Balanced Strands for Asymmetric, Edge-graceful Spiders

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This note is an extension of [4], wherein is shown a relation between the dual notions of graceful and edge-graceful graphs. In particular, this note proves two graceful conjectures raised in [4], and then utilizes the result to edge-gracefully label certain trees not previously known to be edge-graceful.

Strands.

Let P_n be a path of $n+1$ vertices and n edges. We say that r *threads* n if the vertices of P_n can be labeled using all of the integers from 0 to n, with r as the label of one end vertex, so that the set of edge labels is 1 through n, where the label of edge uv is the magnitude of the difference between the vertex labels u and v. (This notion of threading is simply a special case of the notion of *graceful* graphs as coined by Golomb in [1].) What we shall show in this section is that r threads n for all integers r and n with $0 \leq r \leq n$. Just to pose this problem in a more interesting context, imagine having $n+1$ pearls labeled 0 through n and ask the question, *Is it possible to thread all of these pearls onto a chain starting with any pearl so that the set of all label differences between adjacent pearls is the set of integers 1 through n?* For example, figure 1*a* shows that for pearls labeled 0 through 9, the chain can start with 2 or 7; figure 1*b* is the same sequence of pearls viewed as the path P_9.

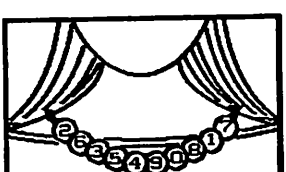

Figure 1a. A strand of pearls.

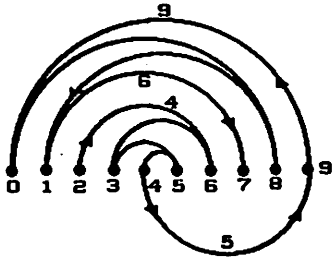

Figure 1b. The same strand.

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We adopt the following notation. If r threads n, let S be the corresponding sequence of vertex labels along P_n, beginning with r. We say that S is a *strand* for r threads n. From figure 1, a strand for 2 threads 9 is $\{2,6,3,5,4,9,0,8,1,7\}$. For simplicity, a strand is considered both as a string of integers and as a set of integers. If $a_1b_1a_2b_2\ldots c$ is a sequence within a string, then $a_1a_2a_3\ldots$ and $b_1b_2b_3\ldots$ are consecutive integers and c is either a_j or b_j for some $j \geq 3$. Let S and T be strings of integers with r being the last term of S and the first term of T; let T' be the string obtained from T by deleting its first term; then S *join* T, denoted $S \vee T$, is the concatenation of S followed by T'. We adopt the convention that r **T** n **E** m means that r threads n and ends on m. Furthermore, r **T** n and r threads n are used interchangeably.

Lemma 1. *If* r **T** n **E** m *then* $n-r$ **T** n **E** $n-m$.

Proof: Let $S = \{r, s_1, s_2, \ldots, s_{n-1}, m\}$ be a strand for r **T** n **E** m. Then

$$n - S = n - \{r, s_1, \ldots, s_{n-1}, m\} = \{n-r, n-s_1, \ldots, n-s_{n-1}, n-m\}$$

is clearly a strand for $n-r$ **T** n **E** $n-m$. □

Two strands related as in the proof of this lemma are said to be *inverses* of each other. We denote the inverse of the strand S as S^{-1}. Since r **T** n **E** m *iff* m **T** n **E** r, then two strands are said to be *reverses* of each other if one strand is the other written in reverse order. The reverse of the strand S is denoted $\overline{S}$. So the inverse of the above strand for 2 **T** 9 **E** 7 is $\{7,3,6,4,5,0,9,1,8,2\}$, and its reverse is $\{7,1,8,0,9,4,5,3,6,2\}$. The terms *inverse* and *reverse* are analogously used with respect to threads. We say that r **T** n is a *standard* thread if $r \leq \frac{n}{2}$; and a *standard* strand is a strand for a standard thread. Since the inverse of any nonstandard strand is standard, we will couch our strand generating algorithms in terms of standard strands. We say that a strand is *primitive* if it or its inverse starts or ends on 0. The primitive strands are thus the unique strands for 0 **T** $2m$ **E** m and 0 **T** $2m+1$ **E** $m+1$, along with their inverses and reverses.

At this point we ask a natural question, *What are the possible ends of strands?* After generating many possible strands for various values of r and n, one is led to the following conjecture.

Conjecture 2. Knowing the End from the Beginning for r threads n. *The following table indicates the possible ends for all standard strands.*

start at r	end at, with $n = 2k$	end at, with $n = 2k+1$	valid for*
0	k	$k+1$	$k \geq 0$
1	$k-1,\ k+1$	$k,\ k+2$	$k \geq 1$
2	$k-2,\ k,\ k+2$	$k-1,\ k+1,\ k+3$	$k \geq 2$
3	$k-3,\ k-1,\ \ldots,\ k+3$	$k-2, k,\ \ldots,\ k+4$	$k \geq 3$
4	$k-4, k-2,\ \ldots,\ k+4$	$k-3, k-1,\ \ldots,\ k+5$	$k \geq 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

*Except when the end <u>is</u> the beginning.

For example, if this table is to be believed, the strands for 3 **T** 8 must end in 1, 5, or 7, whereas the strands for 3 **T** 9 must end in either 2, 4, 6, or 8. And such strands exist, as indicated below.

$$3\ \mathbf{T}\ 8 \begin{cases} 3-5-4-7-0-8-2-6-1 \\ 3-0-8-1-7-2-6-4-5 \\ 3-6-2-4-5-0-8-1-7 \end{cases}$$

$$3\ \mathbf{T}\ 9 \begin{cases} 3-5-6-1-7-4-8-0-9-2 \\ 3-5-2-6-7-1-8-0-9-4 \\ 3-4-9-0-8-1-7-5-2-6 \\ 3-5-4-7-1-6-2-9-0-8 \end{cases}$$

We content ourselves with proving only the extremes of conjecture 2; and the following strand generating algorithm will be our means to that end. Let $[a,b]$ denote the set of integers from a to b, inclusively. We say that S is a *strand* for $[a,b]$ starting at d and ending at c if $c,d \in [a,b]$ and $S-a$ is a strand for $d-a$ threads $b-a$ and c is the last term of S. The *inverse* of such a strand is naturally taken as $(S-a)^{-1}+a$. With this extended strand definition, the reader may be tempted to pick up an old strand of pearls for $[a,b]$ and commence stringing more pearls onto the strand, adding (next to c) first a pearl with integer label less than a or more than b to ultimately construct a new strand for $[a',b']$ where $a' < a$ and $b < b'$. Using the terminology of the American folk dance whereby dancers are instructed to circle left or right in ever new, but familiar patterns, we so name the following resultant algorithm.

Theorem 3. The Allemande Left & Right Algorithm. *Let S be a strand for the interval $[a, b]$, starting at d and ending at c. Then there is a strand for the interval $[a', b']$ starting at d and ending at c', as given by any of the rules below.*

1. **Go left, finish left.** $\begin{cases} a' = a - b + c - 1, \\ b' = 2b - c, \\ c' = a - 1. \end{cases}$

2. **Go left, finish right.** $\begin{cases} a' = a - b + c - 1, \\ b' = 2b - c + 1, \\ c' = b + 1. \end{cases}$

3. **Go right, finish left.** $\begin{cases} a' = 2a - c - 1, \\ b' = b - a + c + 1, \\ c' = a - 1. \end{cases}$

4. **Go right, finish right.** $\begin{cases} a' = 2a - c, \\ b' = b - a + c + 1, \\ c' = b + 1. \end{cases}$

Proof: Let S be a strand for $[a, b]$ starting at d and ending at c. Consider case 1. Let $T = \{c, a-b+c-1, 2b-c, a-b+c, 2b-c-1, \ldots, a-2, b+1, a-1\}$. Then $S \vee T$ is a strand for $[a', b']$ starting at d and ending at c', where a', b', c' are as given in case 1. See figure 2. That is, we continue "threading" onto S by first going *left* from c, and then hopping back and forth over S until finishing one unit to the *left* of S. For notational purposes, let $\widehat{\mathbf{LL}}(S) = S \vee T$. Then the strands resulting from each of the other cases are defined as follows.

$$\widehat{\mathbf{LR}}(S) = S \vee \{c, a - b + c - 1, 2b - c + 1, a - b + c, 2b - c, \ldots, b + 1\}$$

$$\widehat{\mathbf{RL}}(S) = S \vee \{c, b - a + c + 1, 2a - c - 1, b - a + c, 2a - c, \ldots, a - 1\}$$

$$\widehat{\mathbf{RR}}(S) = S \vee \{c, b - a + c + 1, 2a - c, b - a + c, 2a - c + 1, \ldots, b + 1\} \square$$

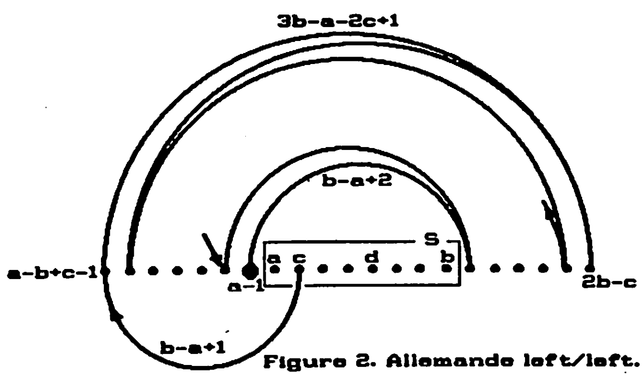

Figure 2. Allemande left/left.

Figure 1 is a good example of this algorithm; take S as $\{2,6,3,5,4\}$, a strand for the interval $[2,6]$, starting at 2 and ending at 4; then $\widehat{\mathbf{RR}}(S)$ is the strand for 2 threads 9 as given in figure 1.

Unfortunately, applying the allemande algorithm to standard strands results in nonstandard strands. So in order to apply the allemande algorithm systematically so as to generate standard strands, some shifting, inverting and reversing is called for. In particular, the following strands defined below are all standard strands, where S is a standard strand for r **T** n.

- $\mathbf{LL}(S) = \overline{\widehat{\mathbf{LL}}(\overline{S}) + n - r + 1}$
- $\mathbf{LR}(S) = \overline{\widehat{\mathbf{LR}}(\overline{S}) + n - r + 1}^{-1}$
- $\mathbf{RL}(S) = \overline{\widehat{\mathbf{RL}}(\overline{S}) + r + 1}$
- $\mathbf{RR}(S) = \overline{\widehat{\mathbf{RR}}(\overline{S}) + r}^{-1}$

To understand why such string manipulation is helpful, consider **LR** as defined above. Since we only know the beginning of S, namely r, rather than the end, and since we wish to apply the allemande algorithm, we take $\overline{S}$, which ends with r. Applying $\widehat{\mathbf{LR}}$ to this strand yields a strand for $[-n+r-1, 2n-r+1]$ starting at some place in S and ending at $n+1$. Reversing this strand yields a strand for the same interval, but which starts at $n+1$. Then shifting by $n-r+1$ to the right yields a strand for $2n-r+2$ **T** $3n-2r+2$. This strand is still nonstandard; but the inverse yields a standard strand for $n-r$ **T** $3n-2r+2$. Similar reasoning motivates each of the other three formulas.

As an example, let's apply **LR** to $\{1,2,0,3\}$, a strand for 1 **T** 3 **E** 3. First reverse: $\{3,0,2,1\}$; then apply $\widehat{\mathbf{LR}}$, concatenating a label 4 units to the left of 1, etc.: $\{3,0,2,1,-3,6,-2,5,-1,4\}$; then add 3: $\{6,3,5,4,0,9,1,8,2,7\}$; then reverse: $\{7,2,8,1,9,0,4,5,3,6\}$; and finally invert: $\{2,7,1,8,0,9,5,4,6,3\}$, which is a strand for 2 **T** 9 **E** 3.

Although all this reversing, shifting, and inverting is somewhat cumbersome, the above terminology makes the proofs of the following two theorems amazingly smooth.

Theorem 4. *For each nonnegative integer m, and for all integers k with $k \geq m$, there are strands for*

$$m \ \mathbf{T} \ 2k \ \mathbf{E} \ k+m,$$
$$\text{and}$$
$$m \ \mathbf{T} \ 2k+1 \ \mathbf{E} \ k+m+1.$$

Proof: It is clear that the theorem is true when $m = 0$. Assume that the theorem is true for all integers m with $0 \leq m \leq q$. We shall show that the theorem is true when $m = q + 1$. By our math induction hypothesis, the following statements are true.

$$\left\{\begin{array}{l} \quad q \;\; \mathbf{T} \;\; 2q+1 \;\; \mathbf{E} \;\; 2q+1 \\ \left\{\begin{array}{cccc} 0 & \mathbf{T} & 2q+2 & \mathbf{E} & q+1 \\ 0 & \mathbf{T} & 2q+3 & \mathbf{E} & q+2 \end{array}\right. \\ \left\{\begin{array}{cccc} 1 & \mathbf{T} & 2q+4 & \mathbf{E} & q+3 \\ 1 & \mathbf{T} & 2q+5 & \mathbf{E} & q+4 \end{array}\right. \\ \qquad\qquad \vdots \\ \left\{\begin{array}{cccc} q & \mathbf{T} & 4q+2 & \mathbf{E} & 3q+1 \\ q & \mathbf{T} & 4q+3 & \mathbf{E} & 3q+2 \end{array}\right. \end{array}\right. .$$

By reversing each of the above statements (except for the initial statement) and then by inverting, this sequence of statements is equivalent to the following sequence of statements, which are indexed in pairs of statements (except for the initial statement).

$$\left\{\begin{array}{ll} (*) & \quad q+1 \;\; \mathbf{T} \;\; 2q+1 \;\; \mathbf{E} \;\; 0 \\ (0) & \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 2q+2 & \mathbf{E} & 2q+2 \\ q+1 & \mathbf{T} & 2q+3 & \mathbf{E} & 2q+3 \end{array}\right. \\ (1) & \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 2q+4 & \mathbf{E} & 2q+3 \\ q+1 & \mathbf{T} & 2q+5 & \mathbf{E} & 2q+4 \end{array}\right. \\ \vdots & \qquad\qquad \vdots \\ (q) & \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 4q+2 & \mathbf{E} & 3q+2 \\ q+1 & \mathbf{T} & 4q+3 & \mathbf{E} & 3q+3 \end{array}\right. \end{array}\right.$$

Applying **RR** to statement (*) and **LL** to the first part of statement (0) yields

$$(q+1) \quad \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 4q+4 & \mathbf{E} & 3q+3 \\ q+1 & \mathbf{T} & 4q+5 & \mathbf{E} & 3q+4 \end{array}\right. .$$

Applying **RL** to the statements of (0) above yields

$$(q+2) \quad \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 4q+6 & \mathbf{E} & 3q+4 \\ q+1 & \mathbf{T} & 4q+7 & \mathbf{E} & 3q+5 \end{array}\right. .$$

Applying **RL** to the statements of (1) above yields

$$(q+3) \quad \left\{\begin{array}{ccccc} q+1 & \mathbf{T} & 4q+7 & \mathbf{E} & 3q+5 \\ q+1 & \mathbf{T} & 4q+8 & \mathbf{E} & 3q+6 \end{array}\right. .$$

In like fashion, continue applying **RL** to the statements in succession, and the resultant cascade of threads demonstrates that the theorem is true when $m = q + 1$. □

Theorem 4 proves conjecture 2 of [4]; in that paper, the above result is essentially and painstakingly proven up to $m = 20$. For convenience we summarize the constructive process as given in the above theorem.

Corollary 5. The Standard Strand Algorithm.

1. Applying **RR** to the primitive strand for r **T** $2r - 1$ **E** 0 results in a strand for r **T** $4r$ **E** $3r$.

2. Applying **LL** to the primitive strand for r **T** $2r$ **E** $2r$ results in a strand for r **T** $4r + 1$ **E** $3r + 1$.

3. Applying **RL** to a standard strand for r **T** $2m$ **E** $m + r$ results in a strand for r **T** $2m + 2r + 2$ **E** $m + 2r + 1$.

4. Applying **RL** to a standard strand for r **T** $2m + 1$ **E** $m + r + 1$ results in a strand for r **T** $2m + 2r + 3$ **E** $m + 2r + 2$.

Example. We illustrate the use of the above algorithm by constructing a strand for 12 **T** 40. We work the algorithm in reverse, and at times, stop to reverse or invert a strand so as to be able to continue working in reverse.

By theorem 4, we know a strand for such ends in 32: 12 **T** 40 **E** 32.

By reversing and inverting, this is equivalent to 8 **T** 40 **E** 28.

By the corollary, such a thread arises from **RL**(8 **T** 22 **E** 19).

By reversing and inverting, 8 **T** 22 **E** 19 is equivalent to 3 **T** 22 **E** 14.

By the corollary, such a thread arises from **RL**(3 **T** 14 **E** 10).

By the corollary, such a thread arises from **RL**(3 **T** 6 **E** 6). (Note that 3 **T** 6 **E** 6 is a primitive strand, (and is equivalent to 0 **T** 6 **E** 3).

Reversing this reasoning, that is, computing

$$\overline{(\mathbf{RL}(\overline{(\mathbf{RL}(\mathbf{RL}(3T6E6)))^{-1}}))^{-1}},$$

yields the strand

$\{\underline{12,28,11,29,10,30,9,31},\underline{16,24,15,25,14,26,13,27},\underline{20,19,21,18,22,17,23},$
$\underline{0,40,1,39,2,38,3,37},\underline{4,36,5,35,6,34,7,33,8,32}\}.$

The underlining in the above strand is to underscore the natural structure of the strand; note that the third underlined string is a copy (shifted to the right by 17) of the strand for 3 **T** 6 **E** 6. As the reader can verify, all edge labels, $[1, 40]$, occur.

Let S be a standard strand for r threads n. We say that S is a *balanced* strand if the integers in the sequence of S alternate in belonging to the two sets $[0, \lfloor \frac{n}{2} \rfloor]$ and $[\lfloor \frac{n}{2} \rfloor + 1, n]$. We say that a nonstandard strand for r threads n is *balanced* if its inverse is balanced. For example, the strand of figure 1 is balanced. As the reader may show, there is no balanced strand for 1 threads 4. In fact,

Conjecture 6. *There are no balanced strands for r* **T** *$4r$ for any positive integer r.*

However, the remaining threads all have balanced strands.

Theorem 7. *For each nonnegative integer m, and for all integers k with $k \geq m$, there are balanced strands for*

$$\begin{array}{lllll} m & \mathbf{T} & 2k & \mathbf{E} & k-m, \qquad \text{(except when } k = 2m), \\ & & \text{and} & & \\ m & \mathbf{T} & 2k+1 & \mathbf{E} & k+m+1. \end{array}$$

Proof: It is clear that the theorem is true when $m = 0$. Assume that the theorem is true for all integers m with $0 \leq m \leq q$. We shall show that the theorem is true when $m = q + 1$. As in theorem 4, by our math induction hypothesis and reversing and inverting there are balanced strands for each of the following threads.

$$\left\{ \begin{array}{ll} (0) & \left\{ \begin{array}{ccccc} q+1 & \mathbf{T} & 2q+2 & \mathbf{E} & 0 \\ q+1 & \mathbf{T} & 2q+3 & \mathbf{E} & 2q+3 \end{array} \right. \\ (1) & \left\{ \begin{array}{ccccc} q+1 & \mathbf{T} & 2q+4 & \mathbf{E} & 1 \\ q+1 & \mathbf{T} & 2q+5 & \mathbf{E} & 2q+4 \end{array} \right. \\ \vdots & \qquad\qquad \vdots \\ (q) & \left\{ \begin{array}{ccccc} q+1 & \mathbf{T} & 4q+2 & \mathbf{E} & q \\ q+1 & \mathbf{T} & 4q+3 & \mathbf{E} & 3q+3 \end{array} \right. \end{array} \right.$$

Applying **RR** to the balanced strand corresponding to the first statement of (0) yields a balanced strand for

$$(q+1) \quad \left\{ \begin{array}{ccccc} & & & & \\ q+1 & \mathbf{T} & 4q+5 & \mathbf{E} & 3q+4 \end{array} \right. .$$

(The first part of the above set of statements is purposely left blank; see conjecture 6.)

Applying **RR** and **RL** to a balanced strand corresponding to the second statement of (0) above yields balanced strands for

$$(q+2) \quad \left\{ \begin{array}{ccccc} q+1 & \mathbf{T} & 4q+6 & \mathbf{E} & q+2 \\ q+1 & \mathbf{T} & 4q+7 & \mathbf{E} & 3q+5 \end{array} \right. .$$

Applying **RR** and **RL** to a balanced strand corresponding to the second statement of (1) above yields balanced strands for

$$(q+3)\quad \begin{cases} q+1 & \mathbf{T} & 4q+7 & \mathbf{E} & q+3 \\ q+1 & \mathbf{T} & 4q+8 & \mathbf{E} & 3q+6 \end{cases} .$$

In like fashion, continue applying **RR** and **RL** to a balanced strand corresponding to the second statement of each set in this sequence, and the resultant cascade of threads demonstrates that the theorem is true when $m = q + 1$. □.

Theorem 7 proves conjecture 10 of [4] (modulo conjecture 6 of this paper); in [4], balanced threads are referred to as *very graceful paths* and the conjecture is proven only up to $m = 3$ for the integer pairs, m **T** $2k + 1$. For convenience we summarize the construction process of the above theorem.

Corollary 8. The Balanced Strand Algorithm.

1. Applying **RR** to the primitive strand for r **T** $2r$ **E** 0 results in a balanced strand for r **T** $4r + 1$ **E** $3r + 1$.
2. Applying **RR** to a balanced standard strand for r **T** $2m+1$ **E** $m+r+1$ results in a balanced strand for r **T** $2m + 2r + 2$ **E** $m + 1$.
3. Applying **RL** to a balanced standard strand for r **T** $2m+1$ **E** $m+r+1$ results in a balanced strand for r **T** $2m + 2r + 3$ **E** $m + 2r + 2$.

Example. We illustrate the use of the above algorithm by constructing a balanced strand for 12 **T** 40. In the example following corollary 5, note that the given strand is not balanced, because of the sequence 27, 20, 19 within the strand.

By theorem 7, we know a balanced strand for 12 **T** 40 can end on 8: 12 **T** 40 **E** 8.

Reversing yields 8 **T** 40 **E** 12.

A balanced strand for such arises by applying **RR** to a balanced thread for 8 **T** 23 **E** 20.

Reversing and inverting yields 3 **T** 23 **E** 15.

A balanced strand for such arises by applying **RL** to a balanced strand for 3 **T** 15 **E** 11,

which arises by applying **RL** to the primitive strand for 3 **T** 7 **E** 7.

Reversing the above reasoning, that is, computing

$$\overline{\mathbf{RR}((\overline{\mathbf{RL}(\mathbf{RL}(3\mathrm{T}7\mathrm{E}7))})^{-1})},$$

yields the desired balanced strand:

$$\{\underline{12,29,11,30,10,31,9,32},\underline{16,25,15,26,14,27,13,28},\underline{20,21,19,22,18,23,17,24},$$
$$\underline{0,40,1,39,2,38,3,37,4,36,5,35,6,34,7,33,8}\}.$$

Spiders.

A graph with p vertices and q edges is said to be *edge-graceful* if the edges can be labeled 1 through q so that the vertices are labeled 0 through $p-1$, where the label of a vertex is the sum modulo p of the labels of all edges incident to that vertex. The conjecture on which we dwell is

Conjecture 9. Lee's Conjecture [2]. *All trees of odd order are edge-graceful.*

As shown in [3], a tree of order $2n+1$ is edge-graceful, if its edges can be labeled using all of the integers from ± 1 to $\pm n$, so that its vertices are labeled using all the integers from 0 to $\pm n$, where the label of a vertex is simply the sum of the labels of all edges incident to that vertex. It is this condition which shall be used herein to establish the edge-gracefulness of various trees. A *spider* is a tree with at most one vertex of degree more than two (called the *core*). The path from the core to any extremal vertex is called a *leg* of the spider. The *length* of a leg is the number of edges in the leg. As shown in [3], there are several edge-graceful invariant operations with respect to trees; applying these operations to edge-graceful spiders leads to establishing the edge-gracefulness of a more general class of trees. So in some respects, Lee's conjecture would be well-advanced if we could prove that all spiders of odd order are edge-graceful. As shown in [3] and [5], any odd ordered spider whose legs all have the same length is edge-graceful. However, what about asymmetric spiders, *i.e.*, spiders whose leg lengths differ? As a contribution toward establishing the edge-gracefulness of these spiders, we offer the following reassuring theorem. We intersperse its proof with examples. But first we state a lemma needing no proof.

Lemma 10. Let S be a sequence of integer labels along the edges of a path such that the terms of the sequence alternate in belonging to two sets, A and B. Let L be the set of interior vertex labels, whereby a vertex label is calculated by adding the labels of the two edges incident to that vertex. Let q be any integer. Finally, let S' be a new sequence obtained from S by replacing each label b from B with $b+q$. *Then the set of interior vertex labels of S' is simply $L+q$.*

Theorem 11. *All three legged spiders of odd order are edge-graceful.*

Proof: Let us denote the edge labelings on the three legs as the sequences E_1, E_2, and E_3, where the initial integer in a sequence is the label of the outer edge of the spider's leg and the final integer is the label of the edge incident with the core of the spider. Similarly, let the vertex labelings on the three legs, from the outer vertex to the core vertex be the sequences V_1, V_2, and V_3. Furthermore, let us agree that $|E_1| \leq |E_2| \leq |E_3|$. Since the number of edges must be even, we have five cases to consider. Since these are all similar, let us examine case 1 in the most detail.

Case 1. Suppose that the lengths of the three legs are $2k+1$, $2m$, and $2n+1$, where

$$2k+1 < 2m < 2n+1.$$

We shall label the edges of this spider with all of the integers ± 1 to $\pm(k+m+n+1)$ so that its vertices are labeled using all of the integers from 0 to $\pm(k+m+n+1)$. To do this, let

$$E_1 = -\{k+m+n+1, -1, k+m+n, -2, \ldots, -k, m+n+1\},$$

$$E_3 = \{k+m+n+1, -1, k+m+n, -2, \ldots, -n, k+m+1\}.$$

The remaining $2m$ unused edge labels are

$$A = [k+1, k+m] \text{ and } B = -[n+1, m+n].$$

We shall start E_2 with $-(m+n-k)$ and end it with m so that the core vertex is labeled $k+m-n$. Hence the vertices are labeled

$$V_1 = \{-(k+m+n+1), -(k+m+n), \ldots, -(m+n-k+1), k+m-n\},$$

$$V_3 = \{k+m+n+1, k+m+n, \ldots, k+m-n+1, k+m-n\}.$$

The unused vertex labels are thus $[-(m+n-k), k+m-n]$, not counting the core vertex. Let $A' = A-(k+1)$ and $B' = B-(m-n-1)$, so that $A' = [0, m-1]$ and $-B' = [m, 2m-1]$. Let S be a balanced strand for $2m-k-1$ threads $2m-1$ ending at $m-k-1$. (Note that S is the reverse of a standard strand.) Let S' be a strand of integers obtained from S by replacing each label from $-B'$ with its negative. If we think of S' as being the edge labels along P_{2m}, then the interior vertex labels of this path are precisely $-[1, 2m-1]$. Now let E_2 be that sequence of integers obtained from S' by replacing each integer label from A' and B' with its corresponding label from A and B, respectively; that is, if a' is a label from A', replace it with $a = a' + (k+1)$, and if b' is a label from B', replace it with $b = b' + (m-n-1)$. Then the edge labels of E_2 are precisely $A \cup B$, as desired. By Lemma 10, the interior vertices of V_2 are the integers

$[-(m+n-k-1), k+m-n-1]$. Note also that the outer edge of E_2 is labeled $-(m+n-k)$ and the edge incident with the core is labeled m, which means that the outer vertex of the second leg and the core vertex are labeled $-(m+n-k)$ and $k+m-n$, respectively, so that V_2's integer sequence uses every vertex label in $[-(m+n-k), k+m-n]$, as desired. We reduce the other cases to the problem of finding the analogous balanced strands S.

Example for Case 1. Consider a spider with legs of lengths 7, 10, and 17. Thus $k=3$, $m=5$, and $n=8$, which means that

$$E_1 = \{-17, 1, -16, 2, -15, 3, -14\},$$

and

$$E_3 = \{17, -1, 16, -2, 15, -3, 14, -4, 13, -5, 12, -6, 11, -7, 10, -8, 9\}.$$

Since the core vertex is $k+m-n=0$, then

$$V_1 = \{-17, -16, -15, -14, -13, -12, -11, 0\}.$$

and

$$V_3 = \{17, 16, 15, 14, \ldots, 2, 1, 0\}.$$

Now E_2 must start with -10 and end with 5; and the unused edge labels are

$$U = \{4, 5, 6, 7, 8, -9, -10, -11, -12, -13\}.$$

Placing these labels on the right edges of E_2 corresponds to finding a balanced strand for 6 **T** 9 **E** 1. To find a balanced strand for 1 **T** 9 **E** 6, start with 1 **T** 2 **E** 0, and apply **RR**, (yielding $\{1,5,0,3,2,4\}$), and then apply **RL**, getting

$$\{1, 8, 0, 9, 3, 7, 2, 5, 4, 6\}.$$

Reverse this strand, obtaining

$$S = \{6, 4, 5, 2, 7, 3, 9, 0, 8, 1\},$$

and then write it in terms of the corresponding unused edge labels. That is, identify $\{0, 1, \ldots, 9\}$ with U; then let E_2 be that permutation of U corresponding to S, which gives

$$E_2 = \{-10, 8, -9, 6, -11, 7, -13, 4, -12, 5\}.$$

Hence $V_2 = \{-10, -2, -1, -3, -5, -4, -6, -9, -8, -7, 0\}$, as desired. So this spider is edge-graceful.

Case 2. Suppose that the lengths of the three legs are $2k < 2m < 2n$, respectively. Let

$$E_1 = -\{k+m+n, -1, k+m+n-1, -2, \ldots, m+n+1, -k\}$$

and

$$E_3 = \{k+m+n, -1, k+m+n-1, -2, \ldots, k+m+1, -n\}.$$

Then V_1 and V_2 are the same as in case 1, excluding the first terms. Furthermore, A and B are exactly the same as in case 1. Start and end E_2 on $-(m+n-k)$ and m, respectively. Hence the balanced thread which generates E_2 is exactly the same as in case 1.

Example for Case 2. Consider a spider with legs of length 6, 8, and 12, so that $k = 3$, $m = 4$, and $n = 6$, making

$$E_1 = \{-13, 1, -12, 2, -11, 3\}$$

and

$$E_3 = \{13, -1, 12, -2, 11, -3, 10, -4, 9, -5, 8, -6\}.$$

Since the core has label $k + m - n = 1$, then

$$V_1 = \{-13, -12, -11, -10, -9, -8, 1\}$$

and

$$V_3 = \{13, 12, 11, \ldots, 3, 2, 1\}.$$

Since E_2 must start and end with $-(m+n-k) = -7$ and $m = 4$, respectively, and since the unused edge labels are

$$\{4, 5, 6, 7, -7, -8, -9, -10\},$$

then S can be taken as a balanced thread for 4 **T** 8 **E** 0, which is

$$\{4, 5, 3, 6, 2, 7, 1, 8, 0\}.$$

Write this in terms of the unused edge labels:

$$E_2 = \{-7, 7, -8, 6, -9, 5, -10, 4\},$$

and it follows that this spider is edge-graceful.

Case 3. Suppose that the lengths of the three legs are $2k < 2m + 1 < 2n + 1$. Let

$$E_1 = -\{k+m+n+1, -1, k+m+n, -2, \ldots, m+n+2, -k\}$$

and

$$E_3 = \{k+m+n+1, -1, k+m+n, -2, \ldots, -n, k+m+1\}.$$

This time V_1 is the same as in case 1, excluding the second to the last term; V_3 is the same; and the unused edge labels are

$$A = [k+1, k+m] \text{ and } B = -[n+1, m+n+1].$$

In order for the core vertex to have label $k+m-n$, E_2 must end with $-(k+n+1)$. Furthermore, $A' = A - (k+1)$ and $B' = B - (m-n-1)$. So we want S to be the reverse of the balanced strand for $k+m$ **T** $2m$. But this is equivalent to finding its inverse: $m-k$ **T** $2m$. Since such a strand exists and ends in k by theorem 7, then take S as the balanced strand for $2m-k$ **T** $2m$ **E** $k+m$.

Example for Case 3. Consider a spider with legs of lengths 2, 7, and 13, so that $k=1$, $m=3$, and $n=6$, making $E_1 = \{-11, 1\}$, and

$$E_3 = \{11, -1, 10, -2, \ldots, 6, -6, 5\}.$$

So $V_1 = \{-11, -10, -2\}$ and

$$V_3 = \{11, 10, 9, \ldots, 1, 0, -1, -2\}.$$

Since E_2 must end with $-(k+n+1) = -8$ and since the unused edge labels are $\{2,3,4,-7,-8,-9,-10\}$, S is the reverse of a balanced strand for 4 **T** 6. To find such a strand, first find 2 **T** 6 **E** 1: start with 1 **T** 2 **E** 0, apply **RL** to obtain

$$\{1,5,0,6,3,4,2\}.$$

Invert: $\{5,1,6,0,3,2,4\}$; and write in terms of the unused edge labels: $E_2 = \{-9,3,-10,2,-7,4,-8\}$, and it follows that this spider is edge-graceful.

Case 4. Suppose that the lengths of the three legs are $2k+1 < 2m+1 < 2n$. Let

$$E_1 = -\{k+m+n+1, -1, k+m+n, -2, \ldots, -k, m+n+1\}$$

and

$$E_2 = \{k+m+n+1, -1, k+m+n, -2, \ldots, -m, k+n+1\}.$$

This time we shall take the core vertex to have label $k+n-m$ so that

$$V_1 = \{-(k+m+n+1), -(k+m+n), \ldots, -(m+n-k+1), k+n-m\}$$

and

$$V_2 = \{k+m+n+1, k+m+n, \ldots, k+n-m\}.$$

The unused edge labels are

$$A = [k+1, k+n] \text{ and } B = -[m+1, m+n].$$

In order for the core vertex to have label $k+n-m$, E_3 must end with n. So S must be the reverse of a balanced strand for $n-k-1$ **T** $2n-1$ **E** $2n-k-1$.

Example for Case 4. Suppose that the lengths of the three legs are 5, 9, and 14, so that $k=2$, $m=4$, and $n=7$. Let $E_1 = \{-14, 1, -13, 2, -12\}$ and $E_2 = \{14, -1, 13, -2, \ldots, -4, 10\}$. Then $V_1 = \{-14, -13, \ldots, -10, 5\}$ and $V_2 = \{14, 13, 12, \ldots, 5\}$. Since E_3 must end with $n = 7$, and since the unused edge labels are $[3, 9] \cup -[5, 11]$, S is the reverse of a balanced strand for 4 **T** 13 **E** 11. To find such a strand, we shall first find 2 **T** 13 **E** 9: start with a balanced strand for 1 **T** 3 **E** 3, namely $\{1, 2, 0, 3\}$, apply **RL** to get 1 **T** 7 **E** 5; reverse and invert to obtain 2 **T** 7 **E** 6, then apply **RL** to obtain $\{2, 11, 1, 12, 0, 13, 5, 8, 6, 7, 3, 10, 4, 9\}$; invert: $S = \{11, 2, 12, 1, 13, 0, 8, 5, 7, 6, 10, 3, 9, 4\}$; now write in terms of the unused edge labels:

$$E_3 = \{-9, 5, -10, 4, -11, 3, -6, 8, -5, 9, -8, 6, -7, 7\},$$

and it follows that this spider is edge-graceful.

Case 5. Suppose that two of the three legs have the same length. Let us consider a more general problem. Consider a spider with four legs, with two legs of length m and two legs of length n, where $m \le n$. Consider the case when m is even. Let

$$E_1 = \{m+n, -1, m+n-1, -2, \ldots, n+\frac{m}{2}+1, -\frac{m}{2}\}.$$

Let $E_2 = -E_1$. If the core is to be labeled 0, then $V_1 = \{m+n, m+n-1, \ldots, n+1, 0\}$, and $V_2 = -V_1$. The unused edge label are $\pm[\frac{m}{2}+1, n+\frac{m}{2}]$. Let S be a strand, (not necessarily balanced), for $n - \frac{m}{2} - 1$ **T** n. Let $S' = S + \frac{m}{2} + 1$. Finally let E_3 be that string obtained from S' by multiplying every other term by -1. Let $E_4 = -E_3$. It follows that this four legged spider is edge-graceful. Now imagine that legs three and four together form a violin bow and legs one and two together form a violin string. Draw the bow across the midvertex of the string, stopping whenever a vertex from the bow coincides with the midvertex of the string—the resultant four legged spider is edge graceful. (See the *Cut and Paste Algorithm* of [3] for a formalization of this operation.) Keep drawing the bow across the string, until the four legged spider becomes a three legged spider. This spider is edge-graceful! The same conclusion follows if the bow is formed by legs one and two together and the string is formed by legs three and four together. Furthermore, the same argument applies if m is odd.

Example for Case 5. Consider a spider with leg lengths of 8, 8, and 10. Let us first consider a spider with leg lengths of 5, 5, 8, and 8. Let $E_1 = \{13, -1, 12, -2, 11\}$ and $E_2 = -E_1$. Since the core is to be labeled 0, then $V_1 = \{13, 12, 11, 10, 9, 0\}$ and $V_2 = -V_1$. The unused edge labels are $\pm[3, 10]$ and the unused vertex labels are $\pm[1, 8]$. Let S be a strand for $[3, 10]$ starting at 8; to find this strand we shall find 5 threads 7, or 2 **T** 7 **E** 6. The inverse of its reverse is 1 **T** 7 **E** 5, which is $\{1, 6, 0, 7, 3, 4, 2, 5\}$, (see the previous example), and its reverse is 5 threads 7. Add 3 to this strand to yield S; alternate signs to yield $E_3 = \{8, -5, 7, -6, 10, -3, 9, -4\}$. So $V_3 = \{8, 3, 2, 1, 4, 7, 6, 5\}$. Let $E_1' = E_3$, $E_2' = -E_3$, and E_3' be the concatenation of E_1 and the reverse of E_2. Hence this three legged spider with legs labeled E_1', E_2', and E_3' is edge-graceful. □

Corollary 12. *Any four legged spider of odd order with two legs of equal length is edge-graceful.*

Proof: See Case 5 of the above theorem. □

The main result of [4] is a rendition of the above corollary, but only for spiders whose two short legs of equal length have length no more than 41. Furthermore, useage of balanced strands enables proposition 19 of [4] to be broadened, giving a condition for many multi-legged spiders to be edge-graceful. But a general procedure to edge-gracefully label all assymetric odd ordered spiders remains an open question, even for four legs.

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