An Invariant For One-Factorizations Of The Complete Graph

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1 Introduction

A *one-factor* (or a *perfect matching*) of the complete graph on $2n$ vertices, K_{2n}, is a collection of edges which contain each vertex precisely once. A *one-factorization* (OF) is a partition of the set of edges of K_{2n} into edge-disjoint one-factors. A perfect *one-factorization* (P1F) is a one-factorization in which the union of any two distinct one-factors forms a Hamiltonian cycle. Recent surveys are by Mendelsohn and Rosa [16] and Wallis [23] on OF and Seah [18] on P1Fs.

Two OF of K_{2n} are said to be *isomorphic* if there exists a bijection of the vertex set which maps one-factors to one-factors. Given two OFs of K_{2n}, a problem is to identify whether they are isomorphic. Currently the best known algorithm for testing isomorphism of one-factorizations is subexponential [4]. Therefore one attempts to reduce the magnitude of the problem by using easily computable *isomorphism invariants.* One such frequently used invariant, the so-called *d-tables structure* or *cycle structure*, is quite sensitive for "random" OFs but fails to distinguish between P1Fs (cf. [4], [8], [16]). More recently, a new invariant, the so-called *train invariant*, has been introduced and explored in the paper by Dinitz and Wallis [9] (see also [8] and [22]). In this paper we define another invariant, the *tricolour invariant*, and present a number of results relating to it. The structure of the remainder of the paper is as follows. In Section 2 the invariant is defined and a theorem is proved about its form for the series of one-factorizations usually denoted by GK_{2n}. The performance of the invariant in distinguishing between non-isomorphic OFs of K_8 and K_{10} and P1Fs of K_{2n}, $6 \le n \le 11$ is described in Section 3. In Section 4 we relate the invariant to Steiner triple systems. Finally in Section 5 we use it to construct an anti-Pasch STS(39) containing an STS(19) subsystem in response to a question posed by Griggs and Murphy [13].

2 Tricolour

Let the vertex set of the complete graph K_{2n} be V and denote the one-factors of an OF by F_i, $i = 0, 1, 2, \ldots, 2n-2$. Define a mapping $f\colon \binom{V}{3} \to \binom{Z_{2n-1}}{3}$ as follows.

For $a, b, c \in V$, $a \ne b \ne c \ne a$, suppose that the edges $ab \in F_i$, $bc \in F_j$, $ca \in F_k$. Trivially $i \ne j \ne k \ne i$ and we set $f(\{a,b,c\}) = \{i,j,k\}$. Let $N(\{i,j,k\}) = \#\{\{a,b,c\} : f(\{a,b,c\}) = \{i,j,k\}\}$. Finally for $n \in Z$, $0 \le n \le \max(N) = M$ (say) define the *tricolour vector* $(v_0, v_1, v_2, \ldots, v_m)$ by $v_n = \#\{\{i,j,k\} : N(\{i,j,k\}) = n\}$.

The (non-negative) integer v_0 will be called the *tricolour number* of the one-factorization.

Using the above notation and definitions we now establish the following.

Theorem. *The tricolour vector of the one-factorization GK_{2n} is given by*

(i) $v_0 = 0$, $v_1 = 2(2n-1)(n-1)(n-3)/3$, $v_2 = (2n-1)(n-1)$, $v_i = 0$ *for* $i \ge 3$, *if 3 does not divide* $2n-1$,

(ii) $v_0 = 0$, $v_1 = 2(2n-1)(n-2)^2/3$, $v_2 = (2n-1)(n-2)$, $v_3 = 0$, $v_4 = (2n-1)/3$, $v_i = 0$ *for* $i \ge 5$, *if 3 divides* $2n-1$..

Proof: Let the vertex set of the complete graph K_{2n} be $\{\infty, 0, 1, 2, \ldots, 2n-2\} = V$ and the one-factors F_i, $i = 0, 1, 2, \ldots, 2n-2$ be given by $(\infty, i), (1+i, 2n-2+i), (2+i, 2n-3+i), \ldots, (n-1+i, n+i)$, all arithmetic being modulo $2n-1$. Denote the one-factor containing an edge uv by $\overline{uv}$ and for $a, b, c \in V$, $a \ne b \ne c \ne a$, let $\{\overline{ab}, \overline{ac}, \overline{bc}\} = S$.

The proof proceeds by identifying which further triples $u, v, w \in V$, $u \ne v \ne w \ne u$ with $\{\overline{uv}, \overline{uw}, \overline{vw}\} = T$ satisfy $S = T$. There are three cases to consider

1. $u = x$, $v = y$, $w = z$, where $\{a,b,c\} \cap \{x,y,z\} = \varnothing$. For $\infty \in \{a,b,c,x,y,z\}$ then if $S = T$ it follows wlog that $a+b = x+y$, $a+c = x+z$, $b+c = y+z$ leading to $a = x$, $b = y$, $c = z$, a contradiction. Wlog if $a = \infty$ then if $S = T$ it follows that $x+y = 2c$, $x+z = 2b$, $y+z = b+c$ leading to $x = (b+c)/2$, $y = (3c-b)/2$, $z = (3b-c)/2$ whence $x \ne y \ne z \ne x$.

2. $u = a$, $v = y$, $w = z$, where $\{b,c\} \cap \{y,z\} = \varnothing$. In this case $\{\overline{ay}, \overline{az}, \overline{yz}\} = T \neq S$.

3. $u = a$, $v = b$, $w = z \neq c$. In this case $\{\overline{ab}, \overline{az}, \overline{bz}\} = T$ so $S = T$ iff $\overline{ac} = \overline{bz}$ and $\overline{az} = \overline{bc}$. Again this sub-divides into three cases.

   a . If $\infty \notin \{a,b,c,z\}$ then $a+c = b+z$ and $a+z = b+c$ leading to $c = z$, a contradiction.

   b . If a or b (wlog a) $= \infty$ then $b+z = 2c$ and $b+c = 2z$ leading to $3(c-z) = 0$.

   c . If c or z (wlog c) $= \infty$ then $b+z = 2a$ and $a+z = 2b$ leading to $3(a-b) = 0$.

We can thus distinguish two situations.

If 3 does not divide $2n-1$, the cases 3b. and 3c. above also lead to contradictions and the only triples of vertices which map under the function f to the same set of one-factors are $\{\infty, b, c\}$ and $\{(b+c)/2, (3c-b)/2, (3b-c)/2\}$.

Hence $v_2 = (2n-1)(2n-2)/2 = (2n-1)(n-1)$ and $v_1 = 2n(2n-1)(2n-2)/6 - 2v_2 = 2(2n-1)(n-1)(n-3)/3$.

Since $\sum v_i = \binom{2n-1}{3}$ then all other $v_i = 0$.

If 3 divides $2n-1$, solutions are obtained in the cases 3b. and 3c. above. Let $2n-1 = 3m$. Then in case 3b., $z = c+m$ or $c+2m$ giving $b = c+2m$ or $c+m$ respectively. Hence the triples of vertices $\{\infty, \alpha, \alpha+m\}$, $\{\infty, \alpha, \alpha+2m\}$, $\{\infty, \alpha+m, \alpha+2m\}$, $\alpha = 0,1,2,\ldots,m-1$ map under the function f to the same set of one-factors. In case 3c. then $a = b+m$ or $b+2m$ giving $z = b+2m$ or $b+m$ respectively. This leads to the triple of vertices $\{\alpha, \alpha+m, \alpha+2m\}$, $a = 0,1,2,\ldots,m-1$ mapping under the function f to the same set of one-factors as the three triples above.

Hence $v_4 = m = (2n-1)/3$. Further in this case $v_2 = (2n-1)(2n-2)/2 - 3m = (2n-1)(n-2)$ and $v_1 = 2n(2n-1)(2n-2)/6 - 2v_2 - 4v_4 = 2(2n-1)(n-2)^2/3$.

Since $\sum v_i = \binom{2n-1}{3}$ then all other $v_i = 0$. This completes the proof of the theorem.

3 Statistics

For K_2, K_4 and K_6 there is a unique one-factorization to within isomorphism; it belongs to the series GK_{2n} and therefore its tricolour vector is as given in the theorem above. There are precisely 6 pairwise non-isomorphic OFs of K_8 [7] and these are given in a compact notation by Wallis [23]. For completeness we also list the OF as well as its tricolour vector. Note that not only the tricolour vector but even the tricolour number is a complete invariant in this case

#1	1234567	2134657	3124756	4152637	5142736	6172435	7162534
	Vector: 28 0 0 0 0 0 0 0 7						
#2	1234567	2134657	3124756	4152637	5142736	6172534	7162435
	Vector: 24 0 0 0 8 0 0 0 3						
#3	1234567	2134657	3124756	4162537	5172634	6142735	7152436
	Vector: 18 0 8 0 8 0 0 0 1						
#4	1234567	2134657	3124756	4162735	5172634	6142537	7152436
	Vector: 22 0 0 0 12 0 0 0 1						
#5	1234567	2134657	3142756	4162537	5172634	6123547	7152436
	Vector: 9 8 12 0 6						
#6	1234567	2143657	3162547	4172635	5123746	6152734	7132456
	Vector: 0 14 21 (GK_8)						

For K_{10} the number of pairwise non-isomorphic OFs is 396 [11], [12]. The tricolour vectors are listed in Appendix 1, ordered for ease of comparison as in [11] and also as in Dinitz and Wallis [9] where the indegree sequences of the train invariant are listed. In this case the tricolour vector is not a complete invariant. There are 51 pairs, 8 triples and 2 sets of four pairwise non-isomorphic OFs which have the same tricolour vectors. These are identified in Appendix 1. This gives a sensitivity for the invariant of 323/396 (approximately 0.816). Nevertheless the tricolour invariant does distinguish between the only two one-factorizations, #16 and #26, which the train invariant does not. The two invariants together are complete for OFs of K_{10}.

For K_{12} the number of pairwise non-isomorphic OFs has recently been enumerated [25]. There are precisely 526,915,620 of them! However for $n \geq 7$, the exact number of pairwise non-isomorphic OFs of K_{2n} is unknown (but large; cf., e.g., [3]). At this point we encounter the combinatorial explosion and so restrict our attention mainly to perfect one-factorizations. There are precisely 5 pairwise non-isomorphic P1Fs of K_{12} [17]. Again we list each of these in compact form with, in order to aid identification, the order of its automorphism group and its tricolour vector.

#1	123456789TEW	132W4T59687E	142537698ETW	152T3W486E79
	1628354W7T9E	172E38465T9W	1823495E6107W	19263E475W8T
	1T27394E586W	1E293T45678W	1W24365789TE	
	Aut = 1		Vector: 31 67 48 19	
#2	123456789TEW	132W4T59687E	14237698ETW	15283W4679TE
	1627394W5E8T	17234E5T6W89	1824356E7T9W	192E3T485W67
	1T2638457W9E	1E2T3649578W	1W293E37586T	
	Aut = 5		Vector: 45 40 60 20	

#3	123456789TEW	132W4T59687E	1425376T8W9E	152936487WTE
	1623475E8T9W	172E3T456W89	182T394E5W67	19273E4658TW
	1T263W49578E	1E28354W697T	1W24385T6E79	
	Aut= 10 (2 fixed points)		Vector: 30 70 45 20	
#4	123456789TEW	132W4T59687E	142539678WTE	1527384W6T9E
	162E3W49578T	1726354E89TW	18293T456E7W	1924375T6W8E
	1T2836475E9W	1E23485W697T	1W2T3E465879	
	Aut = 55 (1 fixed point)		Vector: 55 0 110	
#5	123456789TEW	132W4T59687E	1426375E89TW	1523486W79TE
	16293E457W8T	172E38465T9W	182T394E5W67	1925364W7T8E
	1T273W49586E	1E243T57698W	1W2835476T9E	
	Aut = 110 (GK_{12})		Vector: 0 110 55	

Again not only the tricolour vector but also the tricolour number is a complete invariant. However, it is interesting that a different OF which was discovered by Cameron [3] and is uniform in that the union of any two one-factors forms two 6-cycles, has the same tricolour vector: 55 0 110 as OF #4 above. Cameron's one-factorization is

123456789TEW	132E4W586T79	1428375T6E9W	152T3W48697E
1623457W89TE	17293E465W8T	1826394E57TW	19243T5E678W
1T2W3547689E	1E2538496W7T	1W27364T598E	

For K_{14} the number of pairwise non-isomorphic Plus with non-trivial automorphism group has been determined by Seah and Stinson [19], [20]; there are precisely 21 of them. We list below the tricolour vectors of the one-factorizations preserving the numbering of the latter given by Seah and Stinson. Note that #1-20 are to be found in [19] and #21 in [20]. Once again the tricolour vector is a complete invariant.

OF number	Tricolour vector				
1	104	0	182		
2	74	72	130	8	2
3	98	60	86	36	6
4	44	132	98	12	
5	62	102	104	18	
6	60	106	102	18	
7	68	106	80	30	2
8	60	116	84	24	2
9	108	40	90	48	
10	96	84	54	36	16
11	98	54	92	42	
12	78	86	96	18	8
13 (GK_{14})	0	208	78		
14	60	112	90	24	
15	60	136	42	48	

OF number	Tricolour vector				
16	66	110	79	28	3
17	60	120	80	20	6
18	73	101	76	33	3
19	77	87	95	21	6
20	59	120	83	18	6
21	65	98	107	12	4

Finally, for $8 \leq n \leq 11$, we tested the tricolour invariant using the exhaustive computer search for P1Fs having either one or two fixed points with a cyclic automorphism on the remaining points which was done by Anderson [1]. The results are given below, the numbering being as in tables 2 and 3 of Anderson's paper.

Value of 2n	P1F number	Tricolour vector					
16	15-1	135	150	105	60	5	
	14-1(Kotzig)	140	154	91	56	14	
18	17-1(GK_{18})	0	544	136			
	17-2	187	255	153	85		
	16-1(GK_{18})	0	544	136			
	16-2	144	288	216	32		
	16-3	144	304	184	48		
	16-4	208	224	152	96		
	16-5	192	256	152	64	16	
20	19-1(GK_{20})	0	798	171			
	19-2	342	342	171	0	114	
	19-3	399	171	342	0	0	57
	19-4	285	456	19	190	19	
	19-5	228	418	266	38	19	
	19-6	285	342	266	38	38	
	19-7	361	266	190	114	38	
	18-1(GK_{20})	0	798	171			
	18-2	252	384	261	54	18	
	18-3	306	252	351	54	6	
	18-4	270	366	225	108		
	18-5	180	492	243	54		
	18-6	324	294	243	72	36	
	18-7	252	348	315	54		
	18-8	288	342	225	108	6	
	18-9	270	330	297	72		
	18-10	198	438	297	36		

Value of 2n	P1F number	Tricolour vector					
22	21-1	420	448	294	168		
	21-2	441	420	315	126	28	
	21-3	399	504	273	126	28	
	20-1	280	680	290	60	0	20
	20-2	340	560	310	120		
	20-3	360	500	390	60	20	
	20-4	540	280	350	100	40	20
	20-5	440	420	350	80	20	20
	20-6	360	580	290	40	40	20
	20-7	340	620	250	80	20	20
	20-8	380	580	230	60	80	
	20-9	280	600	430	0	20	
	20-10	420	500	270	80	40	20
	20-11	400	520	250	120	40	
	20-12	360	520	330	120		
	20-13	440	480	210	160	40	
	20-14	400	460	330	140		
	20-15	340	520	410	40	20	
	20-16	380	520	290	120	20	
	20-17	440	480	290	20	80	20

4 Steiner triple systems

Consider a Steiner triple system, STS(v), defined on the set $\{0, 1, 2, \ldots, v-1\} = V$. It is well-known (cf., e.g., [15]) that this yields a one-factorization of the complete graph K_{v+1} defined on the vertex set $V \cup \{\infty\}$. Again denoting the one-factors by F_i, $i \in V$, if $\{x, y, z\}$ is a block of the STS(v) then the edges $xy \in F_z$, $yz \in F_x$, $zx \in F_y$, and the edges $\infty x \in F_x$.

For such a Steiner one-factorization, the operation of the function f specified by the tricolour invariant is as follows.

(i) if $\{x, y, z\}$ is a block of the STS(v) then $f(\{x, y, z\}) = \{x, y, z\}$.

(ii) if $\{x, y, z\}$ is a non-block of the STS(v) and $\infty \notin \{x, y, z\}$ then $f(\{x, y, z\}) = \{a, b, c\}$ where $\{x, y, c\}$, $\{x, b, z\}$, $\{a, y, z\}$ are blocks of the STS(v),

(iii) $f(\{\infty, l, m\}) = \{l, m, n\}$ where $\{l, m, n\}$ is a block of the STS(v).

If (iii) is ignored, the function f is nothing other than the well-known transformation which produces a directed graph known as the *train* of the STS(v) and goes back to White [24] in 1913 (see also [5] and [21]). Although trains are in general a fairly sensitive invariant (for example they successfully distinguish all eighty STS(15)s) they are cumbersome; the directed graphs contain $\binom{v}{3}$ vertices. To help overcome this problem, Colbourn, Colbourn and Rosenbaum [5] introduced the *compact train*, a set of ordered triples (p, q, r), each triple meaning that the train contains r components with p vertices, q of which have indegree zero (after eliminating the unique directed cycle from each component). This much simpler invariant which is just a summary of train structure also distinguishes the eighty STS(15)s with the exceptions of #6 and #7 in the standard listing as given for example in [15].

The tricolour invariant is similarly a summary of train structure. Indeed we can define two tricolour invariants for Steiner triple systems. The first is the tricolour invariant of the one-factorization induced by the Steiner triple system calculated as described in section 2. We will call this the OF *tricolour vector*. The second is the tricolour invariant of the Steiner triple system itself obtained as noted above by ignoring blocks $\{\infty, l, m\}$ and which we will call the STS *tricolour vector*. However for both vectors the tricolour number is the same. It is easily seen that in computing these invariants

(i) if $\{x, y, z\}$ is a block of the STS(v) then $\#\{F_x, F_y, F_z\} = n+$ Indegree of $\{x, y, z\}$ in the train where $n = 4$ for the one-factorization induced by the STS(v) and $n = 1$ for the STS(v) itself,

(ii) if $\{x, y, z\}$ is a non-block of the STS(v) then $\#\{F_x, F_y, F_z\} =$ Indegree of $\{x, y, z\}$ in the train.

This leads to different OF and STS tricolour vectors as the following results for $v = 7$, 9 and 13 show.

STS(7)									
OF tricolour vector	28	0	0	0	0	0	0	0	7
STS tricolourvector	28	0	0	0	0	7			
STS(9)									
OF tricolour vector	0	72	0	0	12				
STS tricolour vector	0	84							
STS(13); cyclic									
OF tricolour vector	130	52	78	0	0	13	0	13	
STS tricolour vector	130	52	91	0	13				
STS(13); non-cyclic									
OF tricolour vector	104	84	72	0	3	15	7	1	
STS tricolour vector	104	87	87	7	1				

The tricolour vectors of the eighty STS(15)s, ordered as in [15], are given in Appendix 2. They form a complete invariant in distinguishing the systems. Indeed the first three components are enough to form a complete invariant. In Appendix 3, the tricolour vectors of all non-isomorphic cyclic STS(v) for $v = 19$, 21, 25 and 27 are given, ordered as in [6]. They do not distinguish between two of the STS(21)s, #1 and #3, but are otherwise complete.

5 Anti-Pasch Steiner triple systems

An *anti-Pasch Steiner triple system* is one in which there is no collection of four blocks whose union has cardinality six. Such a collection must be isomorphic to $\{a,b,c\}$, $\{a,y,z\}$, $\{x,b,z\}$, $\{x,y,c\}$ and is known as a *Pasch configuration* or *quadrilateral*. Anti-Pasch STS(v) are known to exist for all $v \equiv 3 \pmod 6$, [2], [14] and it is conjectured for all $v \equiv 1 \pmod 6$, $v \geq 19$ also though results for this residue class are still fragmentary. In their study of anti-Pasch STS(19)s, Griggs and Murphy [13] proved that the unique STS(9), which is anti-Pasch, could not be embedded in an anti-Pasch STS(19). This is in contrast to the situation for general STS(v) for which the classical result of Doyen-Wilson [10] states that any STS(v) can be embedded in an STS(u) for all admissible $u \geq 2v+1$. Griggs and Murphy asked for an example of an anti-Pasch STS(2v+1) containing a (necessarily anti-Pasch) STS(v). Two of the four cyclic STS(19)s are anti-Pasch [13] and below we exhibit embedding of each of them in an anti-Pasch STS(39).

Consider an STS(v) on a base set V whose elements will be referred to as *undashed*. Suppose further that the STS(v) is embedded in an STS(2v+1) on a base set U. Let $D = U \setminus V$; the elements of D will be referred to as *dashed*. Then it is easy to see that every block of the STS(2v+1) comprises either three undashed elements (if it is a block of the sub-STS(v)) or one undashed element and two dashed elements. Further the set of pairs of dashed elements occurring with each undashed element form a one-factor of the complete graph on $|D|$ vertices and the collection of all these one-factors form a one-factorization. In such a system, a quadrilateral can appear in three possible formats of dashed and undashed elements as follows.

(i) $\{a,b,c\}$, $\{a,y,z\}$, $\{x,b,z\}$, $\{x,y,c\}$,

(ii) $\{a,b,c\}$, $\{a,y',z'\}$, $\{b,x',z'\}$, $\{c,x',y'\}$,

(iii) $\{a,b',c'\}$, $\{a,y',z'\}$, $\{x,b',z'\}$, $\{x,c',y'\}$.

Now case (i) will not occur if the STS(v) is anti-Pasch. Case (iii) can be avoided by using a perfect one-factorization of K_{v+1} (or any OF where the union of any two one-factors contains no 4-cycle). Consider case (ii) and suppose that we attempt to construct the required structure beginning with the perfect one-factorization. To succeed, the triples of one-factors which are counted by the tricolour number must contain an anti-Pasch STS(v). Hence, for example, the one-factorization GK_{2n} which is perfect when $2n-1$ is prime will never suffice since we have proved in section 2 that $v_0 = 0$. One-factorizations which would appear to have the greatest chance of success are those with the largest tricolour number.

We wish to embed the two anti-Pasch cyclic STS(19)s in anti-Pasch STS(39)s. It is natural therefore to use a perfect one-factorization of K_{20} which also has an automorphism of order 19. From our results in section 3 the one with the largest tricolour number is #19-3 with $v_0 = 399$ and indeed we discover that this is successful.

The details are as follows.

Let $V = Z_{19} = \{0, 1, 2, \ldots, 18\}$ and $D = \{\infty', 0', 1', \ldots, 18'\}$.

One-factors F_i, $i = 0, 1, 2, \ldots, 18$ on the complete graph K_{20} on the vertices D are given by $\{\infty' i', (1+i)'(3+i)', (2+i)'(7+i)', (4+i)'(12+i)', (5+i)'(15+i)', (6+i)'(18+i)', (8+i)'(9+i)', (10+i)'(16+i)', (11+i)'(14+i)', (13+i)'(17+i)'\}$, addition being modulo 19. Let $\{x, y', z'\}$ be a triple of the STS(39) when $y'z' \in F_x$.

The two anti-Pasch cyclic STS(19)s are generated by the starter blocks

(i) $\{0, 1, 4\}$, $\{0, 2, 12\}$, $\{0, 5, 13\}$ and

(ii) $\{0, 1, 8\}$, $\{0, 2, 5\}$, $\{0, 4, 13\}$ (the Netto system) respectively.

It is left as an easy exercise for the reader that either of these sets of triples can be adjoined to those already constructed without introducing quadrilaterals. Hence both anti-Pasch cyclic STS(19)s can be embedded in an anti-Pasch STS(39).

Appendices

For economy of space, Appendices 1, 2 and 3 are not attached to this paper but are available from the first author upon request.

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