–(iv). We define for $\theta = 1, 2, \ldots, n$,

$$\begin{aligned}\mathcal{A}_\theta = \ & \{\langle (i,i),(j,j)\rangle : 1 \le i,j \le n, S(i,j) = k(\theta)\} \cup \\ & \{\langle (i,(i+1)_n),(j,(j+1)_n)\rangle : 1 \le i,j \le n, S(i,j) = k((\theta-2)_n)\} \cup \\ & \{\langle (\theta,\theta),((\theta-2)_n,(\theta-1)_n)\rangle\},\end{aligned} \tag{35}$$

and for $k = n+1, n+2, \ldots, 2n-3$,

$$\mathcal{A}_k = \{\langle ((k-n+t+2)_n,(k-n+t+2)_n),(t,(t+1)_n)\rangle : 1 \le t \le n\}, \tag{36}$$

where $S(i,j)$, for $i \neq j$, is the entry of cell (i,j) in $S(n; (\frac{n-1}{2})^{(n)})$, whose existence is established in Lemma 9, and $k(\theta)$ is as defined following the lemma. Property (III) and the fact that $k(\theta)$ does not appear in the θ^{th} row or column of $S(n; (\frac{n-1}{2})^{(n)})$, i.e., $S(i,j) \neq k(\theta)$ if $i = \theta$ or $j = \theta$, imply that the cells in any pairs of cells in $\mathcal{A}_k$, for $1 \leq k \leq n$, are distinct. Also, the cells (θ, θ) and $((\theta-2)_n, (\theta-1)_n)$ lie in different rows and columns. Combining this with an argument similar to that presented for the $\mathcal{A}_k$'s defined in (33) and (34), we conclude that $\mathcal{A}_1, \ldots, \mathcal{A}_{2n-3}$, as defined in (35) and (36), satisfy conditions (i)–(iv). Therefore, we can construct $2n-3$ MOPLS's of odd order n that are $2n$-compatible. By deleting the entries of any set of compatible cells from all squares, we obtain $2n-3$ MOPLS's of odd order n that are p-compatible for any $p < 2n$. Hence, $M_n(p) \geq 2n-3$ for $p \leq 2n$ if n is odd.

Now, notice that the expression of m in (32) gives

$$m - n + 1 = \left\lfloor \frac{(2n-p)(2n-p-1)}{2(p-n)} \right\rfloor = \left\lfloor \frac{(2n-p)^2 - (2n-p)}{2(p-n)} \right\rfloor .$$

If $p \leq 2n$, then $\lfloor (2n-p)(2n-p-1)/(2(p-n)) \rfloor$ is nonnegative, and hence, $m \geq n-1$. On the other hand, if $(2n-p)^2 < p$, then $\lfloor ((2n-p)^2 - (2n-p))/(2(p-n)) \rfloor \leq \lfloor ((p-1)-(2n-p))/(2(p-n)) \rfloor = 0$, and hence $m \leq n-1$. Therefore, $m = n-1$, which implies that $M_n(p) \geq 2n-3 = m+n-2$, if $(2n-p)^2 < p \leq 2n$.

We conclude that $M_n(p) \geq m+n-2$ for all $n+1 \leq p \leq 2n$. From (32), it follows that

$$m + n = \left\lfloor \frac{p(p-1)}{2(p-n)} \right\rfloor .$$

Thus,

$$M_n(p) \geq \left\lfloor \frac{p(p-1)}{2(p-n)} \right\rfloor - 2.$$

On the other hand, it can be shown that the upper bound in Theorem 2 reduces to the above lower bound. Hence, $M_n(p) = \lfloor p(p-1)/(2(p-n)) \rfloor - 2$ and the bound in Theorem 2 is strict in the case $n+1 \leq p \leq 2n$. □

Based on the proof of Theorem 15, we summarize the following method to construct $M_n(p)$ MOPLS's of order n that are p-compatible, where $n+1 \leq p \leq 2n$:

- If n is even or n is odd and $p \leq (2n-p)^2$:

  1. Find $\lambda_1, \ldots, \lambda_m$ satisfying the conditions stated in Lemma 14.

2. Construct the arrays $S(n; \lambda_1, \ldots, \lambda_m)$ and $S(p-n; p-n-\lambda_1, \ldots, p-n-\lambda_m)$ as shown in the proofs of Lemmas 9, 11, and 12.
3. Determine $\mathcal{A}_k$ for $k = 1, 2, \ldots, M_n(p)$, as given in (33) and (34), where $S(i,j)$ and $S'(i,j)$ are the entries of cell (i,j) in $S(n; \lambda_1, \ldots, \lambda_m)$ and $S(p-n; p-n-\lambda_1, \ldots, p-n-\lambda_m)$, respectively.
4. Based on $\mathcal{A}_1, \ldots, \mathcal{A}_{M_n(p)}$, construct $M_n(p)$ MOPLS's of order n that are p-compatible as explained in the beginning of this section.

- If n is odd and $p > (2n-p)^2$:

  1. Construct the array $S(n; (\frac{n-1}{2})^{(n)})$ as shown in the proof of Lemma 9.
  2. Determine $\mathcal{A}_k$, for $k = 1, 2, \ldots, M_p(n) = 2n-3$, as given in (35) and (36), where $S(i,j)$ is the entry of cell (i,j) in $S(n; (\frac{n-1}{2})^{(n)})$ and $k(\theta)$ is given in (20).
  3. Based on $\mathcal{A}_1, \ldots, \mathcal{A}_{M_n(p)}$, construct $M_n(p)$ MOPLS's of order n that are $2n$-compatible as explained in the beginning of this section.
  4. If $p < 2n$, delete the entries of $2n-p$ compatible cells from the above MOPLS's to obtain $M_n(p)$ MOPLS's of order n that are p-compatible.

For example, in case $n = 3$ and $p = 4$, the construction yields the sets $\mathcal{A}_1, \ldots, \mathcal{A}_4$ given in (14) and the four MOPLS's shown in Figure 3. In case $n = 4$ and $p = 8$, we obtain the sets $\mathcal{A}_1, \ldots, \mathcal{A}_5$ given in (13) and the five MOPLS's shown in Figure 2.

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