The Rotation Index Of A Graph

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ABSTRACT. A *rooted graph* is a pair (G, x), where G is a simple undirected graph and $x \in V(G)$. If G is rooted at x, its *k-th rotation number* $h_k(G, x)$ is the minimum number of edges in a graph F of order $|G| + k$ such that for every $v \in V(F)$ we can find a copy of G in F with the root vertex x at v; any such F of size $h_k(G, x)$ is called a *minimal graph*. In this paper we prove that if (G, x) is a rooted graph with $d_G(x) = d$ then

$$\rho(G, x) = \lim_{k \to \infty} \frac{h_k(G, x)}{k}$$

exists and satisfies $d/2 \leq \rho(G, x) \leq d$, where $\rho(G, x)$ is termed the *rotation index* of (G, x). We obtain the precise value of this parameter for several classes of rooted graphs and describe the asymptotic behaviour of corresponding minimal graphs.

1 Introduction

A *rooted graph* is a pair (G, x), where G is an undirected graph without loops or multiple edges, and x is any vertex of G. If (G, x) and (F, y) are rooted graphs, then (G, x) is a *rooted subgraph* of (F, y) if there is an injection f from $V(G)$ to $V(F)$ such that $f(x) = y$ and $f(a)f(b) \in E(F)$ whenever $ab \in E(G)$. This property is denoted by $(G, x) < (F, y)$; we call (G, x) a *homogeneous rooted subgraph* of (F, y) if $(G, x) < (F, y)$ for each $y \in V(F)$, and write $(G, x) < F$.

Let k be a non-negative integer. The *k-th rotation number* $h_k(G, x)$ is the smallest number of edges in a graph F of order $|G| + k$ such that $(G, x) < F$.

Thus informally, the parameter $h_k(G,x)$ is the minimum size of a graph of order $|G|+k$ such that for every $v \in V(F)$ we can find a copy of G in F with the root vertex x at v. Following the convention of [3], [9]–[11], any graph F of order $|G|+k$ with the property $(G,x) < F$ is called a *feasible graph* for $h_k(G,x)$. A feasible graph of minimum size is termed a *minimal graph* for $h_k(G,x)$, whilst the complement of a minimal graph is an *extremal graph* for $h_k(G,x)$.

Note that when $k = 0$ the definition given becomes that of the *rotation number* (in which case we abbreviate $h_0(G,x)$ to $h(G,x)$), which was introduced in [5] and determined for various different types of graphs in [1], [2], [4], [6]–[12]. Generalized rotation numbers were introduced in [3] and calculated for complete bipartite graphs $G = K_{n,r}$ with $n \geq r+k$ and root x in the vertex class of order n, thereby extending theorems of [1], [9], [10].

As indicated above, previous research in this area has focussed upon bounding $h_k(G,x)$ for specific types of rooted graphs, and classifying the associated minimal graphs (or the complementary extremal graphs), for values of $k < |G|$. However, if k takes values outside this range, then in some cases we can expect the behaviour of $h_k(G,x)$, and the related minimal graphs, to differ considerably from that described in earlier work. For instance, if (G,x) is connected and $k < |G|$, then any minimal graph for $h_k(G,x)$ is connected also, as each vertex therein must appear in a subgraph isomorphic to G. Evidently the same argument does not apply for $k \geq |G|$, an observation which prompted the current investigation, since it suggests that for larger k there may be disconnected minimal graphs for $h_k(G,x)$.

The aim of this paper is to examine the growth of $h_k(G,x)$ as $k \to \infty$ for any rooted graph (G,x). In the next section we obtain general upper and lower bounds for the parameter

$$\rho(G,x) = \lim_{k\to\infty} \frac{h_k(G,x)}{k}.$$

We term this quantity the *rotation index* of (G,x), and illustrate our study by computing the exact value of this parameter for several types of rooted graphs featured in related papers (Theorems 4–7). The constructive nature of the proof techniques yields structural information about the corresponding minimal graphs.

2 General Results

Let (G,x) be a rooted graph. In this section we prove a basic result (Theorem 3) which describes the asymptotic behaviour of $h_k(G,x)$. Our proofs are achieved via an examination of the properties of feasible graphs for $h_k(G,x)$. We start with the following elementary lemma.

Lemma 1. *If H is any feasible graph, then for all $k \geq |H| - |G|$,*

$$h_k(G,x) \leq \alpha + k\frac{e(H)}{|H|}$$

for some constant α.

Proof: Suppose H is feasible. For each $i = 1, \ldots, |H|-1$ choose a further feasible graph F_i of order $|H|+i$, and let $F_0 = H$. For $k \geq |H| - |G|$, we have $|G| + k = q|H| + i$, where $q \geq 1$ and $0 \leq i \leq |H| - 1$. The graph F which is the disjoint union of F_i and $q-1$ copies of H is feasible for $h_k(G,x)$, and so

$$\begin{aligned} h_k(G,x) &\leq e(F) \\ &= e(F_i) + (q-1)e(H) \\ &= e(F_i) + \left[\frac{(|G|+k-i)}{|H|} - 1\right] e(H) \\ &\leq e(F_i) + |G|\frac{e(H)}{|H|} + k\frac{e(H)}{|H|}. \end{aligned}$$

Thus $h_k(G,x) \leq \alpha + k\frac{e(H)}{|H|}$ where $\alpha = \max_i\{e(F_i)\} + |G|\frac{e(H)}{|H|}$. □

Corollary 2. *Let H be a feasible graph for $h_k(G,x)$; then*

$$\lim_{k\to\infty} \sup \frac{h_k(G,x)}{k} \leq \frac{e(H)}{|H|}.$$

Proof: It is easily seen from Lemma 1 that $h_k(G,x)/k \leq \alpha/k + e(H)/|H|$, which implies the result. □

Theorem 3. *Let (G,x) be a rooted graph with $d_G(x) = d$; then the rotation index*

$$\rho(G,x) = \lim_{k\to\infty} \frac{h_k(G,x)}{k}$$

exists and satisfies $d/2 \leq \rho(G,x) \leq d$.

Proof: Clearly $\lim_{k\to\infty} \inf h_k(G,x)/k = \lim_{k\to\infty} \inf h_k(G,x)/(|G|+k)$. Note that if $\psi > \lim_{k\to\infty} \inf h_k(G,x)/(|G|+k)$ then there is a feasible graph H with $e(H)/|H| < \psi$. By Corollary 2, we have $\lim_{k\to\infty} \sup h_k(G,x)/k \leq e(H)/|H| < \psi$, so $\lim_{k\to\infty} \sup h_k(G,x)/k \leq \lim_{k\to\infty} \inf h_k(G,x)/k$ and hence the limit exists.

Now if H is feasible, each vertex has degree at least d, so $e(H) \geq (|G| + k)d/2$, and thus $\rho(G,x) \geq d/2$. On the other hand, if H is feasible for

$h_k(G,x)$ then we can construct a graph H' which is feasible for $h_{k+1}(G,x)$, with $e(H') = e(H) + d$, as follows. Choose from H a rooted subgraph (G', x') isomorphic to (G,x), then let $V(H') = V(H) \cup \{v'\}$ and $E(H') = E(H) \cup \{vv'; v \in \Gamma_{G'}(x')\}$. Obviously H' is feasible, and by induction $h_k(G,x) \leq h_0(G,x) + kd$, so $\rho(G,x) \leq d$. □

3 Particular Examples

The general upper and lower bounds for $\rho(G,x)$ obtained in Theorem 3 are employed heavily in this section, where we focus on particular classes of rooted graphs from the references, and determine the exact values of the associated rotation indices.

Our proof strategy involves deriving a lower bound for the size of any feasible graph for $h_k(G,x)$, and then citing one with precisely this number of edges (or one with size tending asymptotically to this number). The first example arose from work of [9] and [10].

Theorem 4. *If $G = K_{n,r}$, where $n \leq r$ and x is any vertex of the n-set, then*

$$\rho(G,x) = \frac{r}{2}.$$

Proof: Each vertex of a feasible graph for $h_k(G,x)$ has degree at least r, so by Theorem 3 we have $\rho(G,x) \geq r/2$. In addition the graph $K_{r,r}$ is feasible for $h_{r-n}(G,x)$, so

$$\rho(G,x) \leq \frac{e(K_{r,r})}{|K_{r,r}|} = \frac{r}{2},$$

which proves the result. □

The rooted graphs of Theorem 5 were investigated in [7].

Theorem 5. *If $G = K_{1,n} \cup K_{1,m}$, where $n > m$ is even and x is the centre of $K_{1,n}$, then*

$$\rho(G,x) = \frac{n}{2}.$$

Proof: Each vertex of a feasible graph for $h_k(G,x)$ has degree at least n, so Theorem 3 implies $\rho(G,x) \geq n/2$.

By Theorem 2.1.1 from [7], $h(G,x) = n(n+m+2)/2$. An example of a corresponding minimal graph $\tilde{H}$ is the n-regular graph with vertex set $V = \{0, 1, \ldots, n+m+1\}$ and edge set

$$E = \left\{ij;\ i - \frac{n}{2} \leq j \leq i + \frac{n}{2}\right\}$$

with addition mod$(n+m+2)$. Denote the *closed neighbourhood* of a vertex $i \in V$ by $\overline{\Gamma}(i) = \{i\} \cup \{j \in V; ij \in E\}$, and write $\tilde{H}[X]$ for the subgraph of

$\tilde{H}$ induced by $X \subseteq V$. To see that $\tilde{H}$ is feasible for $h(G,x)$, observe that $\tilde{H}[\overline{\Gamma}(i)]$ contains a copy of $K_{1,n}$ centred at i, whilst $\tilde{H}[V - \overline{\Gamma}(i)]$ contains a copy of $K_{1,m}$ centred at $i + \lceil (n+m+2)/2 \rceil$. We have $|\tilde{H}| = n+m+2$ and $e(\tilde{H}) = n(n+m+2)/2$, so

$$\rho(G,x) \leq \frac{e(\tilde{H})}{|\tilde{H}|} = \frac{n}{2},$$

and the theorem is proved. □

It is a straightforward matter to adapt the proof of Theorem 5 to yield a similar result for the case when n is odd. The next theorem extends work on rotation numbers from [8].

Theorem 6. *If $G = C_3 \cup C_{n-3}$ $(n > 6)$ and x is a vertex of C_3, then*

$$\rho(G,x) = 1.$$

Proof: Each vertex of a feasible graph for $h_k(G,x)$ has degree at least two, so $\rho(G,x) \geq 1$ by Theorem 3. Furthermore, it is clear that we can create a feasible graph for $h_k(G,x)$ from the disjoint union of a feasible graph for $h(G,x)$ and a feasible graph for $h_{k-3}(C_3,x)$.

Now any minimal graph for $h(G,x)$ has at most $(3n+4)/2$ edges (see [8], Theorems 1–5). For example, if $n \equiv 0 \pmod 6$ then we can construct a minimal graph $\hat{H}$ of size $3n/2$ as follows. Take a set of $m = n/3$ vertices $\{0,\ldots,m-1\}$, and construct a 3-regular graph by adding edges

$$\{0(m/2)\} \cup \{i(m-i);\ 1 \leq i \leq m/2-1\} \cup \{i(i+1)(\bmod m);\ 0 \leq i \leq m-1\}.$$

Finally replace each vertex with a C_3 such that each vertex therein is adjacent to a different neighbouring edge of the original, so the resultant graph $\hat{H}$ is 3-regular and thus of the required size. To see that $\hat{H}$ is feasible for $h(G,x)$, each vertex appears in a C_3 corresponding to one of the original m vertices, whilst it is easily checked that in all cases the graph induced on the remaining $n-3$ vertices is hamiltonian.

Also, it is obvious that any minimal graph for $h_{k-3}(C_3,x)$ has size k if $k \equiv 0 \pmod 3$ and size $k+1$ otherwise. Writing $k = 3\alpha + \beta$ $(0 \leq \beta \leq 2)$, the graph comprising the disjoint union of $(\alpha - 1)$ C_3 and a minimal graph for $h_\beta(C_3,x)$ attains this bound. Hence we have

$$\rho(G,x) \leq \lim_{k\to\infty} \frac{(3n+4)/2 + (k+1)}{n+k} = 1,$$

which achieves the result. □

It should be apparent that Theorem 6 can be generalized to any rooted graph of the type $G = C_3 \cup G_{n-3}$ (n fixed, $n > 6$), where G_{n-3} is any graph of order $n-3$ and x is a vertex of C_3. For it is always possible to form a feasible graph for $h_k(G,x)$ from the disjoint union of a minimal graph for $h(G,x)$ and a minimal graph for $h_{k-3}(C_3,x)$. Since K_n is feasible for $h(G,x)$ for any rooted graph (G,x) of order n, then

$$1 \le \rho(G,x) \le \lim_{k\to\infty} \frac{\binom{n}{2}+k+1}{n+k} = 1.$$

In contrast to the examples above, our final class of rooted graphs (G,x) with $d_G(x)=d$ satisfy $\rho(G,x) > d/2$; they were studied in [3].

Theorem 7. *If $G = K_{n,r}$, where $n > r$ and x is any vertex of the n-set, then*

$$\rho(G,x) = r\left(1-\frac{r}{2n}\right).$$

Proof: Let H be a feasible graph for $h_k(G,x)$. Consider every copy of $K_{n,r}$ contained by H as a subgraph, and denote by S the set of vertices of H which appear in at least one of the independent r-sets. Write $s=|S|$. We proceed by deriving two different lower bounds for $e(H)$.

Each vertex of S has degree at least n, whilst each vertex of $V(H)-S$ has degree at least r, whence

$$e(H) \ge s\frac{n}{2} + (n+r+k-s)\frac{r}{2} = \frac{1}{2}\left[s(n-r)+r(n+r+k)\right] = f_1(s).$$

In addition, each vertex of S has degree at least r in S, whilst those of $V(H)-S$ have at least r neighbours in S, so

$$e(H) \ge s\frac{r}{2} + (n+r+k-s)r = \frac{r}{2}\left[2(n+r+k)-s\right] = f_2(s).$$

Therefore $e(H) \ge \min_s \max\{f_1(s), f_2(s)\}$. Since the minimum occurs when $f_1(s)=f_2(s)$, eliminating s between these two functions gives $ne(H) \ge r(2n-r)(n+r+k)/2$, so $e(H)/(n+r+k) \ge r(1-r/2n)$ and

$$\rho(G,x) \ge r\left(1-\frac{r}{2n}\right).$$

We establish equality for this last lower bound by constructing the following feasible graph H^*. Let $k=n-r$. Let V be a set of $n+r+k=2n$ vertices, and for disjoint sets $X,Y \subseteq V$, let $K_{X,Y}$ denote the complete bipartite graph with independent sets X and Y. To form H^*, choose $N_1,N_2 \subseteq V$ such that $|N_1|=|N_2|=n$ and $V=N_1\cup N_2$. Then take $R_1 \subseteq V-N_1 \subseteq N_2$ with $|R_1|=r$, and $R_2\subseteq V-N_2 \subseteq N_1$ with $|R_2|=r$; then $V(H^*)=V$ and

$$E(H^*) = K_{N_1,R_1} \cup K_{N_2-R_1,R_2} = K_{N_2,R_2} \cup K_{N_1-R_2,R_1}.$$

We have $|H^*| = 2n$ and $e(H^*) = r(2n - r)$, so

$$\rho(G,x) \leq \frac{e(H^*)}{|H^*|} = r\left(1 - \frac{r}{2n}\right),$$

which completes the proof. □

4 Concluding Remarks

1. The proof of Lemma 1 shows that if $h_{k^*}(G,x)/(|G| + k^*) = \rho(G,x)$ for some finite integer k^*, then for $k \geq k^*$, writing $|G| + k = \alpha(|G| + k^*) + \beta$ $(0 \leq \beta \leq |G| + k^* - 1)$, we can construct a feasible graph for $h_k(G,x)$ from the disjoint union of $\alpha - 1$ copies of a minimal graph for $h_{k^*}(G,x)$ and a minimal graph for $h_{k^*+\beta}(G,x)$. Clearly in the case $\beta = 0$ this is simply α disjoint copies of a minimal graph for $h_{k^*}(G,x)$, which satisfies $h_k(G,x)/(|G| + k) = \alpha h_{k^*}(G,x)/\alpha(|G| + k^*) = \rho(G,x)$ and is therefore minimal. Applying this observation to examples from Section 3 indicates the asymptotic behaviour of corresponding minimal graphs for $h_k(G,x)$.

2. In the light of Theorem 7, a natural question to ask is whether there are other obvious examples of rooted graphs (G,x) with $d_G(x) = d$ satisfying $\rho(G,x) > d/2$. Our examination of related papers has failed to yield any. It remains an interesting open problem to find classes of rooted graphs with this property.

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