Classification of 2-quasi-invariant subsets

Leonid Brailovsky   Dmitrii V. Pasechnik
Cheryl E. Praeger
Department of Mathematics
University of Western Australia
Nedlands, Perth, WA 6009, Australia

ABSTRACT. Let G be a group acting on a set Ω. A subset (finite or infinite) $A \subseteq \Omega$ is called *k-quasi-invariant*, where k is a non-negative integer, if $|A^g \setminus A| \leq k$ for every $g \in G$. In previous work of the authors a bound was obtained, in terms of k, on the size of the symmetric difference between a k-quasi-invariant subset and the G-invariant subset of Ω closest to it. However, apart from the cases $k = 0, 1$, this bound gave little information about the structure of a k-quasi-invariant subset. In this paper a classification of 2-quasi-invariant subsets is given. Besides the generic examples (subsets of Ω which have a symmetric difference of size at most 2 with some G-invariant subset) there are basically five explicitly determined possibilities.

1 Introduction and results

For a group G acting on a set Ω the G-invariant subsets of Ω are the unions of G-orbits in Ω. In [1] the notion of a quasi-invariant subset was introduced, in order to solve a group theoretic question in [2]: $A \subseteq \Omega$ was defined to be *quasi-invariant* if its image under every element of G differed from A by at most one point, that is if $|A^g \setminus A| \leq 1$ for each $g \in G$. It was shown in [1] that quasi-invariant subsets of Ω are precisely the invariant subsets, and invariant subsets with one point added or removed. This concept prompted a natural generalization, also given in [1]: for a positive integer k, a subset $A \subseteq \Omega$ is said to be *k-quasi-invariant* if

$$|A^g \setminus A| \leq k \text{ for every } g \in G. \qquad (1)$$

Clearly, a subset of Ω which has symmetric difference of size at most k with some invariant set is a k-quasi-invariant subset. We call such subsets *generic*. Thus, in view of [1], all 0-quasi-invariant subsets and 1-quasi-invariant subsets are generic. It was asked in [1] whether all k-quasi-invariant subset are generic for every $k > 1$.

In [3] examples were given of several families of non-generic k-quasi-invariant subsets and a bound was obtained on the size of the symmetric difference between a k-quasi-invariant subset and the G-invariant subset of Ω closest to it. More precisely, for a k-quasi-invariant subset $A \subseteq \Omega$ we considered $d(A) = \min |A \Delta N|$, where N runs over all G-invariant subsets of Ω. It was proved that $d(A)$ is bounded by a subquadratic function of k for both finite and infinite k-quasi-invariant subsets A. However, the arithmetical results of [3] do not give enough structural information about k-quasi-invariant subsets to allow a characterization of these sets for any $k > 1$. In this paper we give a complete classification of 2-quasi-invariant subsets as a further step towards understanding the structure of a k-quasi-invariant subset for general k.

The classification of 2-quasi-invariant subsets A is achieved by first showing that there exists a G-invariant subset $N' \subseteq \Omega$ such that $|A \Delta N'| \leq 3$, and then, in the case where $|A \Delta N'| = 3$, describing the set $A' = A \Delta N'$. Note that A' is a 2-quasi-invariant subset as well, and that $d(A') = |A'|$. It can be obtained from A by the following "surgery" on A:

> - *for every G-orbit Ω_i, such that $|A \cap \Omega_i| > |\Omega_i \setminus A|$, replace $A \cap \Omega_i$ by $\Omega_i \setminus A$. In particular, remove from A every G-orbit Ω_i which is contained in A.*

This surgery does not change the quasi-invariance of a subset and our classification will be done modulo it. The basic problem is that of classifying the 2-quasi-invariant subsets A of Ω such that $|A| = d(A) = 3$. This is done by examining the possible configurations formed by the set of images of A under the action of G, that is

$$\mathcal{F} = \mathcal{F}(A) =_{def} \{A^g | g \in G\}.$$

The key step in our analysis is to obtain an upper bound on $|\Omega|$. The results of [3] imply that A intersects nontrivially at most 3 G-orbits Ω_i, and that if $|A \cap \Omega_i| \geq 3$ then $|\Omega_i| < 8$. However if A contains only 1 or 2 points of Ω_i then the results in [3] give no bound on $|\Omega_i|$; indeed they do not even guarantee finiteness. Our main theorem which gives the classification of 2-quasi-invariant subsets is the following.

Theorem 1.1 *Let G be a group acting on a set Ω and let A be a 2-quasi-invariant subset of Ω. Then either*

1. *A is generic, that is the symmetric difference between A and some G-invariant subset is at most 2, or*

1	2	3
1	6	4
1	6	5
2	4	5
3	2	6
3	4	5

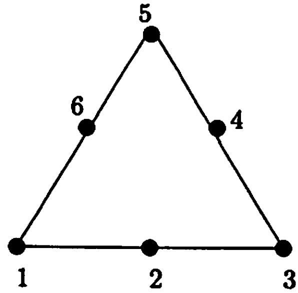

Figure 1: An example of 2-quasi-invariant family.
Left: the list of elements of $\mathcal{F}$.
Right: elements are depicted by straight lines and smooth curves.

2. *the set A' obtained by the surgery above has size* 3 *and, if Ω' is the smallest G-invariant subset containing A', then the pair $\mathcal{F}(A')$, Ω' is one of those in List* 1.2.

List 1.2 *Examples of families $\mathcal{F}(A)$ for* 2*-quasi-invariant subsets A of size 3 of a set Ω:*

1. *four lines skew to a given point of the projective plane Ω of order* 2;
2. *the six triples shown in Figure* 1 *with $\Omega = \{1, \dots, 6\}$;*
3. *ten triples forming an orbit of the natural action of $PSL_2(5)$ on the* 3*-subsets of the projective line Ω of order* 5;
4. *the seven lines of the projective plane Ω of order* 2;
5. *six of the lines of the projective plane Ω of order* 2.

Note that in all but the last case the setwise stabilizer $G = G\{\mathcal{F}\}$ of $\mathcal{F}$ in $Sym(\Omega)$ is transitive on Ω, while in case 5 the setwise stabilizer of $\mathcal{F}$ has orbits of sizes 3 and 4 in Ω. Let H be a subgroup of G acting transitively on $\mathcal{F}$. Then in all but the first case the number of H-orbits on Ω coincides with the corresponding number of G-orbits. In the first case there exist H having 2 or 3 orbits on Ω. In fact, 3 is the maximal possible number of orbits on Ω [3, Theorem B]. The first case of the list is a member of the infinite family of examples of k-quasi-invariant subsets presented in [3] to demonstrate the sharpness of the bound $2k-1$ on the number of the orbits.

In view of our discussion about 2-quasi-invariant subsets, Theorem 1.1 will follow automatically from the following.

Theorem 1.3 *Let G be a group acting on Ω and let A be a 2-quasi-invariant subset of Ω with $|A| = d(A) \geq 3$. Then $|A| = d(A) = 3$ and $\mathcal{F} = \mathcal{F}(A)$ is one of the 5 families in List 1.2.*

The rest of the paper is devoted to the proof of Theorem 1.3. The proof will be carried out separately for the cases of transitive and intransitive action of G on Ω.

It should be noted that, though the present proof is computer-free, the initial one used the CAYLEY system [4, 5] for algebraic computations to determine the possibilities for $\mathcal{F}(A)$ in the case $|A| = 3$, $|\Omega| = 6$, and for some preliminary experimentation.

2 Transitive case

We consider first the case when the group G has only one orbit on Ω. Then we use the resulting classification for the general case. We will show that all but example 5 of List 1.2 appear in this case, and that there are no others. We remind the reader that a family $\mathcal{F}$ of subsets of a set is called *self-intersecting* (respectively, *i-intersecting* if each pair $F, F' \in \mathcal{F}$ satisfies $F \cap F' \neq \emptyset$ (respectively, $|F \cap F'| \geq i$), cf. [6, 7].

The following result is a slight reformulation of a statement from [3].

Result 2.1 *[3, Corollary 2.3] Let A be a k-quasi-invariant subset of a G-orbit Ω such that $|\Omega \setminus A| \geq |A|$. Then $|A| \leq 2k - 1$.* □

It immediately implies the following.

Lemma 2.2 *Let A be a 2-quasi-invariant subset of a G-orbit Ω satisfying the condition of Theorem 1.3. Then $|A| = 3$.* □

Our next task is to find the possible values for $|\Omega|$. We use the following result, which is a slight modification of [3, Theorem 2.1].

Proposition 2.3 *Let A be a finite k-quasi-invariant subset of cardinality $n > k$ of an orbit Ω of G, and let $\mathcal{F} = \mathcal{F}(A)$. Then*

$$|\Omega| \leq \frac{n^2}{n - k(1 - 1/|\mathcal{F}|)}. \tag{2}$$

Proof. By [3, Theorem 2.1], Ω is finite. Hence, we may assume that G is finite, by considering the constituent of G acting on Ω instead of G if necessary. Fix some point $\omega \in \Omega$ and let $H = G_\omega$ denote the stabilizer of ω in G. Since the action of G is transitive, we may identify Ω with the set of left cosets of H in G in such a way that the image of gH under $x \in G$ is $x^{-1}gH$. Then $A = \{g_1H, \dots, g_nH\}$, for some $g_1, \dots, g_n \in G$. By (1), for every $g \in G$ there exist at least $n-k$ ordered pairs (g_i, g_j) such that

$$g \in g_iHg_j^{-1}.$$

Also, for the elements of the setwise stabilizer X of A in G, the number of those pairs equals n. Hence we obtain, by counting the triples (g, g_i, g_j) with $g_iH = gg_jH$,

$$|G \setminus X|(n-k) + |X|n \le n^2|H|,$$

whence

$$\frac{|G|}{|H|}(n-k) + k\frac{|X||G|}{|H||G|} \le n^2.$$

Keeping in mind that $\frac{|G|}{|H|} = |\Omega|$, and $\frac{|G|}{|X|} = |\mathcal{F}|$, we derive

$$|\Omega|(n-k) + k|\Omega|/|\mathcal{F}| \le n^2,$$

and (2) follows. □

Since Ω is finite and $\mathcal{F}$ is a 1-intersecting family, we may apply the following well-known result of Hilton and Milner.

Result 2.4 [8] *Let $\mathcal{F}$ be a nontrivial 1-intersecting family of subsets of size n of a finite set Ω of size greater than $2n$. Then*

$$|\mathcal{F}| \le \binom{|\Omega|-1}{n-1} - \binom{|\Omega|-n-1}{n-1} + 1. \tag{3}$$

Moreover, the equality is possible only if, for some $X_1 \subset \Omega$, $|X_1| = n$, and $x \in \Omega \setminus X_1$,

$$\mathcal{F} = \mathcal{F}_1 = X_1 \cup \{X \subset \Omega \mid x \in X,\ |X| = n,\ X \cap X_1 \ne \emptyset\},$$

or $n \le 3$, and for some $X_1 \subset \Omega$, $|X_1| = 3$,

$$\mathcal{F} = \mathcal{F}_2 = \{X \subset \Omega \mid |X| = n,\ |X \cap X_1| \ge 2\}.$$

□

Using (3) for $n = 3$, which is the case we are considering in Theorem 1.3, we obtain $|\mathcal{F}| \le 3|\Omega| - 8$. It is easy to check that the extremal families $\mathcal{F}_1$ and $\mathcal{F}_2$ cannot admit a group transitive on Ω. Hence we have

$$|\mathcal{F}| \le 3|\Omega| - 9. \tag{4}$$

Using (2), we obtain

$$3|\Omega|^2 - 34|\Omega| + 81 \le 0.$$

Solving the latter inequality, we have the following.

Corollary 2.5 *Let A be a 2-quasi-invariant subset of size 3 of an orbit Ω of G. Then $|\Omega| \leq 7$.* □

Since, by our assumptions, $|A| \leq |\Omega \setminus A|$, we have $|\Omega| \geq 6$. Hence we have only two possibilities for $|\Omega|$.

Case $|\Omega| = 7$. By transitivity of G, in this case $K \cong Z_7 \leq G$ acts on Ω. Without loss of generality, $K = \langle g \rangle$, where $g = (1, 2, \ldots, 7)$, and $\{1, 2\} \subset A$. It is straightforward to check that all the ways of adjoining to $\{1, 2\}$ the third element in order to get a suitable 3-element subset A of Ω lead either to a contradiction or to the configuration of images isomorphic to $PG(2, 2)$. For instance, if we take $A = \{1, 2, 3\}$, then $A^{g^3} = \{4, 5, 6\}$, and $A \cap A^{g^3} = \emptyset$, a contradiction. By (4), $|\mathcal{F}| \leq 12$. So K cannot have more than one orbit on $\mathcal{F}$, since all such orbits are of size 7. Therefore $|\mathcal{F}| = 7$ and the whole group $G \leq Aut(PG(2, 2))$. Hence we have Case 4 of the List 1.2.

Case $|\Omega| = 6$. First, the following classical result gives $|\mathcal{F}| \leq 10$. (Result 2.4 is not applicable here, since it needs $|\Omega| > 2n$.)

Result 2.6 *[6, Erdös-Ko-Rado Theorem] Let $\mathcal{F}$ be a self-intersecting family of subsets of size n of a set Ω of size at least $2n$. Then $|\mathcal{F}| \leq \binom{|\Omega|-1}{n-1}$.*
□

In our case $\mathcal{F} = \{A^g \mid g \in G\}$. By Lemma 2.2 we have $|A| = 3$, and we call the elements of $\mathcal{F}$ *triples*. We will directly reconstruct all the configurations satisfying the following necessary combinatorial conditions, which come from transitivity of G on Ω and $\mathcal{F}$.

1. The number $s(\omega)$ of triples through a point $\omega \in \Omega$ is a constant $s = s(\omega) = |\mathcal{F}|/2$ not depending on the choice of ω, and

2. the number of triples intersecting a given triple in two points is a constant k_2.

Without loss of generality, $\Omega = \{1, \ldots, 6\}$, $A = \{1, 2, 3\}$ (or, for brevity, $A = 123$).

Note that condition 1 implies that $|\mathcal{F}|$ is even.

Assume $k_2 = 0$. Clearly, $\mathcal{F}$ contains a triangle $\mathcal{T}$, that is, three triples having empty common intersection. Without loss of generality, we may renumber the elements of Ω in such a way that $\mathcal{T} = \{123, 145, 246\} \subseteq \mathcal{F}$. Since the number of triples in $\mathcal{T}$ through 1 is bigger than that through 3, there exists $X \in \mathcal{F} \setminus \mathcal{T}$ such that $3 \in X$. Since $k_2 = 0$, $\{1, 2, 3\} \cap X = \{3\}$ so $1, 2 \notin X$; also if $4 \in X$ then $X \cap \{1, 4, 5\} = X \cap \{2, 4, 6\} = \{4\}$, whence $5, 6 \notin X$ implying that $|X| \neq 3$. Hence $4 \notin X$ so that $X =$

356. Since there was only one possibility for X, $s = s(3) = 2$, and the resulting $\mathcal{F} = \mathcal{T} \cup \{X\}$ is as in Case 1 of the List 1.2. The group $G = \langle (2,3)(4,5), (1,5)(2,6), (1,3)(4,6) \rangle$ acts transitively on both Ω and $\mathcal{F}$.

Assume $k_2 = 1$. Without loss of generality, $A = 123$ and $124 \in \mathcal{F}$. By (1), neither 456 nor 356 can belong to $\mathcal{F}$. Then, using our assumption on k_2, we see that only $156, 256, 345, 346$ remain as candidates for membership of $\mathcal{F}$. Adding all of them, we have Case 2 of the List 1.2, and the group $\langle (1,2), (1,3,6)(2,4,5) \rangle$ acts transitively on Ω and $\mathcal{F}$. On the other hand, if we wish to add to $\mathcal{F}$ only some of them, we have to add two, since $|\mathcal{F}|$ is even. But there is no way to do this, with $s(i)$ remaining a constant for $i = 1, \ldots, 6$.

In the remainder of the section, $k_2 \geq 2$. The next two lemmas immediately imply $k_2 \leq 3$. Note that $s = |\mathcal{F}|/2 \leq 5$.

Lemma 2.7 *Let A, B and C be three distinct triples in $\mathcal{F}$. Then at least one of the sets $A \cap B$, $A \cap C$ and $B \cap C$ has size 1. It follows that $|A \cap B \cap C| \leq 1$.*

Proof. Note that, by (1), all the sets $A \cap B$, $A \cap C$, $B \cap C$ are not empty. We may assume that at least one pair of the triples intersects in a set of size 2, say $|A \cap B| = 2$. Suppose first that $A \cap B \subset C$. Without loss of generality, $A = 123$, $B = 124$, $C = 125$. Thus, by transitivity of G on Ω, for each point $a \in \Omega$ there exists a non-empty subset $X_a \subseteq \Omega$ such that for any $b \in X_a$ there exist at least three triples containing $\{a, b\}$. Since $s \leq 5$, X_6 does not contain 1 or 2. Thus we may assume that $3 \in X_6$. Since by (1) each of the three triples containing $\{3, 6\}$ must intersect B and C, each must contain either 1 or 2. Hence there are only two possible triples containing 3 and 6, contradicting the fact that $3 \in X_6$. It follows that $A \cap B \not\subset C$, whence $|A \cap B \cap C| \leq 1$.

To complete the proof, suppose, contrary to the statement of the lemma, that $|A \cap B| = |A \cap C| = |B \cap C| = 2$. We may assume without loss of generality that $A = 123$, $B = 124$, $C = 234$. By (1), $456, 356, 156 \notin \mathcal{F}$. By the already proved part of the lemma, 125, 126, 235, 236, 245, 246 $\notin \mathcal{F}$, as well. The remaining triples are 134, 135, 136, 145, 146, 256, 345, 346. If $256 \in \mathcal{F}$ then $s = s(2) = 4$, for there are no other triples which remain as candidates for membership of $\mathcal{F}$ and contain 2. So, to obey $s(5) = s(6) = 4$, all the remaining triples through 5 and/or 6 belong to $\mathcal{F}$, forcing $|\mathcal{F}| = 10$, but then $s = |\mathcal{F}|/2 = 5 \neq s(2)$. Thus $256 \notin \mathcal{F}$, and so $s = s(2) = 3$. By the same argument, to obey $s(5) = s(6) = 3$, all the remaining triples apart from 256 which contain 5 or 6 belong to $\mathcal{F}$, contradicting $|\mathcal{F}| = 2s = 6$. □

Lemma 2.8 *Let $B, C \in \mathcal{F}$ such that $|A \cap B| = |A \cap C| = 2$. Then $(A \cap B \cap C) \cup (B \setminus A) \cup (C \setminus A) \notin \mathcal{F}$.*

Proof. By Lemma 2.7, we may assume that $A = 123$, $B = 124$, $C = 235$. So the triple under question is 245. Assume to the contrary that $245 \in \mathcal{F}$. Then $s = s(2) \geq 4$.

Consider triples containing 6. By Lemma 2.7, the triples 126, 236, 246, 256 do not belong to $\mathcal{F}$. As well, by (1) the triples 136, 456, 146, 356 do not belong to $\mathcal{F}$. Only two triples containing 6, namely 346 and 156, remain as candidates for membership of $\mathcal{F}$, contradicting $s \geq 4$. □

Now we are able to reach our goal and complete the consideration of the case $|\Omega| = 6$. Assume $k_2 = 2$ first. Considering Lemma 2.7, without loss of generality, $123, 124, 235 \in \mathcal{F}$. Hence $s = s(2) \geq 3$. Consider triples containing 6. By (1), one has $456, 356, 146 \notin \mathcal{F}$. Since $k_2 = 2$, one has $126, 136, 236 \notin \mathcal{F}$, for the only triples intersecting 123 in two points are 124 and 235. The remaining triples through 6 are 156, 246, 256, 346. The triples 246 and 256 cannot both belong to $\mathcal{F}$, for if they do then $s(2) \geq 5$, while $s(6) \leq 4$, contradiction. Next, if exactly one of 246, 256 belongs to $\mathcal{F}$, then $s(2) \geq 4$, $s(6) \leq 3$, a contradiction, as well. Finally, if neither of them belongs to $\mathcal{F}$, then $s(6) \leq 2$, contradicting $s = s(2) \geq 3$.

It remains to consider the case $k_2 = 3$. Using Lemmas 2.7 and 2.8 we see that there is no loss of generality in the assumption that 123, 124, 135, 236 belong to $\mathcal{F}$. It is straightforward to check, using (1), Lemmas 2.7 and 2.8, that only 6 other triples, namely 146, 156, 245, 256, 345, 346 remain as candidates for membership of $\mathcal{F}$. If not all of them lie in $\mathcal{F}$ then we have a contradiction to $k_2 = 3$. Thus, we get Case 3 of the List 1.2. The group $\langle (1,2)(5,6), (2,3)(4,5), (3,4)(5,6) \rangle$ acts transitively on Ω and $\mathcal{F}$.

The consideration of the transitive case is now complete. We summarize our conclusions in the following proposition.

Proposition 2.9 *Let G be a group acting on Ω transitively and let A be a 2-quasi-invariant subset of Ω with $|A| = d(A) \geq 3$. Then $|A| = d(A) = 3$ and $\mathcal{F} = \mathcal{F}(A)$ is one of the first 4 families in List 1.2.* □

3 Intransitive case

Suppose that the group G acts on Ω with orbits Ω_i, $i \in I$, where $|I| \geq 2$, and that $A \subseteq \Omega$ is 2-quasi-invariant. We remind the reader that we assume $|A| = d(A) \geq 3$. As in [3], we shall say that A intersects a G-orbit Ω_i *properly* if $A \cap \Omega_i \neq \emptyset$ and $\Omega_i \not\subseteq A$, and we shall call such sets $A \cap \Omega_i$ *orbit segments* of A.

The number of G-orbits in Ω that intersect A properly will be denoted by $m = m(A)$. We may relabel the orbits so that $A_i = A \cap \Omega_i$, for $i = 1, \ldots, m$, are the proper intersections of A with G-orbits. Since $|A| = d(A)$, $|A_i| \leq |\Omega_i \setminus A|$ for $i = 1, \ldots, m$. The following results were proved in [1] and [3].

Result 3.1 *[3, Theorem B] Let A be a k-quasi-invariant subset of* Ω. *Then* $m(A) \leq 2k-1$. □

Result 3.2 *[3, Proposition 4.1] Let A be a disjoint union of* B_1 *and* B_2, *where* B_i *is a non-empty union of the orbit segments of A, for* $i = 1, 2$. *Then each* B_i *is a* $k(B_i)$*-quasi-invariant subset, where* $k(B_1)$, $k(B_2)$ *are such that*

$$k(B_1) + k(B_2) \leq 3k/2.$$

□

Result 3.3 *([1], cf. also [3, Theorem C]) Let A be a 1-quasi-invariant subset of* Ω *with respect to a group G, such that, for any orbit* Ω_i *of G in* Ω, $|A \cap \Omega_i| \leq |\Omega_i \setminus A|$. *Then* $|A| \leq 1$. □

The next lemma deals with ordinary graphs. Our terminology is fairly standard. A graph $\Gamma = \Gamma(V, E)$ is a pair consisting of a *vertex set* V and an *edge set* $E \subseteq \{\{v, w\} \mid v, w \in V\}$. The automorphism group of Γ consists of all permutations of V preserving E as a set. Γ is said to be *edge- (respectively vertex-) transitive* if its automorphism group is transitive on E (respectively on V). Given $v \in V$, we denote by $\Gamma(v)$ the set $\{u \in V \mid \{u, v\} \in E\}$ of neighbours of v. The complete n-vertex graph is denoted by K_n, the n-cycle by C_n. A graph $\Gamma = \Gamma(V, E)$ is called a *claw* with the *center* c if $c \in V$ and $E = \{\{c, v\} \mid v \in V \setminus \{c\}\}$.

Lemma 3.4 *Let* $\Gamma = \Gamma(V, E)$ *be an edge-transitive graph with* $|V| \geq 4$ *such that, for any* $\{u, v\} \in E$, *there exists* $w = w_{u,v} \in V$ *such that* $\{v, w\} \in E$, *and any* $e \in E$ *intersects* $\{u, v, w\}$ *nontrivially. Then* $\Gamma \cong K_4$ *or* C_4.

Proof. First, observe that Γ is connected. Let Γ be a complete graph. Then clearly $|V| \leq 4$, so $\Gamma \cong K_4$.

Thus, we may assume the existence of $x \in V \setminus \Gamma(v)$. Since each edge through x should contain either u or w, one has $\Gamma(x) \subseteq \{u, w\}$. Suppose first that Γ is vertex-transitive. Then $|\Gamma(x)| = |\Gamma(v)| \geq 2$. So Γ has valency 2, whence Γ is an ordinary polygon. The only polygon satisfying the conditions of the lemma is the quadrangle, so $\Gamma \cong C_4$. Now suppose that Γ is not vertex-transitive. Then as Γ is edge-transitive it is bipartite, with the parts V_1, V_2 being the orbits of the automorphism group in V. Since x and v are in the same part, say, V_1, we have for any $y \in V_1$ that $\Gamma(y) = \{u, w\}$. So $V_2 = \{u, w\}$, and $E = \{\{a, b\} \mid a \in V_1, b \in V_2\}$. Now it is clear that Γ satisfies the conditions of the lemma if and only if $|V_1| = 2$, and $\Gamma \cong C_4$. □

We return to the proof of Theorem 1.3. We proceed by a number of steps.

Lemma 3.5 $|A| = 3$.

Proof. Result 3.1 immediately implies that $m \leq 3$. We will consider the two cases, $m = 2$ and $m = 3$, separately.

Two-orbit case. We may assume that A is the union of two orbit segments A_1 and A_2 such that $|A_1| \geq |A_2|$. By Result 3.2, $k(A_1) + k(A_2) \leq 3$. Therefore $k(A_1) = 2$, $k(A_2) = 1$. Hence by Proposition 2.9 $|A_1| \leq 3$, and by Result 3.3, $|A_2| = 1$.

Assume that $|A_1| = 3$. By Proposition 2.9, the set of G-images of A_1 gives us one of the first four examples from the List 1.2. It is straightforward to check that each of them satisfies the following property. (Note that in cases 1 and 3 there is actually nothing to check, since there are no D, B such that $|D \cap B| = 2$.)

(*) If D, B are two triples such that $|D \cap B| = 2$ then there exists a triple C such that $|D \cap C| = |B \cap C| = 1$.

Since $k(A_1) = 2$ there is an element $g \in G$ such that $|A_1^g \setminus A_1| = 2$. Then, by (1), $A_2^g = A_2$. On the other hand, if $|A_1^g \setminus A_1| = 1$ then, by (*), there exists $g' \in G$ such that $|A_1^{g'} \setminus A_1| = |A_1^{g'} \setminus A_1^g| = 2$. Hence, by (1), $A_2^{g'} = A_2$, $A_2^{g'} = A_2^g$, whence again $A_2^g = A_2$. Therefore A_2 is G-invariant, so $A_2 = \Omega_2$, which contradicts the assumption that A intersects Ω_2 properly. Hence $|A_1| = 2$. So $|A| = 3$, as required.

Three-orbit case. We assume that A is the union of three orbit segments A_1, A_2 and A_3. By Result 3.2, $k(A_{i_1} \cup A_{i_2}) + k(A_{i_3}) \leq 3$, for any permutation (i_1, i_2, i_3) of $\{1, 2, 3\}$. Therefore, by Result 3.3, $k(A_{i_3}) = 1$. Hence $|A_1| = |A_2| = |A_3| = 1$. So $|A| = 3$. □

In what follows we assume $|A_i| \geq |A_{i+1}|$ for each $1 \leq i < m$.

Lemma 3.6 *For $m = 2$, let $E = \{A_1^g \mid g \in G\}$ and $V = \Omega_1$, and for $m = 3$, let $E = \{A_1^g \cup A_2^g \mid g \in G\}$ and $V = \Omega_1 \cup \Omega_2$. Then*

1. *Each $X \in E$ lies in a unique G-image of A; in particular, $|E| = |\mathcal{F}(A)|$;*

2. *The graph $\Gamma = \Gamma(V, E)$ is isomorphic either to K_4 or to C_4.*

Proof. Pick $X = \{u, v\} \in E$ such that $X \subset A$. Denote $S = A \setminus X$. To show the first part, it is sufficient to show that, if $X \subset A^g$, then $A^g = A$. Observe that, by Result 3.3, X is not 1-quasi-invariant. Thus there exists $h \in G$ such that $X^h \cap X = \emptyset$. By (1), $S^h = S$. Let $g \in G$ such that $X \subset A^g$.

Then $X^g = X$. We have $A^{gh} = X^h \cup S^{gh}$ and, again by (1), $A^{gh} \cap A$ is nonempty, so $S^{gh} = S$. Hence $S^g = S^{h^{-1}} = S$, and so $A^g = A$.

We turn to the second part of the lemma. It suffices to check that Γ satisfies the conditions of Lemma 3.4. Since X is not 1-quasi-invariant, the condition $|V| \geq 4$ is satisfied. Suppose that for any $w \in \Gamma(v)$ there exists $T \in E$ satisfying $T \cap \{u, v, w\} = \emptyset$. We show that this assumption leads to a contradiction.

Our assumption is equivalent to the following. For any $g \in G$ such that $X^g \cap X = \{v\}$ there exists $x \in G$ satisfying $X^x \cap \{u, v, w\} = \emptyset$, where $X^g = \{v, w\}$ and $X^x = T$. By (1), for any such g and x one has $S^g = S^x = S$. Hence for any $g \in G$ such that $X^g \cap X = \{v\}$ one has $S^g = S$. Also, by (1), if for $g \in G$ one has $X^g \cap X = \emptyset$, then $S^g = S$.

We have shown that for any $g \in G$ such that $X^g \cap X \subseteq \{v\}$ one has $S^g = S$. Also, by the first part of the lemma, if $X^g = X$, then $A^g = A$, so $S^g = S$, as well.

Consider the case $m = 2$. Here Γ is vertex-transitive, so for any $g \in G$ such that $X^g \cap X = \{u\}$ we have $S^g = S$. Thus any $g \in G$ such that $A^g \cap A \neq \emptyset$ stabilizes S. By (1), S is G-invariant, a contradiction.

It remains to consider the case $m = 3$. It suffices to prove that for any $g \in G$ such that $X^g \cap X = \{u\}$ we have $S^g = S$, to obtain the required contradiction. We may assume the existence of $g \in G$ such that $X^g \cap X = \{u\}$, otherwise we are done.

We claim that there exists $g' \in G$ such that $X^{g'} \cap X = \{v\}$. Indeed, otherwise the graph Γ is a disjoint union of claws, and the center of each claw is a G-image of u. By (1), if $X \cap X^f = \emptyset$ for some $f \in G$, then $S^f = S$. Since u is not G-invariant, such an element f exists. Consider the claw with center u and edge set $\{X, X^{g_1}, \ldots, X^{g_i}, \ldots\}$. Then $X^{g_i f}$ is in the claw with center u^f and is disjoint from X. Thus $S^{g_i f} = S$. Then $S^{g_i f} = S^f$ and so $S^{g_i} = S$. Hence $S = S^{g_1} = \cdots = S^{g_n} = \ldots$. It follows that S is G-invariant, which is a contradiction proving our claim.

Since the number of G-images of X containing v is equal to the number of those containing v^g, there exist at least two G-images of X containing v^g. Therefore there exists $h \in G$ such that $X^h \cap X = \emptyset$, $v^g \in X^h$. Since $X^g \cap X^h = \{v^g\}$ we have $X \cap X^{hg^{-1}} = \{v\}$, so $S = S^{hg^{-1}}$, that is $S^g = S^h$. Also, since $X^h \cap X = \emptyset$, we have $S^h = S$, whence $S^g = S$. Thus, for any $g \in G$ such that $X^g \cap X \subseteq \{u\}$ one has $S^g = S$. We are done. □

To complete the proof of the theorem, we shall reconstruct the whole set of images of A using the set E.

Let X, X^g be a pair of edges of Γ with empty intersection, $X \subset A$, $S = A \setminus X$. By (1), $S^g = S$. Suppose that X^h is equal to neither X nor X^g. If $S^h = S$ then the intersection of at least 4 images of A (namely, A,

A^g, A^h, $A^{h'}$, where $h' \in G$ is chosen such that $X^h \cap X^{h'} = \emptyset$) would be equal to S. So $|E| \geq 8$, for S is not G-invariant. This is a contradiction to the second part of Lemma 3.6.

Thus, if $|X^h \cap X| = 1$ then $S^h \neq S$. Hence the points in S^G are in a one-to-one correspondence with the pairs of parallel edges of Γ, and we immediately have for $\Gamma \cong K_4$ that $|S^G| = 3$, and $\mathcal{F}(A)$ is as in List 1.2 (5), whereas for $\Gamma \cong C_4$ we have $\mathcal{F}(A)$ as in List 1.2 (1).

The proof of Theorem 1.3 is complete.

References

[1] L. Brailovsky. Structure of quasi-invariant sets. *Arch. Math. (Basel) 59(1992) 322–326.*

[2] L. Brailovsky. On the small squaring and commutativity. *Bull. Lond. Math. Soc.* 25(1993), 330–336.

[3] L. Brailovsky, D.V. Pasechnik and C.E. Praeger. Subsets close to invariant subsets under group actions. *Submitted*

[4] J. Cannon. A language for group theory. *Preprint of University of Sydney*, 1982.

[5] J. Cannon. An introduction to the group theory language Cayley. in: "Computational group theory" (Proc. of LMS Meeting, Durham), Ed. M. Atkinson, 145–183, (Academic Press, 1982).

[6] P. Erdös, C. Ko and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford Ser (2)* 12(1961) 313–320.

[7] P. Frankl and Z. Füredi. Non-trivial intersecting families. *J. Comb. Th. (A)* 41(1986) 150-153.

[8] A.J.W. Hilton and E.C. Milner. Some intersecting theorems for systems of finite sets. *Quart. J. Math. Oxford Ser (2)* 18(1967) 369–384.

[9] C.E. Praeger, On permutation groups with bounded movement. *J. of Algebra* 44(1991) 436–442.

The present address of the first author:
Leonid Brailovsky
Scitex Corporation Ltd.
P.O. Box 330
Herzliya 46103
Isreal

An Invariant For One-Factorizations Of The Complete Graph

Terry S. Griggs
Department of Mathematics and Statistics
University of Central Lancashire
Preston PR1 2HE
United Kingdom

Alex Rosa
Department of Mathematics and Statistics
McMaster University
Hamilton, Ontario
Canada L8S 4K1

1 Introduction

A *one-factor* (or a *perfect matching*) of the complete graph on $2n$ vertices, K_{2n}, is a collection of edges which contain each vertex precisely once. A *one-factorization* (OF) is a partition of the set of edges of K_{2n} into edge-disjoint one-factors. A perfect *one-factorization* (P1F) is a one-factorization in which the union of any two distinct one-factors forms a Hamiltonian cycle. Recent surveys are by Mendelsohn and Rosa [16] and Wallis [23] on OF and Seah [18] on P1Fs.

Two OF of K_{2n} are said to be *isomorphic* if there exists a bijection of the vertex set which maps one-factors to one-factors. Given two OFs of K_{2n}, a problem is to identify whether they are isomorphic. Currently the best known algorithm for testing isomorphism of one-factorizations is subexponential [4]. Therefore one attempts to reduce the magnitude of the problem by using easily computable *isomorphism invariants.* One such frequently used invariant, the so-called *d-tables structure* or *cycle structure*, is quite sensitive for "random" OFs but fails to distinguish between P1Fs (cf. [4], [8], [16]). More recently, a new invariant, the so-called *train invariant*, has been introduced and explored in the paper by Dinitz and Wallis [9] (see also [8] and [22]). In this paper we define another invariant, the *tricolour invariant*, and present a number of results relating to it. The structure of the remainder of the paper is as follows. In Section 2 the invariant is defined and a theorem is proved about its form for the series of one-factorizations usually denoted by GK_{2n}. The performance of the invariant in distinguishing between non-isomorphic OFs of K_8 and K_{10} and P1Fs of K_{2n}, $6 \le n \le 11$ is described in Section 3. In Section 4 we relate the invariant to Steiner triple systems. Finally in Section 5 we use it to construct an anti-Pasch STS(39) containing an STS(19) subsystem in response to a question posed by Griggs and Murphy [13].

2 Tricolour

Let the vertex set of the complete graph K_{2n} be V and denote the one-factors of an OF by F_i, $i = 0, 1, 2, \ldots, 2n-2$. Define a mapping $f\colon \binom{V}{3} \to \binom{Z_{2n-1}}{3}$ as follows.

For $a, b, c \in V$, $a \ne b \ne c \ne a$, suppose that the edges $ab \in F_i$, $bc \in F_j$, $ca \in F_k$. Trivially $i \ne j \ne k \ne i$ and we set $f(\{a,b,c\}) = \{i,j,k\}$. Let $N(\{i,j,k\}) = \#\{\{a,b,c\}\colon f(\{a,b,c\}) = \{i,j,k\}\}$. Finally for $n \in Z$, $0 \le n \le \max(N) = M$ (say) define the *tricolour vector* $(v_0, v_1, v_2, \ldots, v_m)$ by $v_n = \#\{\{i,j,k\}\colon N(\{i,j,k\}) = n\}$.

The (non-negative) integer v_0 will be called the *tricolour number* of the one-factorization.

Using the above notation and definitions we now establish the following.

Theorem. *The tricolour vector of the one-factorization GK_{2n} is given by*

(i) $v_0 = 0$, $v_1 = 2(2n-1)(n-1)(n-3)/3$, $v_2 = (2n-1)(n-1)$, $v_i = 0$ *for* $i \ge 3$, *if 3 does not divide* $2n-1$,

(ii) $v_0 = 0$, $v_1 = 2(2n-1)(n-2)^2/3$, $v_2 = (2n-1)(n-2)$, $v_3 = 0$, $v_4 = (2n-1)/3$, $v_i = 0$ *for* $i \ge 5$, *if 3 divides* $2n-1$..

Proof: Let the vertex set of the complete graph K_{2n} be $\{\infty, 0, 1, 2, \ldots, 2n-2\} = V$ and the one-factors F_i, $i = 0, 1, 2, \ldots, 2n-2$ be given by $(\infty, i), (1+i, 2n-2+i), (2+i, 2n-3+i), \ldots, (n-1+i, n+i)$, all arithmetic being modulo $2n-1$. Denote the one-factor containing an edge uv by $\overline{uv}$ and for $a, b, c \in V$, $a \ne b \ne c \ne a$, let $\{\overline{ab}, \overline{ac}, \overline{bc}\} = S$.

The proof proceeds by identifying which further triples $u, v, w \in V$, $u \ne v \ne w \ne u$ with $\{\overline{uv}, \overline{uw}, \overline{vw}\} = T$ satisfy $S = T$. There are three cases to consider

1. $u = x$, $v = y$, $w = z$, where $\{a,b,c\} \cap \{x,y,z\} = \varnothing$. For $\infty \in \{a,b,c,x,y,z\}$ then if $S = T$ it follows wlog that $a + b = x + y$, $a + c = x + z$, $b + c = y + z$ leading to $a = x$, $b = y$, $c = z$, a contradiction. Wlog if $a = \infty$ then if $S = T$ it follows that $x + y = 2c$, $x + z = 2b$, $y + z = b + c$ leading to $x = (b+c)/2$, $y = (3c-b)/2$, $z = (3b-c)/2$ whence $x \ne y \ne z \ne x$.