An Upper Bound for the Depth-r Interval Number of the Complete Graph

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ABSTRACT. A t-interval representation f of a graph G is a function which assigns to each vertex $v \in V(G)$ a union of at most t closed intervals $I_{v,1}, I_{v,2}, \dots, I_{v,t}$ on the real line so that $f(v) \cap f(w) \neq \emptyset$ if and only if $v, w \in V(G)$ are adjacent. If no real number lies in more than r intervals of the representation, we say, the interval representation has depth r. The least positive integer t for which exists a t–representation of depth r of G is called the depth–r interval number $i_r(G)$. E. R. Scheinerman proved that $i_2(K_n) = \lceil \frac{n}{2} \rceil$ for $n \geq 2$ and that $\left\lceil \frac{n-1}{2(r-1)} + \frac{r}{2n} \right\rceil \leq i_r(K_n) \leq n/(2r-2) + 1 + 2\lceil \log_r n \rceil$. In the following paper we will see by construction that $i_3(K_n) = \left\lceil \frac{n-1}{4} + \frac{3}{2n} \right\rceil$. If $n \geq 5$ this is equal to $\lceil \frac{n}{4} \rceil$. The main theorem is if $n \geq r \geq 4$ then $i_r(K_n) \leq \max \left\{ \left\lceil \frac{n-2}{2(r-1)} + \frac{1}{2} \right\rceil , 2 \right\}$. The difference between the lower and the upper bounds is at most 1.

1 Introduction

The least positive integer t for which exists a t–representation of G is called the interval number $i(G)$. Many authors have studied the interval number, see [1], [3], [5], [6], [7], [8], [12] and [13]. Applications of interval representation, e.g. scheduling, molecular evolution and archeology, can be found in [2], [4] and [10]. The depth–r interval number has been studied by C. Maas [9] and E. R. Scheinerman [11]:

Theorem 1.1 [11] *The depth–2 interval number of the complete graph K_n with $n \geq 2$ is given by $i_2(K_n) = \lceil n/2 \rceil$.* □

Lemma 1.2 [9] *Given $t \geq r$ intervals on the real line $I_1, I_2, \ldots, I_t$. If the intersection of $(r+1)$ of them is always empty then there are at most $(t - \frac{1}{2}r)(r-1)$ pairs $\{I_i, I_j\}$ of pairwise intersecting intervals.* □

Theorem 1.3 *If G has n vertices, $m \geq \frac{1}{2}r(r-1)$ edges and $r \geq 2$ then: $i_r(G) \geq \left\lceil \frac{m + \frac{1}{2}r(r-1)}{(r-1)\cdot n} \right\rceil$*

Proof. Applying lemma 1.2 we deduce that we need at least $\frac{m+\frac{1}{2}r(r-1)}{(r-1)}$ intervals to represent the m edges of G. Thus the depth–r interval number has to be greater than (number of intervals)/(number of vertices) and must be an integer. □

This lower bound for the depth–r interval number is tight, e.g. for the complete Graph and $r = 3$, but there are also graphs with depth–r interval number greater than this bound. Now it's easy to obtain a lower bound for the depth–r interval number of the complete graph K_n:

Corollary 1.4 [11] *Let K_n be the complete graph with $n \geq r \geq 2$. Then:*

$$i_r(K_n) \geq \left\lceil \frac{n-1}{2(r-1)} + \frac{r}{2n} \right\rceil$$

Proof. Note that K_n has $m = \frac{1}{2}n(n-1)$ edges and apply lemma 1.3. □

2 The main theorem and the idea

If $n \leq r$ there are no interesting results about the depth–r interval number of the K_n. E. R. Scheinerman [11] investigated the case $r = 2$ and showed that for general r $i_r(K_n) \leq \frac{n}{2(r-1)} + 2\lceil \log_r n \rceil + 1$. It is also possible to show an exact result, when $r = 3$. The main theorem of this paper refers to $n > r \geq 3$. The idea is to think that the number of intervals, which are assigned to each vertex, is fixed. We make a distiction between two cases: $i_r(K_n) \leq 2$ in lemma 3.1 and $i_r(K_n) \leq t$ with $t \geq 3$ in lemma 3.2. It follows that:

Theorem 2.1 *If $n > r \geq 3$ then*

$$\begin{aligned} \left\lceil \frac{n-1}{2(r-1)} + \frac{r}{2n} \right\rceil \leq i_r(K_n) &\leq \max\left\{ \left\lceil \frac{n-2}{2(r-1)} + \frac{1}{2} \right\rceil, 2 \right\} \\ &= \max\left\{ \left\lceil \frac{n+r-3}{2(r-1)} \right\rceil, 2 \right\} \end{aligned}$$

If $n > r+1$ then the upper bound is $\left\lceil \frac{n-2}{2(r-1)} + \frac{1}{2} \right\rceil = \left\lceil \frac{n+r-3}{2(r-1)} \right\rceil$

Proof. First we show the upper bound:

By lemma 3.1 and lemma 3.2 is shown: for $r \geq 3$ and $n^* = (2t-1)(r-1)+2$ that $i_r(K_{n^*}) \leq t$, if $t \geq 2$. In particular $t = \max\left\{\left\lceil \frac{n-2}{2(r-1)} + \frac{1}{2}\right\rceil,\ 2\right\}$ then

$$\begin{aligned} n^* &= \max\left\{\left(2\left\lceil\frac{n-2}{2(r-1)}+\frac{1}{2}\right\rceil - 1\right)(r-1)+2,\ 3(r-1)+2\right\} \\ &\geq \left(2\left(\frac{n-2}{2(r-1)}+\frac{1}{2}\right)-1\right)(r-1)+2 = n \end{aligned}$$

The K_n is an induced subgraph of the K_n^*, as $n^* \geq n$, and therefore:

$$\begin{aligned} i_r(K_n) \leq i_r(K_{n^*}) \leq t &= \max\left\{\left\lceil\frac{n-2}{2(r-1)}+\frac{1}{2}\right\rceil,\ 2\right\} \\ &= \max\left\{\left\lceil\frac{n+r-3}{2(r-1)}\right\rceil, 2\right\} \end{aligned}$$

If $n > r+1$ then $\frac{n+r-3}{2(r-1)} > \frac{r+r-2}{2(r-1)} = 1$ and $\lceil\frac{n+r-3}{2(r-1)}\rceil \geq 2$, thus $\max\{\lceil\frac{n-2}{2(r-1)}+\frac{1}{2}\rceil,\ 2\} \geq 2$. The lower bound is shown in corollary 1.4. Clearly, the difference between the bounds is at most 1. □

Consider the case with $r = 3$ as fixed:

Corollary 2.2 *If $n \geq 2$ then: $i_3(K_n) = \lceil\frac{n-1}{4}+\frac{3}{2\cdot n}\rceil$ for $n \geq 5$ this is equivalent to:*

$$i_3(K_n) = \left\lceil\frac{n}{4}\right\rceil$$

Proof. Let $n = 2$ and $n = 3$ so clearly $i_3(K_n) = 1$. Applying theorem 2.1 to the K_4 we get: $i_3(K_4) = 2$. For $n > 4$ the upper bound of theorem 2.1 is $\lceil\frac{n+3-3}{2\cdot 2}\rceil = \lceil\frac{n}{4}\rceil$. The lower bound is $\lceil\frac{n}{4}+\frac{3}{2n}-\frac{1}{4}\rceil$. If $0 < \frac{3}{2n} \leq \frac{1}{4}$, or $n \geq 6$, this is equal to $\lceil\frac{n}{4}\rceil$. For $n = 5$, $\lceil\frac{n}{4}+\frac{3}{2n}-\frac{1}{4}\rceil = \lceil\frac{n}{4}\rceil$ also holds . This completes the proof and we have determined exactly the depth–3 interval number of the complete K_n. □

To prove the lemmas, used in the proof of the main theorem, we need some new terminology. These are presented in the next section together with the constructions of depth–r interval representations.

3 Terminology and proofs

For integers $r \geq 2$ an r–level n–chain denotes an arrangement of n closed intervals $I_1, I_2, \ldots, I_n$ with the following properties: i) $I_i \cap I_j \neq \emptyset \Leftrightarrow (|j-i| \leq r-1)$ $(1 \leq i, j \leq n)$, where no intersection is a single point.

ii) The left endpoint of I_1 is lower than the left endpoint of I_n.

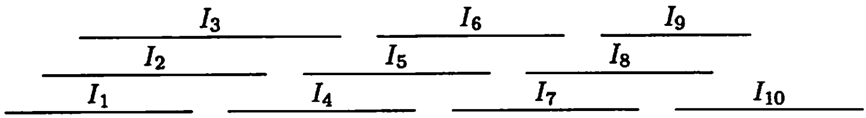

Figure 1: 3-level 10-chain

This arrangement guarantees, that every real number lies in at most r intervals. Figure 1 shows an example.

For $r \geq 3$ let an $(r-1)$–level n–chain of intervals $I_1, I_2, \ldots, I_n$ be given. Let k be an integer with $1 \leq k \leq n$. If in the following description an interval I_x does not exist, we expand the $(r-1)$–level n chain to a $(r-1)$–level $n^\star$–chain and choose an $n^\star$ so large, that I_x exists. After the construction we delete the additional intervals $I_{n+1}, \ldots, I_{n^\star}$.

We define an interval of type1(k) as a small, closed interval which intersects only with $I_k, I_{k+1}, \ldots, I_j$ where $j = \min\{k+r-1, n\}$. The left endpoint of an interval of type1(k) lies between the left endpoint of I_{k+r-2} and the right of I_k. The right endpoint of an interval of type1(k) lies between the left endpoint of I_{k+r-1} and the right of I_{k+1}. In Figure 2 the interval A is an example for an interval of type1(k).

We say that a closed interval which only intersects with $I_k, I_{k+1}, \ldots, I_j$ where $j = \min\{k+r-2, n\}$ is an interval of type2(k). An interval of type2(k) lies between I_k and I_{k+r-2}. Examples are the intervals B in Figure 2.

Finaly we define an interval of type3(k) for $k \geq 2$ as a closed interval which only intersects with $I_k, I_{k+1}, \ldots, I_j$ where $j = \min\{k+r-3, n\}$. It lies between I_{k+r-2} and I_{k-1}. Examples are the intervals C in Figure 2.

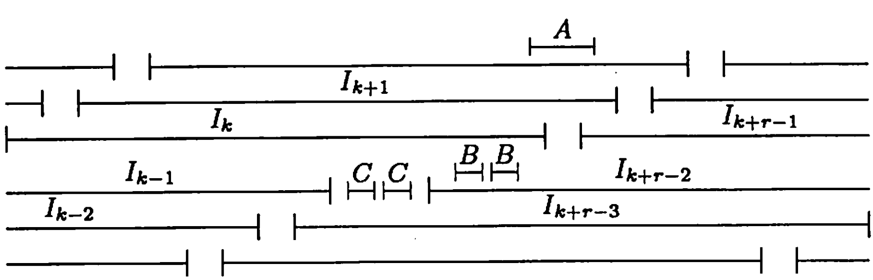

Figure 2: A part of a $(r-1)$–level n–chain with intervals of type1(k), type2(k) and type3(k)

N.B.: Let an $(r-1)$–level n–chain $I_1, \ldots, I_n$ be given. For $1 \leq k \leq n$

it is possible to add intervals A_k of type1(k), so that $A_i \cap A_j = \emptyset$ ($1 \leq i,j \leq n$; $i \neq j$). Additionally it is also legimate to put a finite number of distinct intervals of type2($k+1$) between two intervals of type1(k) and of type1($k+1$), and every real number lies within, at most, r intervals.

Finally we can place a finite number of closed, pairwise disjoint intervals of type3($k+1$) between I_k and I_{k+r-1}. All of these does not intersect with A_k and every real number lies in at most r intervals.

Let f be an interval representation with intervals described in the abstract. We say $I_{v,i}$ and $I_{w,j}$ represent the edge $\{v,w\}$ if $I_{v,i} \cap I_{w,j} \neq \emptyset$. Let $\{v_1,\dots,v_n\}$ always denotes the vertices of the K_n. For $j \in$ IN we let denote the j–successor of the vertex v_i the vertex v_k with $k = i+j$ and $k \leq n$.

To present examples for the constructions, used in the lemmas, we introduce a new graphical method, to describe a t–interval representation. Figure 3 shows two equivalent graphical illustrations. An expression under a vertical line ($D_{i,j}, \dots D_{k,l}, E_{m,n}, \dots E_{o,p}$) denotes an interval of type1($\cdot$), which intersects with the same intervals as the vertical line. An expression between two vertical lines and above the r–level n–chain (F_1, F_2) denotes an interval of type2($\cdot$), also intersecting with intervals, which intersecting with the part of the real line from the left to the right vertical line. Figures 4, 5 and 6 show examples of the construction, one for lemma 3.1 and two for lemma 3.2 respectively. In this figures we use the same notations for the intervals as in the proofs above.

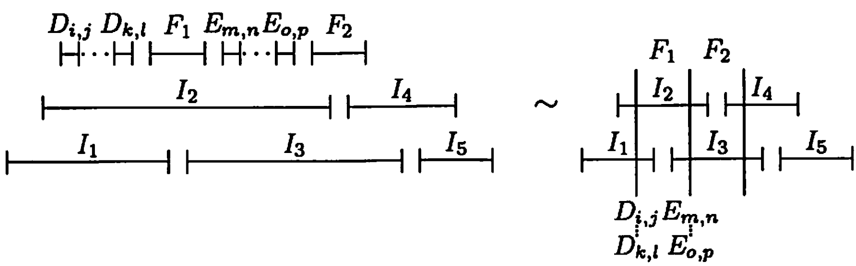

Figure 3: Two equivalent illustrations

Using these preliminaries, we can now construct the depth–r interval representations to complete the proof of our main theorem.

Lemma 3.1 *If $r \in$ IN, $r \geq 3$ and $n = 3(r-1)+2$. Then:*

$$i_r(K_n) \leq 2$$

Be aware, that this is analog to Lemma 3.2 with $t = 2$.

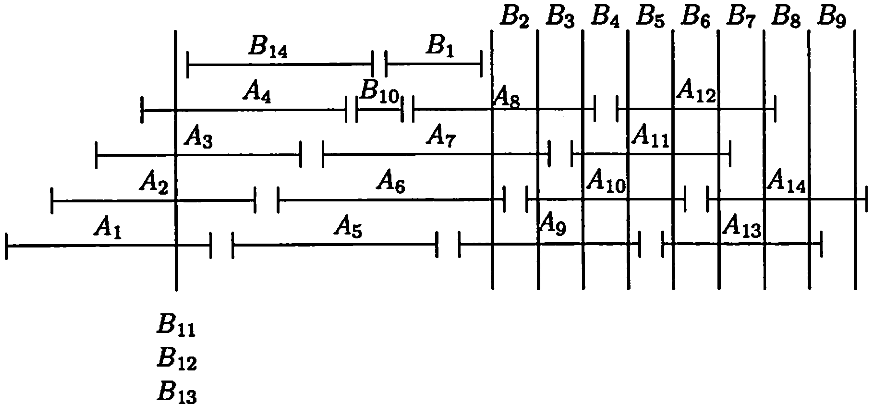

Figure 4: Example for $i_5(K14) \leq 2$

Proof. (by construction)

Figure 4 shows an example of this construction.

step 1: We construct an $(r-1)$–level n–chain of intervals A_i $(1 \leq i \leq n)$ and assign to each vertex v_i the interval A_i. Hence all edges $\{v_i, v_j\}$ with $i+1 \leq j \leq i+r-2 \leq n$ are represented $(1 \leq i \leq n)$.

step 2: B1: Choose B_1 as a closed interval with its left endpoint between the right endpoint of A_{r-1} and the left endpoint of $A_{2(r-1)}$; the right endpoint of B_1 lies between the left endpoint of $A_{2(r-1)+1}$ and the right endpoint of A_{r+1}. B_1 intersects with the same intervals of the $(r-1)$–level n–chain as an interval of type1$(1+(r-1))$, but the left endpoint of B_1 is lower than the left endpoint of $A_{2(r-1)}$. For $(2 \leq i \leq n-r)$ let B_i be an interval of type $1(i+(r-1))$.

B2: For $i = n-r+1 = 2(r-1)+2$ choose closed intervals B_i with left endpoint between the right endpoint of A_{r-1} and the left endpoint of B_1; the right endpoint of B_i lies between the left endpoint of B_1 and the left endpoint of $A_{2(r-1)}$. B_i is similar to an interval of type3(r), which intersects additionally with B_1.

B3: For $(n-r+2 \leq i \leq n-1)$ let B_i be an interval of type2(1).

B4: B_n should intersect only with A_l $(1 \leq l \leq 2(r-1)-1)$ and with B_{n-r+1}. The left endpoint of B_n lies between the righthand limit of $B_{n-r+2}, \ldots, B_{n-1}$ and the right endpoint of A_1. The right endpoint of B_n lies between the left endpoint of B_{n-r-1} and the left endpoint of B_1.

We assign to each vertex v_i the interval B_i $(1 \leq i \leq n)$. By this construction the following edges are represented:

(B1) For $(1 \leq i \leq n)$ the edges $\{v_i, v_j\}$ $(i+r-1 \leq j \leq k)$ with $k = \min\{i + 2(r-1), n\}$ are represented.

(B2) For $(r \leq i \leq 2(r-1)-1)$ and $i = 1$ the edges $\{v_{2(r-1)+2}, v_i\}$.

(B3) For $(1 \leq i \leq r-1)$ the edges $\{v_i, v_j\}$ with $n-r+2 = 2(r-1)+3 \leq j \leq n-1$.

(B4) For $(1 \leq i \leq 2(r-1)-1)$ and $i = n-r+1$ the edges $\{v_i, v_n\}$.

We assigned 2 intervals to each vertex of the K_n and clearly each real number lies in at most 2 of the intervals of the representation. As in lemma 3.2 it is possible to show, that all edges are represented. □

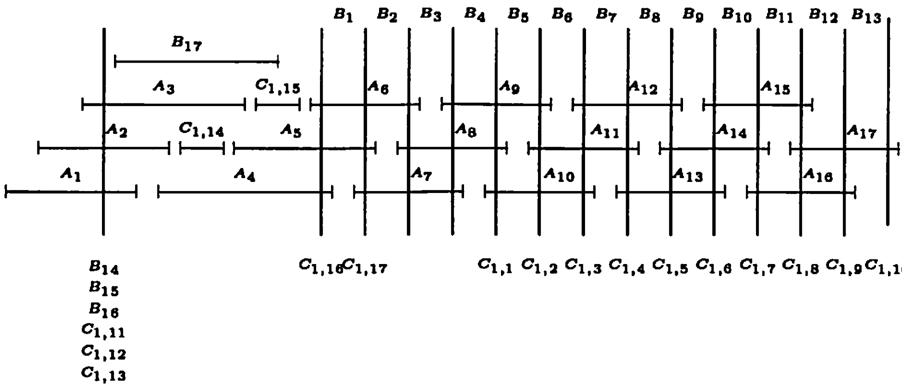

Figure 5: Example for $i_4(K_{17}) \leq 3$

Lemma 3.2 *Let* $r, t \in I\!N, r \geq 3,\ t \geq 3$ *and* $n = (2t-1)(r-1)+2$, *then:*

$$i_r(K_n) \leq t$$

Proof. (by construction)

Figures 5 and 6 show examples of this construction.

step 1: We construct an $(r-1)$–level n–chain of intervals A_i $(1 \leq i \leq n)$ and assign to each vertex v_i the interval A_i.

(A1) The edges between v_i with $1 \leq i \leq n-1$ and the j–successors $(1 \leq j \leq (r-2))$ are represented.

step 2: B1: For $(1 \leq i \leq n-r)$ let B_i be an interval of type1$(i+(r-1))$.

B2: For $(n-r+1 \leq i \leq n-1)$, or $((2t-2)(r-1)+2 \leq i \leq n-1)$ let B_i be an interval of type2(1).

B3: B_n should intesect only with A_l $(1 \leq l \leq 2(r-1)-1)$ and with B_{n-r+1}. The left endpoint of B_n lies between the right-most point of

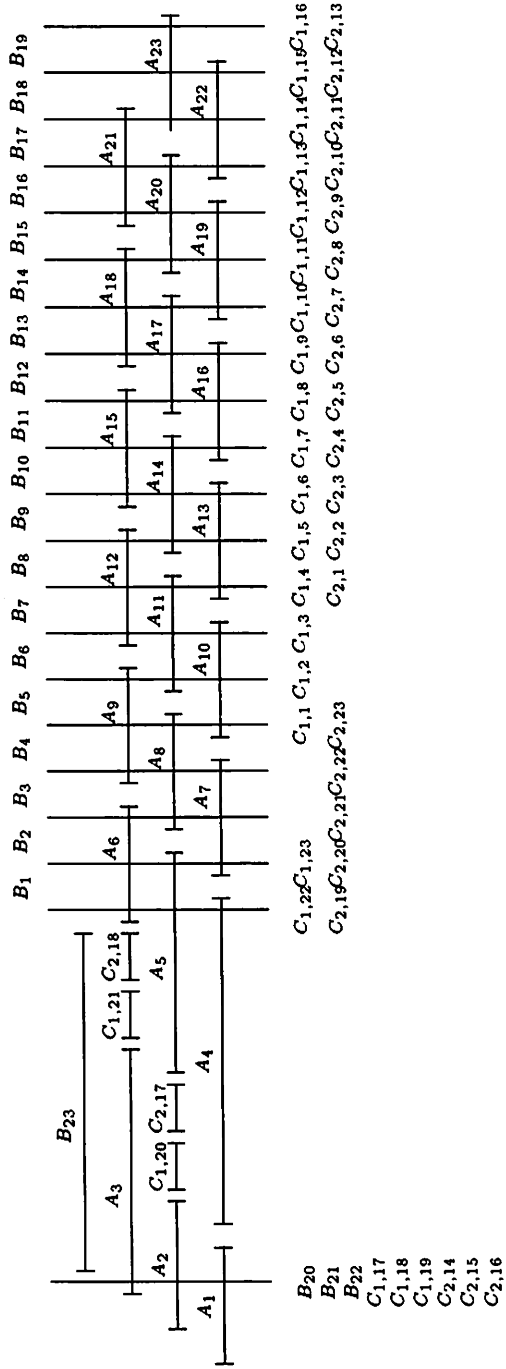

Figure 6: Example for $i_4(K_{23}) \leq 4$

$B_{n-r+1}, \dots, B_{n-1}$ and the right endpoint of A_1. The right endpoint of B_n lies between the right endpoint of A_{r-1} and the left endpoint of $A_{2(r-1)}$.

We assign to each vertex v_i the interval B_i. By this construction the following edges are represented:

(B1) For $(1 \leq i \leq n-r)$ the edges between v_i and the j–successors with $r-1 \leq j \leq 2(r-1)$. i.e. for all i all edges are represented between v_i and the j–successors $(r-1 \leq j \leq 2(r-1))$ with exception of the edge $\{v_{n-(r-1)}, v_n\}$.

(B2) For $(1 \leq i \leq r-1)$ the edges $\{v_i, v_j\}$ with $(2t-2)(r-1)+2 \leq j \leq n-1$.

(B3) For $(1 \leq i \leq 2(r-1)-1)$ the edges $\{v_i, v_n\}$.

step 3: We still have $t-2$ intervals per vertex as yet unallocated. Now we will assign $t-2$ intervals to each vertex. We denote them using $C_{l,i}$ $(1 \leq l \leq t-2)$ $(1 \leq i \leq n)$. The index i indicates the vertex, to which we will assign this interval.

C1: For $(1 \leq l \leq t-2)$ and $(1 \leq i \leq n-(l+1)(r-1)-1)$ let $C_{l,i}$ be an interval of type2$(i+(l+1)(r-1)+1)$.

C2: For $(1 \leq l \leq t-2)$ let the next $r-1$ intervals $C_{l,i}$ with $n-(l+1)(r-1) \leq i \leq n-(l+1)(r-1)+r-2 = n-l(r-1)-1$ be intervals of type2(1). i.e. for $(1 \leq l \leq t-2)$ and $((2t-l-2)(r-1)+2 \leq i \leq (2t-l-1)(r-1)+1)$ let $C_{l,i}$ be an interval of type2(1).

C3: For $(1 \leq l \leq t-2)$ let the next $r-2$ intervals $C_{l,i}$ with $i = (2t-l-1)(r-1)+2+j$ with $0 \leq j \leq r-3$ be of type3$(3+j)$, which also intersect with B_n.

C4: For $(1 \leq l \leq t-2)$ and $((2t-l)(r-1)+1 \leq i \leq n)$, i.e. for $i = (2t-l)(r-1)+1+j$ with $0 \leq j \leq (l-1)(r-1)+1$ and $(1 \leq l \leq t-2)$ let $C_{l,i}$ be an interval of type2$(r+j)$.

For $(1 \leq l \leq t-2)$ and $(1 \leq i \leq n)$ we assign the intervals $C_{l,i}$ to each vertex v_i. Hence the following edges are represented:

(C1) Between v_i with $1 \leq i \leq n-(l+1)(r-1)-1$ and the j–successors with $(l+1)(r-1)+1 \leq j \leq (l+2)(r-1)$ and $(1 \leq l \leq t-2)$. I.e. for all i the edges between v_i and the j–successors are represented by $(l+1)(r-1)+1) \leq j \leq (l+2)(r-1)$ and $(1 \leq l \leq t-2)$.

As the upper bound+1 with $l = l^*$ equals the lower bound with $l = l^*+1$, all edges between v_i and the j–successors $2(r-1)+1 \leq j \leq t(r-1))$ are represented.

(C2) For $(1 \leq i \leq r-1)$ the edges $\{v_i, v_j\}$ with $(2t-l-2)(r-1)+2 \leq j \leq (2t-l-1)(r-1)+1$ and $(1 \leq l \leq t-2)$. As the lower bound with $l = l^*$ equals the upper bound+1 with $l = l^*+1$, all edges between v_i and v_j with $((2t-(t-2)-2)(r-1)+2 = t(r-1)+2 \leq j \leq (2t-1-1)(r-1)+1 = (2t-2)(r-1)+1$ are represented.

(C3) An interval A_k intersects with an interval of type3$(3+j)$ if $k-r+3 \leq 3+j \leq k$. For $(1 \leq l \leq t-2)$ $C_{l,i}$ with $i = (2t-l-1)(r-1)+2+j$ and $(0 \leq j \leq r-3)$ is an interval of type3$(3+j)$. So A_k intersects with $C_{l,i}$ if

i) $i = (2t-l-1)(r-1)+2+j$ and ii) $0 \leq j \leq r-3$ and iii) $k-r+3 \leq 3+j \leq k$ i.e. $k-r \leq j \leq k-3$ and iv) $1 \leq l \leq t-2$ and v) $1 \leq k \leq n$.

i) and ii) can be rephrased as i) $(2t-l-1)(r-1)+2 \leq i \leq (2t-l-1)(r-1)+2+r-3 = (2t-l)(r-1)$.

i) and iii) can be rephrased as ii) $(2t-l-1)(r-1)+2+k-r = (2t-l-2)(r-1)+1+k \leq i \leq (2t-l-1)(r-1)+2+k-3 = (2t-l-1)(r-1)-1+k$.

As i must fulfill $(2t-l-1)(r-1)+2 \leq i$ and $i \leq (2t-l-1)(r-1)-1+k$, k fulfills $k \geq 3$, hence i fulfills $i \leq (2t-l)(r-1)$ and $(2t-l-2)(r-1)+1+k \leq i$, $k \leq 2(r-1)-1$. Thus for $3 \leq k \leq 2(r-1)-1$ and $(1 \leq l \leq t-2)$ the edges between v_k and the j-successors with $(2t-l-2)(r-1)+1 \leq j \leq (2t-l-1)(r-1)-1 = n-l(r-1)-3$ are represented if the index of these vertices is between $(2t-l-1)(r-1)+2$ and $(2t-l)(r-1) = n-(l-1)(r-1)-2$.

Additionally the edges $\{v_i, v_n\}$ with $(2t-l-1)(r-1)+2 \leq i \leq (2t-l)(r-1)$ and $(1 \leq l \leq t-2)$ are represented. In particular $(l=1)$ the edge between v_k with $k = (2t-2)(r-1)+2 = n-r+1$ and v_n is represented.

(C4) For $(1 \leq k \leq n)$ an interval A_k intersects with an interval of type2$(r+j)$, if $k-(r-2) \leq r+j \leq k$, i.e. $k-2r+2 \leq j \leq k-r$. Be aware that $C_{l,i}$ is for $(0 \leq j \leq (l-1)(r-1)+1)$ and $(1 \leq l \leq t-2)$ with $i = (2t-l)(r-1)+1+j$ an interval of type2$(r+j)$, so A_k intersects with $C_{l,i}$, if i) $i = (2t-l)(r-1)+1+j$ ii) $0 \leq j \leq (l-1)(r-1)+1$ iii) $k-2r+2 \leq j \leq k-r$ iv) $1 \leq l \leq t-2$ v) $1 \leq k \leq n$.

As $0 \leq j$ and $j \leq k-r$ $k \geq r$, and as $i = (2t-l)(r-1)+1+j \leq n$ and $k-2r+2 \leq j$, k has to fulfill $(2t-l)(r-1)+1+k-2r+2 \leq n$, i.e. $k \leq (l+1)(r-1)+1$. We can now reformulate condition v) to : v) $r \leq k \leq (l+1)(r-1)+1$. From ii) and iii) we develop $\max\{0, k-2r+2\} \leq j \leq \min\{(l-1)(r-1)+1, k-r\}$ and with i) we imply that $(2t-l)(r-1)+1+\max\{0, k-2r+2\} \leq i \leq (2t-l)(r-1)+1+\min\{(l-1)(r-1)+1, k-r\}$.

To summerize, A_k intersects with $C_{l,i}$, if i) $(2t-l)(r-1)+1+\max\{0, k-2r+2\} \leq i \leq (2t-l)(r-1)+1+\min\{(l-1)(r-1)+1, k-r\} = \min\{n, (2t-l)(r-1)+1+k-r\} = \min\{n, (2t-l-1)(r-1)+k\}$ ii) $r \leq k \leq (l+1)(r-1)+1$ iii) $1 \leq l \leq t-2$

First let us consider the case $r \leq k \leq 2r-3$: $\max\{0, k-2r+2\} = 0$ and $\min\{n, (2t-l-1)(r-1)+k\} \leq (2t-l-1)(r-1)+2r-3 = (2t-l+1)(r-1)-1 < n$, hence condition i) is equivalent to: i) $(2t-l)(r-1)+1 \leq i \leq (2t-l-1)(r-1)+k$ i.e. for $(1 \leq l \leq t-2)$ and $(r \leq k \leq 2r-3 = 2(r-1)-1)$ the edges between v_k and the j-successors with $j \leq (2t-l-1)(r-1)$ are represented, if the index of these vertices is greater than or equal to $(2t-l)(r-1)+1$.

Now consider the case $2r-2 \le k \le (l+1)(r-1)+1$:

$\max\{0, k-2r+2\} = k-2r+2$ so condition i) is equivalent to:

i) $(2t-l)(r-1)+1+k-2r+2 = (2t-l-2)(r-1)+1+k \le i \le \min\{n, (2t-l-1)(r-1)+k\}$ i.e. for $2r-2 \le k \le (l+1)(r-1)+1$ and $(1 \le l \le t-2)$ the edges between v_k and the j–successors with $(2t-l-2)(r-1)+1 \le j \le (2t-l-1)(r-1)$ are represented.

Think of $k \ge 2r-2$ as fixed and choose l^* with $l^*(r-1)+1 < k \le (l^*+1)(r-1)+1$. Consequently, for all l with $l^* \le l \le t-2$ $2r-2 \le k \le (l+1)(r-1)+1$, and therefore the edges between v_k and the j–successors $(2t-l-2)(r-1)+1 \le j \le (2t-l-1)(r-1)$ $(l^* \le l \le t-2)$ are represented. The lower bound with $l = l^\times$ is equal to the upper bound+1 with $l = l^\times + 1$ and $(l^* \le l \le t-2)$. This implies that all edges between v_k and the j–successors with $(2t-(t-2)-2)(r-1)+1 = t(r-1)+1 \le j \le (2t-l^*-1)(r-1)$ are represented. The $((2t-l^*-1)(r-1))$–successor of v_k has index $k+(2t-l^*-1)(r-1) > l^*(r-1)+1+(2t-l^*-1)(r-1) = (2t-1)(r-1)+1 = n-1$. As for $2r-2 \le k \le n$ the edges between v_k and all j–successors with $j \ge t(r-1)+1$ are represented. By defining the type of intervals, we used for the construction, it is clear, that each real number lies in at most r intervals of the t–representation, therefore it remains to be proved that all edges are represented by intersecting intervals:

From (A1), (B1), (C3) and (C1) we can immediately conclude, that for $(1 \le i \le n)$ all edges between v_i and the j–successorrs with $1 \le j \le t(r-1)$ are represented. We will now see that the edges between the vertices v_i and j–successors with $t(r-1)+1 \le j \le n-i$ for $1 \le i \le n-t(r-1)-1 = (t-1)(r-1)+1$ are represented:

Consider the vertices v_i with $1 \le i \le r-1$: (C2), (B2) and (B3) imply that the vertices between v_i and v_j with $t(r-1)+2 \le j \le n$ are represented.

Consider the vertex v_r: In (C3) we saw that for $1 \le l \le t-2$ the edges between v_r and the j–successors with $(2t-l-2)(r-1)+1 \le j \le (2t-l-1)(r-1)-1$ are represented, if the index is greater than or equal to $(2t-l-1)(r-1)+2$ and lower than or equal to $(2t-l)(r-1)$. Both of these conditions hold. Due to (C4) for $1 \le l \le t-2$ the edges between v_r and the j–successors with $j \le (2t-l-1)(r-1)$ are represented, if the index of these vertices is greater than or equal to $(2t-l)(r-1)+1$ (the $((2t-l-1)(r-1))$–successor fulfill this). To summerize it is shown that the edges between v_r and the j–successors $(2t-l-2)(r-1)+1 \le j \le (2t-l-1)(r-1)$, i.e. $(2t-(t-2)-2)(r-1)+1 = t(r-1)+1 \le j \le (2t-1-1)(r-1) = (2t-2)(r-1)$ are represented. The $((2t-2)(r-1))$–successor of vertex v_r is v_{n-1}. Due to (B3) the edge $\{v_r, v_n\}$ is also represented.

Consider the vertices v_i with $r+1 \le i \le 2(r-1)-1$: In (C3) we saw, that for $1 \le l \le t-2$ the edges between v_i and the j–successors with $(2t-l-2)(r-1)+1 \le j \le (2t-l-1)(r-1)-1$ are represented, if the index is greater than or equal to $(2t-l-1)(r-1)+2$ and lower or equal to $(2t-l)(r-1)$. Due to (C4) for $1 \leq l \leq t-2$ the edges between v_i and the j–successors with $j \leq (2t-l-1)(r-1)$ are represented, if the index of these vertices is greater or equal to $(2t-l)(r-1)+1$ (this is held by the $((2t-l-1)(r-1)-1)$–successors). Therefore the edges between v_i and the j–successors $(2t-l-2)(r-1)+1 \leq j \leq (2t-l-1)(r-1)$, i.e. $(2t-(t-2)-2)(r-1)+1 = t(r-1)+1 \leq j \leq (2t-1-1)(r-1) = (2t-2)(r-1)$ are represented. The index of the $((2t-2)(r-1))$–successors of the vertices v_i are greater than $(2t-2)(r-1)+r+1 = n$. Hence, all edges, which incident with the vertices v_i, are represented.

Consider the vertices v_i with $2(r-1) \leq i$: From (C4) it immediately follows that the edges between v_i and the j–successors with $j \geq t(r-1)+1$ are represented.

Now we have shown, that all edges of the K_n are represented by intersecting intervals. To each vertex t intervals are assigned and this representation has depth r. □

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