The Effects of Vertex Deletion and Edge Deletion on the Clique Partition Number

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1 Introduction

A recent paper by Brigham and Dutton [2] examines the effects of vertex removal and edge removal from a graph G on the clique covering number of G. This paper closely follows theirs except that we will look at the clique partition number of a graph. For a survey of the literature on clique coverings and clique partitions see [5]. See also an early paper on these topics by Orlin [6].

Most of the notation and terminology used in this paper can be found in Bondy and Murty [1]. A graph G has vertex set $V(G)$ and edge set $E(G)$. A complete subgraph of G is called a *clique*. A *clique partition* (respectively, *clique covering*) of G is a set of cliques with the property that each edge of G is contained in exactly (respectively, at least) one of the cliques. The *clique partition number* (respectively, *clique covering number*) is denoted by $cp(G)$ (respectively, $cc(G)$). It is the minimum number of cliques required to partition (respectively, cover) the edge set of G. If $e \in E(G)$, then $G-e$ is the graph with the edge e deleted. If $v \in V(G)$, then $G-v$ is the graph obtained by removing v and all edges incident with v. Another way of decreasing the number of vertices in a graph by one is by identifying two vertices. We say that two vertices are *identified* if they are replaced by a single vertex whose neighbour set is the union of the neighbour sets of the two vertices. Let G_{xy} denote the graph obtained by identifying vertices x and y of G.

Theorem 1.1 *Let x and y be vertices of a graph G such that the distance between x and y is at least 4. Then $cp(G_{xy}) = cp(G)$.*

Proof: The proof is the same as in the clique covering case [2]. □

Corollary 1.2 *If the distance between the vertices x and y of G is at least 4, then $cp(G_{xy}) - cp(G_{xy} - e) = cp(G) - cp(G-e)$ for all edges e in G.*

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2 Preliminary Results

In this section we note some relationships between the effects of vertex deletion and edge deletion on the clique partition number. The analogous statements for the clique covering case are also true [2].

Lemma 2.1 *([6],Remark 2.1) Let v be any vertex of G. Then $cp(G-v) \leq cp(G)$.*

Proof: Let $\mathcal{C}$ be a minimum clique partition of G. Let $\mathcal{C}_v$ be the set of cliques of $\mathcal{C}$ with v deleted from each clique to which it belongs. Then $\mathcal{C}_v$ is a clique partition of $G-v$. Therefore $cp(G-v) \leq |\mathcal{C}_v| \leq |\mathcal{C}| = cp(G)$. □

Lemma 2.2 *Let $e = \{u, v\}$. If $cp(G-e) < cp(G)$, then $cp(G-v) < cp(G)$.*

Proof: The graph $G-v$ is the same as the graph $G-e-v$. Therefore $cp(G-v) = cp(G-e-v) \leq cp(G-e) < cp(G)$ by Lemma 2.1. □

Lemma 2.3 *If $cp(G-e) < cp(G)$ for all edges e, then $cp(G-v) < cp(G)$ for all nonisolated vertices v.*

Proof: Apply Lemma 2.2 to all of the edges of G. □

Theorem 2.4 *If $cp(G-v) = cp(G)$ for all nonisolated vertices v, then $cp(G-e) \geq cp(G)$ for all edges e.*

Proof: The statement of the theorem is a direct consequence of Lemma 2.3. □

The converse of Theorem 2.4 is not true. A counter-example is the graph $G = K_4 \vee {K_3}^c$, the join of K_4 and the complement of a triangle. The graph G has $cp(G) = 6$ and $cp(G-e) = 7$ for all $e \in E(G)$, but $cp(G-v) = 5$ if v is one of the three vertices of degree four.

3 Vertex Removal

We have seen in Lemma 2.1 that the clique partition number of a graph cannot increase when a vertex is deleted. The following theorem places bounds on the possible decrease. The bounds here differ from those in the clique covering case [2].

Theorem 3.1 *For any vertex v, $cp(G) - \rho(v) \leq cp(G-v) \leq cp(G) - k(v)$ where $\rho(v)$ is the degree of v and $k(v)$ is the number of edges incident with v which do not lie in any triangle of G.*

Proof: A minimum clique partition of $G - v$ together with each edge incident with v form a clique partition of G. Thus, $cp(G) \leq cp(G-v)+\rho(v)$. Each edge incident with v which does not lie in a triangle must be a 2-clique in every clique partition of G. Thus, the clique partition number is diminished by at least $k(v)$ when v is deleted. Hence $cp(G - v) \leq cp(G) - k(v)$. □

The following lemma uses the concept of separation as described in [7]. If H is a subgraph of G, then $G\backslash H$ denotes the subgraph of G with the edges of H removed. The subgraph H is said to *separate the cliques* of G if for every clique K of G, either every edge of K lies in H or every edge of K lies in $G\backslash H$. It is sufficient to take K to be a triangle.

Lemma 3.2 *If v does not belong to an induced $K_4 - e$ and each edge incident with v lies in a triangle, then $cp(G - v) = cp(G)$.*

Proof: Let H be the induced subgraph of G whose vertex set is $\{v\} \cup N(v)$ where $N(v)$ is the neighbour set of v. Let uw be any edge of H, $u, w \in N(v)$. Suppose that u and w are adjacent to vertex $x \notin H$. Then the subgraph induced on the vertices u, v, w and x form a $K_4 - e$, a contradiction. Thus every triangle of G lies entirely in H or in $G\backslash H$. This is equivalent to saying that H separates the cliques of G. By Theorem 2.1 of [7], $cp(G) = cp(H)+cp(G\backslash H)$. We note that the deletion of v from $V(G)$ does not affect the value of $cp(G\backslash H)$ and so it is sufficient for our purposes to prove that $cp(H) = cp(H - v)$.

Let C be a minimum clique partition of G. Let v belong to exactly r cliques of C: $C_1, C_2, \ldots, C_r$. Suppose that for some $i \neq j$, every vertex of C_i is adjacent to every vertex of C_j. Then C is not minimal since $C_i \cup C_j$ is one clique. Now suppose that $u \in C_i$ is adjacent to $w \in C_j$, $u, w \in N(v)$. Then without loss of generality, u has a neighbour $y \in C_i$ such that y is not adjacent to w. But then the induced subgraph on v, y, u and w is a $K_4 - e$, a contradiction. Furthermore, since every edge incident with v lies in a triangle, each clique of C containing v has at least three vertices. For $1 \leq i \leq r$, let $C_i{}^*$ denote the clique C_i with v removed. Thus $H - v$ is a disjoint union of cliques, namely the $C_i{}^*$, each of which has at least two vertices. Therefore $cp(H - v) = cp(H) = r$. □

Corollaries 3.3 and 3.4 and Theorem 3.5 follow directly from Lemma 3.2 proved above. Their analogues ([2],Corollaries 2 and 3,Theorem 7) for the clique covering case are also true but the proof employed in [2] is deduced from a theorem ([2],Theorem 6) which does not hold true for clique partitions.

Corollary 3.3 *If $cp(G - v) < cp(G)$, then v belongs to an induced $K_4 - e$ or v is incident with an edge which does not lie in a triangle.*

Corollary 3.4 *If G has no induced $K_4 - e$ and every edge of G lies in a triangle, then $cp(G - v) = cp(G)$ for every vertex of G.*

Proof: Apply Lemma 3.2 to every vertex of G. □

Theorem 3.5 *If G has no induced $K_4 - e$, then $cp(G - v) = cp(G)$ for every vertex v of G if, and only if, every edge of G is contained in a triangle.*

Proof: Let G be a graph with no induced K_4-e. If every edge of G lies in a triangle, then $cp(G-v) = cp(G)$ for every vertex of G (Corollary 3.4). If $e = \{u, v\}$ does not lie in a triangle, then $cp(G - v) < cp(G)$ (Theorem 3.1). □

The following is an example of a graph G having $cp(G - v) < cp(G)$ for all $v \in V(G)$. Let G be the join of a vertex and the odd path, P_{2p-1}. Then $cp(G) = 2p - 1$ while $cp(G - v) \in \{2p - 2, 2p - 3, 2p - 4\}$ depending on which vertex v is deleted.

Theorem 3.6 *For any graph G there are graphs H_1 and H_2 for which G is an induced subgraph of both H_1 and H_2, and*

1. *$cp(H_1) = cp(H_1 - v)$ for all vertices $v \in V(H_1)$; and,*
2. *$cp(H_2) > cp(H_2 - v)$ for all vertices $v \in V(H_2)$.*

Proof:

1. Let $C_1, C_2, \ldots, C_r$ be the cliques of a minimum clique partition of G. To construct H_1, first add r new vertices, $v_1, v_2, \ldots, v_r$. Then let $C_i{}^*$ denote the clique formed by joining v_i to each vertex of C_i. The cliques $C_1{}^*$, $C_2{}^*$, …, $C_r{}^*$ form a minimum clique partition of H_1. It is easy to verify that $cp(H_1 - v) = cp(H_1)$ for each vertex v of H_1.
2. To construct H_2, append a new vertex to each vertex of G. By Lemma 2.2, $cp(H_2 - v) < cp(H_2)$ for each vertex v of H_2. This is the same construction as that in Theorem 8 of [2].

□

As noted in [2] for the clique covering version, Theorem 3.6 has the consequence that there can be no forbidden subgraph for either the case that $cp(G) = cp(G-v)$ for all $v \in V(G)$ or the case that $cp(G) > cp(G-v)$ for all $v \in V(G)$.

Theorem 3.7 *If $cp(G - v) = cp(G)$ for all vertices v, then $|E(G)| \geq 3cp(G)$.*

Proof: Let $\mathcal{C}$ be a minimum clique partition of G. If any clique of $\mathcal{C}$ is an edge uv, then $cp(G-v) < cp(G)$. Consequently, every clique of $\mathcal{C}$ has order at least three. □

Equality holds, for example, when G is a graph whose blocks are all triangles. The comparable result for the clique covering case is $|E(G)| > 2cc(G)$ and no larger constant will do [2].

In the following example from [2], G is a planar polyhedral graph. That is, G is a graph associated with the vertices and edges of a solid convex polyhedron. Such a graph is necessarily planar and 3–connected [3]. The graph G^* is its planar dual, $\kappa(G)$ is the vertex connectivity of G and $\kappa'(G)$ is the edge connectivity of G.

Theorem 3.8 *If G is a planar polyhedral graph, then $cp(G) = |E(G)|$ if, and only if, $\kappa'(G^*) \geq 4$.*

Proof: If $cp(G) = |E(G)|$, then G contains no triangles. Therefore $cc(G) = |E(G)|$ and $\kappa'(G^*) \geq 4$ by Theorem 10 of [2]. In the other direction, if $\kappa'(G^*) \geq 4$, then $cc(G) = |E(G)|$ ([2],Theorem 10). Thus G is triangle-free and so $cp(G) = |E(G)|$. □

Corollary 3.9 *If G is a planar polyhedral graph, then $cp(G) < |E(G)|$ if, and only if, $\kappa'(G^*) = \kappa(G^*) = 3$.*

4 Edge Deletion

Unlike vertex removal, when an edge is deleted from a graph it is possible for the clique partition number to increase, decrease or remain the same. Theorem 4.2 places bounds on the amount of change possible. The changes differ from those of the clique covering case [2].

Lemma 4.1 *([6]Corollary 3.3) Let e be be any edge of K_n, the complete graph on n vertices. Then $cp(K_n - e) = n - 1$, if $n \geq 3$.*

Theorem 4.2 *Let s_i be the order of the smallest clique containing the edge e_i among all of the minimum clique partitions of G. Then $cp(G) + s_i - 2 \geq cp(G - e_i) \geq cp(G) - 1$.*

Proof: For the inequality on the right, a minimum clique partition of the graph $G - e_i$ together with the 2–clique e_i gives a clique partition of G. Thus $cp(G) \leq cp(G - e_i) + 1$. For the inequality on the left, let $\mathcal{C}$ be a minimum clique partition of G such that the edge e_i is contained in clique C of order s_i. Then $G - e_i$ can be partitioned by the cliques of $\mathcal{C} \backslash C$ plus $s_i - 1$ cliques of $C - e_i$ (by Lemma 4.1). Thus $cp(G - e_i) \leq |\mathcal{C}| - 1 + s_i - 1 = cp(G) + s_i - 2$. □

Theorem 4.3 *Let G be a graph on n vertices. Let s_i be the order of the smallest clique containing the edge e_i among all minimum clique partitions of G. Then*

1. $cp(G - e_i) = cp(G) - 1$ *if, and only if,* $s_i = 2$*; and,*

2. $cp(G - e_i) = cp(G) + n - 2$ *if, and only if,* $s_i = n$.

Proof:

1. Suppose $s_i = 2$. Let $\mathcal{C}$ be a minimum clique partition of G such that e_i occurs as a 2–clique, C. The cliques of $\mathcal{C} \backslash C$ form a minimum clique partition of $G - e_i$, so $cp(G - e_i) = cp(G) - 1$. Now let $cp(G - e_i) = cp(G) - 1$. Then the cliques of a minimum clique partition of $G - e_i$ plus the edge e_i form a clique partition of G in which the edge e_i occurs as a 2–clique. It is minimal because $cp(G) = cp(G - e_i) + 1$. Since e_i occurs as a 2–clique, $s_i = 2$.

2. If $s_i = n$, then $G = K_n$, $cp(G) = 1$, and by Lemma 4.1 we have $cp(G - e_i) = n - 1 = cp(G) + n - 2$. In the other direction, $cp(G) + n - 2 = cp(G - e_i) \leq cp(G) + s_i - 2 \leq cp(G) + n - 2$, which implies that $s_i = n$.

□

In [2] it is proved that if $cc(G) = cc(G - e)$ for all edges e, then $|E(G)| > 2cc(G)$. Brigham and Dutton give an example of a graph on nine vertices having this property. If there is a graph G having the property that $cp(G) = cp(G - e)$ for all edges e, then $|E(G)| \geq 3cp(G)$. So far we have no examples of such graphs. It seems that the removal of an edge from a small graph usually results in either a decrease or an increase in the clique partition number. The graph G of Figure 1 has $cp(G) = 10$ ([4],Theorem 1). The minimum clique partition of G is composed of 10 triangles. Let e be the edge indicated in Figure 1. Let d_4 denote the number of K_4's in a minimum clique partition of G. By examining the cases $d_4 = 0, 1, 2$ or 3 it is evident that $cp(G - e) = 10$. A minimum clique partition of $G - e$ uses three K_4's, two K_3's and five K_2's. The graph G has ten edges for which $cp(G - e) = cp(G)$. For the remaining edges of G, $cp(G - e) = 11$ (nine K_3's and two K_2's, for example).

For an example of a graph G such that $cp(G - e) < cp(G)$ for all edges e of G, one need only choose a triangle-free graph. Another class of graphs having this property consists of the wheels, W_n, where n is even and $n \geq 6$. A graph of this type consists of a vertex which is adjacent to each vertex of a cycle, C_{n-1}. For n even and $n \geq 6$, $cp(W_n) = n$ and $cp(W_n - e) = n - 1$ for all e in W_n. If G is a graph with the property $cp(G - e) < cp(G)$ for all $e \in E(G)$, it is necessary and sufficient that for each e there is a minimum

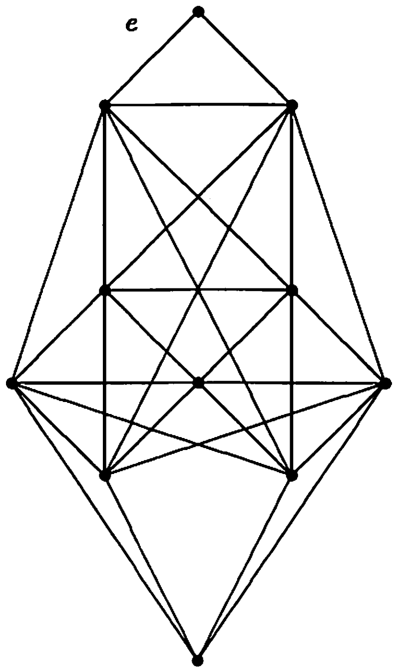

Figure 1. $cp(G - e) = cp(G)$

clique partition of G in which e occurs as a 2–clique (Theorem 4.3). For the clique covering case, Brigham and Dutton [2] conjecture that the only graphs with the property $cc(G - e) < cc(G)$ for all edges e are the triangle-free graphs.

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