A Solution of Dudeney's Round Table Problem for $p^e q^f + 1$

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ABSTRACT. A solution of Dudeney's round table problem is given when n is as follows:
(i) $n = pq + 1$, where p and q are odd primes.
(ii) $n = p^e + 1$, where p is an odd prime.
(iii) $n = p^e q^f + 1$, where p and q are distinct odd primes satisfying $p \geq 5$ and $q \geq 11$, and e and f are natural numbers.

1 Introduction

In 1905, Dudeney [2, problem 273] proposed the Round Table Problem as follows:

"Seat the same n persons at a round table on $(n-1)(n-2)/2$ occasions so that no person shall ever have the same two neighbours twice. This is, of course, equivalent to saying that every person must sit once, and only once, between every possible pair."

The problem proposed by Dudeney is equivalent to asking for a set of Hamilton cycles in the complete graph K_n with the property that every path of length two (2-path) lies on exactly one of the cycles. We call such a set of cycles in K_n a Dudeney set in K_n and denote it by $D(n)$. Clearly $|D(n)| = (n-1)(n-2)/2$.

A Dudeney set in K_n has been constructed when $n = p+1$ (p is prime), $n = 2p$ (p is prime), $n = p^k + 1$ (p is prime), $n = p+2$ (p is prime and 2 is a generator of the multiplicative subgroup of $GF(p)$), and some sporadic cases: $n = 11, 19, 23, 43$ [1,3,5,6]. It is also known that if there is a Dudeney set in K_{n+1}, then there is a Dudeney set in K_{2n}, where $n \geq 2$ [3].

In this paper, we construct a Dudeney set in K_n, where either
(i) $n = pq + 1$, where p and q are odd primes,
(ii) $n = p^e + 1$, where p is an odd prime and e is a natural number, or
(iii) $n = p^e q^f + 1$, where p and q are distinct odd primes satisfying $p \geq 5$ and $q \geq 11$, and e and f are natural numbers.

We note that Dudeney sets in K_n, where $n = p^e + 1$, are constructed in the above (ii) and [6] by different methods.

After we submitted this paper, a Dudeney set in K_n, where n is even, has been constructed in [4] by using the above (ii).

2 Preliminaries

Let p and q be odd prime numbers and e and f be natural numbers. Put $n = p^e q^f + 1 = m + 1$. We denote by $K_n = (V, E)$ the complete graph on n vertices, where $V = \{0, 1, 2, ..., m-1\} \cup \{\infty\} = Z_m \cup \{\infty\}$ is the vertex set (Z_m is the set of integers modulo m), and $E = \{\{a, b\} | a, b \in V, a \not\equiv b \ (mod\ m)\}$ is the edge set.

For each integer x, $0 \leq x \leq m-1$, we define the 1-factor:

$$F_x = \{\{a, b\} \in E \mid a + b \equiv 2x \pmod{m}\} \cup \{\{x, \infty\}\}.$$

We consider the cycle structure of $F_0 \cup F_x$ ($1 \leq x \leq m-1$). The following lemma is easy to prove.

2.1 Lemma([7],p168)
(1) *If* $(x, m) = 1$, *then* $F_0 \cup F_x$ *is a Hamilton cycle of* K_{m+1}.
(2) *If* $(x, m) = m_1 \neq 1$, *then, putting* $m_2 = m/m_1$, *the cycle structure of* $F_0 \cup F_x$ *is*

$$F_0 \cup F_x = C_0 \cup C_1 \cup C_2 \cup \cdots \cup C_{(m_1-1)/2},$$

where the length of C_0 *is* $m_2 + 1$ *and the length of* C_i $(1 \leq i \leq (m_1 - 1)/2)$ *is* $2m_2$.

Using the notation of Lemma 2.1 it is not difficult to see that

$$V(C_0) = \{a \in V \mid a \equiv 0 \ (mod\ m_1)\} \cup \{\infty\}$$

and that we may choose C_i so that

$$V(C_i) = \{a \in V \mid a \equiv \pm i \ (mod\ m_1)\}.$$

Let $V_i = V(C_i)$, $0 \le i \le (m_1 - 1)/2$. For convenience, we extend the subscripts of C and V to all integers:

$$C_i = C_j \text{ (if } i + j = m_1\text{)}, \ C_{i+km_1} = C_i \text{ (for any integer } k\text{)},$$

$$V_i = V_j \text{ (if } i + j = m_1\text{)}, \ V_{i+km_1} = V_i \text{ (for any integer } k\text{)}.$$

We next explain what we mean by exchanging edges between two 1-factors. Let H_1 and H_2 be 1-factors of K_n such that $H_1 \cup H_2$ is not hamiltonian. Let $H_1 \cup H_2$ have cycle structure:

$$H_1 \cup H_2 = C_\alpha \cup C_\beta \cup C_\gamma \cup \cdots \cup C_\omega,$$

and let C be the union of a proper subset of $\{C_\alpha, C_\beta, C_\gamma, \ldots, C_\omega\}$; that is, C is the union of some of the cycles of $H_1 \cup H_2$. We exchange edges of H_1 and H_2 via C to obtain the two new 1-factors H_1' and H_2':

$$H_1' = (H_1 \setminus C) \cup (H_2 \cap C) \ and$$

$$H_2' = (H_2 \setminus C) \cup (H_1 \cap C).$$

3 Exchanging edges

Suppose x is an integer, $(x, m) = m_1 \neq 1$, and $1 \le x \le (m-1)/2$. Then $F_x \cup F_{-x}$ is not a Hamilton cycle. This is easily seen by observing that $F_x \cup F_{-x} \cong F_{2x} \cup F_0$. Thus we have the cycle structure

$$F_x \cup F_{-x} = C_0' \cup C_1' \cup \cdots \cup C_{(m_1-1)/2}',$$

where $V(C_i') = V(C_i) = V_i$, using the notation of Lemma 2.1. Let k be any integer with $1 \le k \le (m_1 - 1)/2$. Let $C_{i_1}', C_{i_2}', \ldots, C_{i_k}'$ be k distinct cycles of $F_x \cup F_{-x}$ not containing the vertex ∞; that is, not equal to C_0'. Put $D = C_{i_1}' \cup C_{i_2}' \cup \cdots \cup C_{i_k}'$. We exchange edges of F_x and F_{-x} via D to obtain the new 1-factor $F_x(D)$:

$$F_x(D) = (F_x \setminus D) \cup (F_{-x} \cap D) \subseteq F_x \cup F_{-x}.$$

Note that when we exchange via D we are only exchanging edges in $V_{i_1}' \cup V_{i_2}' \cup \cdots \cup V_{i_k}'$.

3.1 Proposition *Let $1 \le x \le (m-1)/2$, $(x, m) = m_1 \neq 1$ and $1 \le k \le (m_1 - 1)/2$. Put $m_2 = m/m_1$. Let D be the union of any k distinct cycles* *of $F_x \cup F_{-x}$ not containing the vertex ∞. If an integer y, $1 \le y \le m-1$, satisfies $y \equiv x - 4kx/m_1 \pmod{m_2}$ and $(y, m_1) = 1$, then $F_x(D) \cup F_y$ is not hamiltonian and the length of the cycle containing ∞ is $m_1 + 1$.*

Proof. Note that $\{\infty, x\} \in F_x(D)$. We consider the cycle C_∞ of $F_x(D) \cup F_y$ which contains the vertex ∞. Thus

$$C_\infty = (\infty, x, 2y - x = z_1, w_1, z_2, w_2, \ldots, z_t, w_t = y, \infty)$$

where $\{\infty, x\}, \{z_1, w_1\}, \{z_2, w_2\}, \ldots, \{z_t, w_t\} \in F_x(D)$, $\{x, z_1\}, \{w_1, z_2\}, \{w_2, z_3\}, \ldots, \{w_{t-1}, z_t\}, \{y, \infty\} \in F_y$, and the length of the cycle C_∞ is $2(t+1)$. We have $\infty, x \in V_0$, $z_1, w_1 \in V_{2y}$, $z_2, w_2 \in V_{4y}, \ldots$; in general for $1 \le j \le t$, $z_j, w_j \in V_{2jy}$. Note that to guarantee $w_t = y \in V_y$ we must have $V_{2ty} = V_y$ or $2t \equiv \pm 1 \ (mod\ m_1)$. So we can assume $t \ge (m_1 - 1)/2$.

For $j, 1 \le j \le (m_1 - 1)/2$, the vertex sets V_{2jy} are disjoint. This follows as if $V_{2iy} \cap V_{2jy} \ne \phi, 1 \le i < j \le (m_1 - 1)/2$, then $2iy \equiv 2jy \ (mod\ m_1)$ or $2iy \equiv -2jy \ (mod\ m_1)$. Since $(m_1, 2y) = 1$, we have $i \equiv j \ (mod\ m_1)$ or $i \equiv -j \ (mod\ m_1)$, which contradicts $1 \le i < j \le (m_1 - 1)/2$. Therefore exactly k edges among

$$\{z_1, w_1\}, \{z_2, w_2\}, \ldots, \{z_{(m_1-1)/2}, w_{(m_1-1)/2}\}$$

belong to F_{-x} (as the exchange uses k cycles), and the other edges belong to F_x. Hence

$$w_{(m_1-1)/2} \equiv y + m_1(x - y) - 4kx \equiv y \quad (mod\ m_1 m_2).$$

Thus we have verified that $t = (m_1 - 1)/2$ and the length of C_∞ is $2(t+1) = m_1 + 1$. □

We next exchange the edges of $F_x(D)$ and F_y via the cycle C_∞ (as defined in the preceding proposition), to obtain the two 1-factors

$$F_{x,y}(D) = (F_x(D) \setminus C_\infty) \cup (F_y \cap C_\infty) \quad and$$

$$F^*_{x,y}(D) = (F_y \setminus C_\infty) \cup (F_x(D) \cap C_\infty).$$

Note that $\{\infty, y\} \in F_{x,y}(D)$ and $\{\infty, x\} \in F^*_{x,y}(D)$.

3.2 Proposition *Let $1 \le x \le (m-1)/2$, $(x, m) = m_1 \ne 1$ and $1 \le k \le (m_1 - 1)/2$. Put $m_2 = m/m_1$. Let D be the union of k distinct cycles of $F_x \cup F_{-x}$ not containing the vertex ∞. Let y be any integer with $1 \le y \le m-1$, $y \equiv x - 4kx/m_1 \pmod{m_2}$ and $(y, m_1) = 1$. Then $F_0 \cup F_{x,y}(D)$ is a Hamilton cycle of K_{m+1}.*

Proof. Recall that $F_{x,y}(D)$ is obtained by exchanging the edges of $F_x(D)$ and F_y via the cycle C_∞,

$$C_\infty : (\infty, x, 2y - x = z_1, w_1, z_2, w_2, \ldots, z_{(m_1-1)/2}, w_{(m_1-1)/2} = y, \infty),$$

where $\infty, x \in V_0$ and $z_i, w_i \in V_{2iy} (1 \leq i \leq (m_1 - 1)/2)$ and the length of C_∞ is $m_1 + 1$.

$F_0 \cup F_x(D)$ has cycle structure

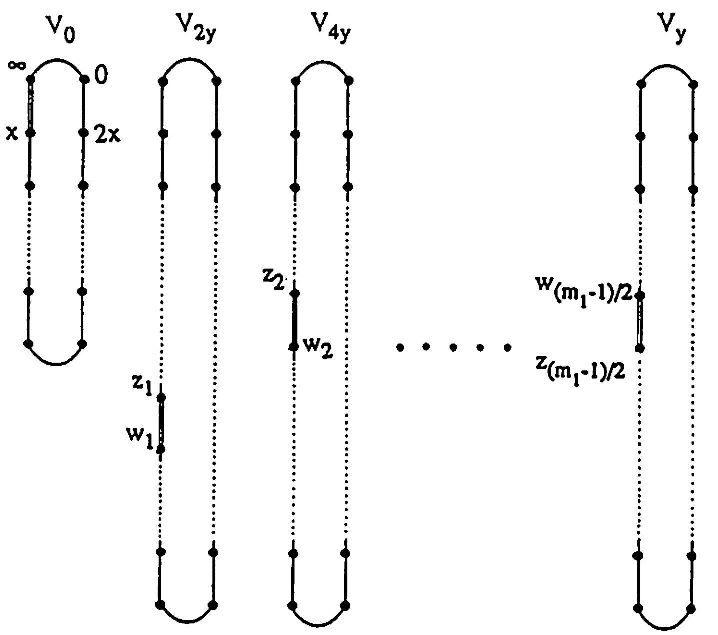
$$F_0 \cup F_x(D) = C_0'' \cup C_1'' \cup \cdots \cup C''_{(m_1-1)/2},$$

where $V(C_i'') = V_i$. (To see this consider $F_0 \cup F_x$, $F_0 \cup F_{-x}$ and $F_x \cup F_{-x}$.)

We consider the cycle C of $F_0 \cup F_{x,y}(D)$ which contains the edge $\{\infty, 0\}$ and determine that

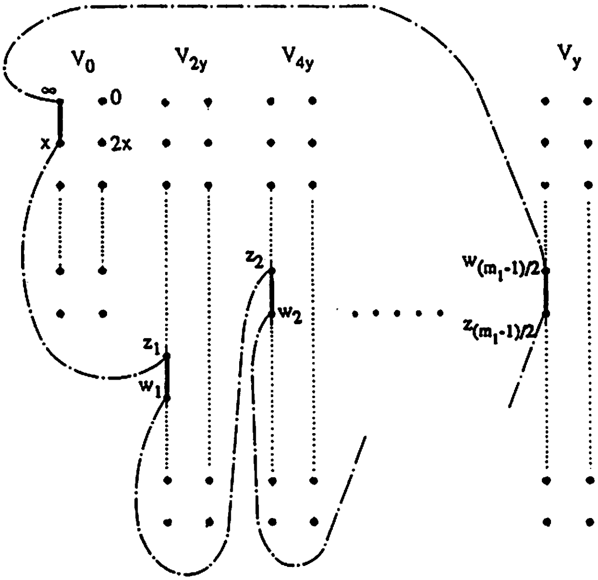
$$C = (\underbrace{\infty, 0, 2x, \ldots, x}_{V_0}, \underbrace{z_1, \ldots, w_1}_{V_{2y}}, \underbrace{z_2, \ldots, w_2}_{V_{4y}}, \ldots, \underbrace{z_{(m_1-1)/2}, \ldots, w_{(m_1-1)/2}}_{V_y}, \infty)$$

which is a Hamilton cycle (Figures 1,2,3). □



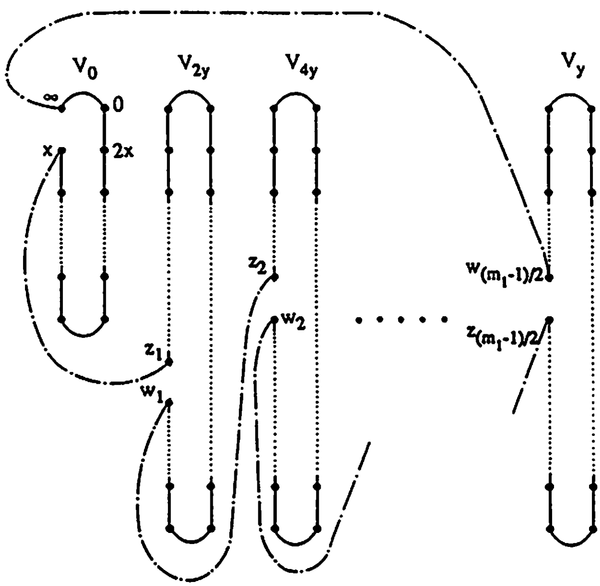
——, ▬▬ are edges of $F_0 \cup F_x(D)$

Figure 1. $F_0 \cup F_x(D)$



—·—, ▬▬ are edges of C_∞

Figure 2. C_∞



——, —·— are edges of $F_0 \cup F_{x,y}(D)$

Figure 3. $F_0 \cup F_{x,y}(D)$

4 Hamiltonicity of $F_0 \cup F^*_{x,y}(D)$

Assuming all notation from the previous sections, we now assign the special value of k and the cycles of D as follows. Since $1 \le x \le (m-1)/2, (x,m) = m_1 \ne 1$ and $m_2 = m/m_1$, we can put $x = im_1$ $(1 \le i \le (m_2-1)/2, (i,m_2) = 1)$.

Put $k = (m_1 + (-1)^{(m_1+1)/2})/4$. Choose $y, 1 \le y \le m-1$ so that $y \equiv (-1)^{(m_1-1)/2} i \ (mod\ m_2)$ and $(y, m_1) = 1$. Since $(i, m_2) = 1, (y, m_2) = 1$ and so $(y, m) = 1$. Therefore $F_0 \cup F_y$ is a Hamilton cycle. Put

$$D = C'_{2y} \cup C'_{6y} \cup C'_{10y} \cup \cdots \cup C'_{(4k-2)y} \quad \text{(a union of } k \text{ cycles).}$$

4.1 Proposition *Assuming all notation and parameters as given above, $F_0 \cup F^*_{x,y}(D)$ is a Hamilton cycle of K_{m+1}.*

Proof. Note that $F^*_{x,y}(D)$ is obtained from F_y by exchanging edges with $F_x(D)$ via the cycle C_∞:

$$C_\infty = (\infty, x, 2y - x = z_1, w_1, z_2, w_2, \ldots, z_t, w_t = y, \infty),$$

where $t = (m_1 - 1)/2$, $\infty, x \in V_0$ and $z_i, w_i \in V_{2iy}(1 \le i \le t)$. By the definition of D, we have

$$\begin{aligned} w_{i-1} + z_i &\equiv 2y \ (mod\ m), \\ z_i + w_i &\equiv (-1)^i 2x \ (mod\ m), \end{aligned}$$

for $i = 1, 2, \ldots, t$, where $w_0 = x$. Inductively, we find

$$\begin{aligned} z_i &\equiv 2iy + (-1)^i x \ (mod\ m), \\ w_i &\equiv -2iy + (-1)^i x \ (mod\ m), \end{aligned}$$

for $i = 1, 2, \ldots, t$. Also

$$w_{i+1} \equiv -z_i - 2y \ (mod\ m),$$

for $i = 1, 2, \ldots, t-1$. Therefore, we have in $F_0 \cup F^*_{x,y}(D)$ the cycle:

$$\begin{aligned} C = (\infty, \quad x, \quad & -x, \quad x + 2y, \quad w_1, \quad z_1, \\ & -z_1, \quad z_1 + 2y, \quad w_2, \quad z_2, \\ & \cdots \\ & \cdots \\ & -z_{t-1}, \quad z_{t-1} + 2y, \quad w_t, \quad z_t, \quad P, \quad \infty), \end{aligned}$$

where P is a path in $F_0 \cup F_y$. The path from ∞ via x to z_t contains all the vertices of the cycle C_∞. Suppose $F_0 \cup F^*_{x,y}(D)$ is not a Hamilton cycle. Then there exists a cycle C' not containing any vertex of C_∞. So every edge of C' is in $F_0 \cup F_y$, but this contracicts the fact that $F_0 \cup F_y$ is a Hamilton cycle. The proof is finished. □

5 Construction of a Dudeney set in K_n

We are now ready to construct Dudeney sets in K_n, where $n = p^e q^f + 1 = m + 1$. Put

$$X = \{x \mid (x, m) \neq 1 \text{ and } 1 \leq x \leq (m-1)/2\}.$$

For each $x \in X$ we will specify k_x, y_x and D_x which will be used to obtain 1-factors $F_{x,y_x}(D_x)$ and $F^*_{x,y_x}(D_x)$. For each $x \in X$, let $(x, m) = m_x \neq 1$, let $m'_x = m/m_x$, and let $x = im_x$, where $1 \leq i \leq (m'_x - 1)/2$.

We determine k_x as

$$k_x = (m_x + (-1)^{(m_x+1)/2})/4.$$

Substituting these values into the congruence equation of Proposition 3.1, we obtain

$$y_x \equiv (-1)^{(m_x-1)/2} i \pmod{m'_x}.$$

In order to employ Proposition 3.1, it is necessary that, as well as satisfying these conditions, y_x be chosen so that

$$1 \leq y_x \leq m - 1, \text{ and} \tag{5.2}$$

$$(y_x, m_x) = 1. \tag{5.3}$$

(Note again that $(y_x, m) = 1$.)

Once we have determined y_x, we will specify D_x as follows:

$$D_x = C'_{2y_x} \cup C'_{6y_x} \cup C'_{10y_x} \cup \cdots \cup C'_{(4k_x-2)y_x}.$$

What remains then, is to determine y_x for each $x \in X$. (Recall that as yet y_x could take on many values.)

Let H be a subset of $Z^*_m = Z_m \setminus \{0\}$. We call H a half-set modulo m if $H \cap (-H) = \phi$ and $H \cup (-H) = Z^*_m$. If H is a subset of Z^*_m such that $H \cap (-H) = \phi$, then, putting $Z = \{1, 2, \ldots, (m-1)/2\} \setminus (H \cup (-H))$, we obtain a half-set $H \cup Z$. In this situation, we say H can be extended by Z to a half-set.

Let σ be the vertex-permutation $\sigma = (0\ 1\ 2\ \ldots\ m-1)(\infty)$, and put $\Sigma = \{\sigma^j \mid 0 \leq j \leq m-1\}$. Clearly σ induces a permutation of the edges of K_{m+1}, and we will also denote this permutation by σ. When $\mathcal{H}$ is a set of 2-factors of K_{m+1}, we define $\Sigma\mathcal{H} = \{H^\tau \mid H \in \mathcal{H}, \tau \in \Sigma\}$.

5.1 Theorem *Assume we have an injection from X into Z^*_m, $x \to y_x$, such that y_x satisfies conditions* (5.1), (5.2) *and* (5.3), *$X \cap Y = \phi$, where $Y = \{y_x \mid x \in X\}$, and $X \cup Y$ can be extended by Z to a half-set in Z^*_m; so*

$X \cup Y \cup Z$ is a half-set modulo m. Put

$$\begin{aligned} \mathcal{H}_1 &= \{F_0 \cup F_{x,y_x}(D_x) \mid x \in X\}, \\ \mathcal{H}_2 &= \{F_0 \cup F^*_{x,y_x}(D_x) \mid x \in X\}, \\ \mathcal{H}_3 &= \{F_0 \cup F_z \mid z \in Z\}, \text{ and} \\ \mathcal{H} &= \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3. \end{aligned}$$

Then $\Sigma\mathcal{H}$ is a Dudeney set of Hamilton cycles in K_n.

Proof. All cycles of $\mathcal{H}$ are Hamilton by Propositions 3.2 and 4.1, and Lemma 2.1 and hence so too are the cycles of $\Sigma\mathcal{H}$. We next verify that every 2-path of K_n lies on precisely one of the cycles in $\Sigma\mathcal{H}$.

Every 2-path of K_n lies on exactly one cycle of

$$\Sigma\{F_0 \cup F_i \mid 1 \le i \le (m-1)/2\}.$$

It is not difficult to see that

$$\Sigma\{F_0 \cup F_w \mid w \in X \cup Y \cup Z\} = \Sigma\{F_0 \cup F_i \mid 1 \le i \le (m-1)/2\}$$

since $X \cup Y \cup Z$ is a half-set and $F_0 \cup F_x = \sigma^x(F_0 \cup F_{-x})$. Note that

$$\begin{aligned} & \Sigma\{F_0 \cup F_w \mid w \in X \cup Y \cup Z\} \\ = \; & \Sigma\{F_0 \cup F_x \mid x \in X\} \cup \Sigma\{F_0 \cup F_{y_x} \mid x \in X\} \cup \Sigma\{F_0 \cup F_z \mid z \in Z\}. \end{aligned}$$

Since $F_x(D_x)$ is obtained from F_x by exchanging edges of F_x with those of F_{-x} in cycles $C_i' \in D$ of $F_x \cup F_{-x}$ (C_i' is given in §3) and $\Sigma(C_i) = \Sigma(C_i^*)$, where C_i is (as given in §2) the cycle of $F_0 \cup F_x$ on vertex set V_i and C_i^* is the cycle of $F_0 \cup F_{-x}$ on vertex set V_i, we have

$$\Sigma\{F_0 \cup F_x(D_x) \mid x \in X\} = \Sigma\{F_0 \cup F_x \mid x \in X\}.$$

As $F_{x,y_x}(D_x)$ and $F^*_{x,y_x}(D_x)$ are obtained by exchanging edges between $F_x(D_x)$ and F_{y_x}, we have

$$F_{x,y_x}(D_x) \cup F^*_{x,y_x}(D_x) = F_x(D_x) \cup F_{y_x}.$$

Hence the 2-paths in $\{F_0 \cup F_{x,y_x}(D_x), F_0 \cup F^*_{x,y_x}(D_x)\}$ are the same as those in $\{F_0 \cup F_x(D_x), F_0 \cup F_{y_x}\}$. So the 2-paths in $\mathcal{H}_1 \cup \mathcal{H}_2$ are the same as those in $\{F_0 \cup F_x(D_x) \mid x \in X\} \cup \{F_0 \cup F_{y_x} \mid x \in X\}$. Therefore every 2-path lies on exactly one cycle of $\Sigma\mathcal{H}$. □

The remainder of the paper will be concerned with showing that the required conditions of Theorem 5.1 can be met whenever $n = p^e q^f + 1$ and n satisfies one of conditions (i), (ii) and (iii) as described in the introduction.

6 Determination of y_x when $n = pq + 1$

Let $n = m + 1 = pq + 1$. Suppose $p = q$. Let $x \in X$. Then $x = ip$ $(1 \le i \le (p-1)/2)$. We determine

$$y_x = \begin{cases} i & \text{if } p \equiv 1 \pmod 4, \text{ and} \\ p - i & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Assume $p < q$. For any $x \in X$, we determine y_x as follows:

(i) $x = ip$ $(1 \le i \le (q-1)/2)$.

If $p \equiv 1 \pmod 4$, put

$$y_x = \begin{cases} i & \text{if } (i,p) = 1, \text{ and} & (6.1) \\ (p-1)q + i & \text{otherwise.} & (6.2) \end{cases}$$

If $p \equiv 3 \pmod 4$, put

$$y_x = \begin{cases} pq - i & \text{if } (i,p) = 1, \text{ and} & (6.3) \\ q - i & \text{otherwise.} & (6.4) \end{cases}$$

(ii) $x = iq$ $(1 \le i \le (p-1)/2)$.

If $q \equiv 1 \pmod 4$, put

$$y_x = p(q-1)/2 + i. \tag{6.5}$$

Then $y_x \equiv i \pmod p$ and $(y_x, q) = 1$. If $q \equiv 3 \pmod 4$, put

$$y_x = p(q+1)/2 - i. \tag{6.6}$$

Then $y_x \equiv -i \pmod p$ and $(y_x, q) = 1$.

We put $Y = \{y_x \mid x \in X\}$ as before.

6.1 Proposition *The set $X \cup Y$ can be extended to a half-set modulo m, that is, there exists a subset Z of Z_m such that $X \cup Y \cup Z$ is a half-set modulo m.*

Proof. It is trivial to see that $X \cup Y = \phi$, and that $X \cup Y$ can be extended a half-set if and only if $(X \cup Y) \cap (-(X \cup Y)) = \phi$. So we will show $(X \cup Y) \cap (-(X \cup Y)) = \phi$. Observe that $(X \cup Y) \cap (-(X \cup Y)) = (X \cap (-X)) \cup (X \cap (-Y)) \cup (Y \cap (-X)) \cup (Y \cap (-Y))$. Since $X \subset [1, (pq-1)/2]$, we have $X \cap (-X) = \phi$. It is clear that $X \cap (-Y) = Y \cap (-X) = \phi$ because $(y_x, m) = 1$. Thus we only have to show $Y \cap (-Y) = \phi$. When $p = q$, this is trivial.

We consider $p < q$. The elements of Y determined by (6.1) and (6.2) lie in $A_1 = [1, (q-1)/2] \cup [(p-1)q + 1, (p-1)q + (q-1)/2]$; so clearly $A_1 \cap (-A_1) = \phi$. Those determined by (6.3) and (6.4) lie in $A_2 = [(q+1)/2, q-1] \cup [pq - (q-1)/2, pq-1]$; so clearly $A_2 \cap (-A_2) = \phi$. Those determined by (6.5) lie in $J_1 = [p(q-1)/2+1, (pq-1)/2]$. Those determined by (6.6) lie in $J_2 = [(pq+1)/2, p(q+1)/2-1]$. Cleary $J_i \cap (-J_i) = \phi$ and $A_i \cap J_j = \phi$ $(i = 1,2; j = 1,2)$. If $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$, we have $Y \subseteq A_1 \cup J_1$. If $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$, $Y \subseteq A_1 \cup J_2$. If $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$, $Y \subseteq A_2 \cup J_1$. If $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$, $Y \subseteq A_2 \cup J_2$. It is easy to check that $Y \cap (-Y) = \phi$. This completes the proof. □

The following theorem is obtained from Theorem 5.1 and Proposition 6.1.

6.2 Theorem *Let p and q be odd primes with $p \leq q$. Put $n = pq+1$. Then there exists a Dudeney set of K_n.*

7 Determination of y_x when $n = p^e + 1$

Let $n = m+1 = p^e+1$, where p is an odd prime and $e \geq 2$. In this case, for any $x \in X$, we can put $x = ip$, where $1 \leq i \leq (p^{e-1}-1)/2$. For each t satisfying $1 \leq t \leq e-1$, put

$$X_t = \{x \in X \mid x = ip^t, (i,p) = 1\}.$$

Then X can be partitioned as $X = X_1 \cup X_2 \cup \cdots \cup X_{e-1}$. Put $a_0 = 0$, and $a_t = p^{e-1} + p^{e-2} + \cdots + p^{e-t} (1 \leq t \leq e-1)$. Define the disjoint intervals $I_t = [a_{t-1}, a_t)$ $(1 \leq t \leq e-1)$, and note that $|I_t| = p^{e-t}$ $(1 \leq t \leq e-1)$. For any $x = ip^t \in X_t$ $(1 \leq t \leq e-1)$, determine $y_x \in I_t$ as follows:

$$y_x = \begin{cases} a_{t-1} + i & \text{if } p^t \equiv 1 \pmod 4\text{; and} \\ a_t - i & \text{if } p^t \equiv 3 \pmod 4. \end{cases}$$

Clearly $(y_x, p^t) = 1$ and

$$y_x \equiv \begin{cases} i & \pmod{p^{e-t}} \quad \text{if } p^t \equiv 1 \pmod 4\text{, and} \\ -i & \pmod{p^{e-t}} \quad \text{if } p^t \equiv 3 \pmod 4. \end{cases}$$

Putting $Y = \{y_x \mid x \in X\}$, we have $Y \subseteq [0, (p^e-1)/2]$ because $a_{e-1} \leq (p^e-1)/2$. Since $X, Y \subseteq [0, (p^e-1)/2]$ and $X \cap Y \neq \phi$, $X \cup Y$ can be extended a half-set modulo p^e. This yields a proof of Theorem 7.1.

7.1 Theorem *Let p be an odd prime and e be an integer, $e \geq 2$. Put $n = p^e+1$. Then there exists a Dudeney set of Hamilton cycles in K_n.*

8 Determination of y_x when $n = p^e q^f + 1$

Let $n = m+1 = p^e q^f + 1$, where p and q are distinct odd primes, $p < q$, and e and f are natural numbers. Put $X = \{x \mid (x,m) \neq 1, 1 \leq x \leq (m-1)/2\}$ as before.

8.1 Lemma *Let $n = m+1 = p^e q^f + 1$, where p and q are odd primes, $p < q$, and $x \in X$. Put $m_1 = (x,m) \neq 1$, $m_2 = m/m_1$ and $x = im_1$, where $1 \leq i \leq (m_2-1)/2$. If y satisfies*

(1) $y \equiv i \pmod{m_2}$ *or* $y \equiv -i \pmod{m_2}$, *and*

(2) $(y, m_1) \neq 1$,

then $(y + m_2, m_1) = 1$.

Proof. First, $(i, m_2) = 1$ by the definition of m_1, so $(y, m_2) = 1$. Put $d = (y, m_1) \neq 1$. Then $(d, m_2) = 1$. Since $x \in X$, we have $m_2 \neq 1$. Thus $d = p^\alpha$ or $d = q^\alpha$ $(\alpha \geq 1)$. Assume $d = p^\alpha$. Since $p \mid y$ (p divides y) and $p \nmid m_2$, we have $p \nmid y + m_2$ and $m_2 = q^\beta$ $(\beta \geq 1)$. Since $(y, m_2) = 1$, we have $q \nmid y$. So $q \nmid y + m_2$. Therefore $(y + m_2, m_1) = 1$. An analogous argument holds in the case $d = q^\alpha$. □

For any integer m_1 such that $m_1 | m$ and $m_1 \neq 1, m$, put

$$X(m_1) = \{x \in X \mid (x, m) = m_1\}.$$

Then

$$X(m_1) = \{x \in X \mid x = im_1, (i, m/m_1) = 1\}.$$

Define three disjoint subsets of X:

$$X_1' = X(p);$$

$$X_2' = X(q); \text{ and}$$

$$X_3' = X \setminus (X_1' \cup X_2').$$

And define three disjoint intervals:

$$I_1' = [0, c_1) \cup [(p-1)c_1, p^e q^f),$$

where $c_1 = p^{e-1}q^f$;

$$I_2' = [((q-1)/2)c_2 - (c_2 - 1)/2, ((q+1)/2)c_2 + (c_2 - 1)/2],$$

where $c_2 = p^e q^{f-1}$; and

$$I_3' = [0, (m-1)/2] \setminus (I_1' \cup I_2').$$

Clearly $X_3' = \bigcup_{\substack{m_1 | m \\ m_1 \neq 1, m, p, q}} X(m_1)$, $|I_1' \cap Z| = 2c_1$, $|I_2' \cap Z| = 2c_2$ and $|I_3' \cap Z| = (m-1)/2 - c_1 - c_2 + 1$, where Z is the set of all integers.

8.2 Lemma *Let $m = p^e q^f$, where p and q are odd primes satisfying $p < q$, $p \geq 5$ and $q \geq 11$, and e and f are natural numbers. Let S be the sum of all divisors of m which are not equal to $1, p^e q^f, p^{e-1}q^f, p^e q^{f-1}$. Then $2S \leq |I_3' \cap Z|$.*

Proof. Since the sum of all divisors of m is $(1+p+p^2+\ldots+p^e)(1+q+q^2+\ldots+q^f) = (p^{e+1}-1)(q^{f+1}-1)/(p-1)(q-1)$, we have to show that

$$2\{((p^{e+1}-1)(q^{f+1}-1)/(p-1)(q-1)) - 1 - p^e q^f - p^{e-1}q^f - p^e q^{f-1}\} \leq (p^e q^f - 1)/2 - p^{e-1}q^f - p^e q^{f-1} + 1.$$

Therefore we have to show that

$$p^{e-1}q^{f-1}g(p,q) + h(p,q) \geq 0,$$

where

$$g(p,q) = p^2q^2 - 3p^2q - 3pq^2 + pq - 2p^2 - 2q^2 + 2p + 2q$$

and

$$h(p,q) = 4p^{e+1} + 4q^{f+1} + 3pq - 3p - 3q + 1.$$

For any p and q with $3 \leq p < q$, we have $h(p,q) \geq 0$. In the equation

$$g(p,q) = 17q^2(p^2-72p/17-48/17)/24+7p^2(q^2-72q/7-48/7)/24+pq+2p+2q$$

the first term is at least 0 when $p \geq 5$ and the second term is at least 0 when $q \geq 11$, and so the lemma follows. □

For any $x = ip \in X_1'$, we have $1 \leq i \leq (c_1 - 1)/2$. We determine $y_x \in I_1'$ as follows. If $p \equiv 1 \pmod 4$, then

$$y_x = \begin{cases} i & \text{if } (i,p) = 1, \text{ and} \\ (p-1)c_1 + i & \text{otherwise.} \end{cases}$$

(Note that if $e \geq 2$, then $(i,p) = 1$, and if $e = 1$ and $(i,p) \neq 1$, then $((p-1)c_1 + i, p) = 1$.) If $p \equiv 3 \pmod 4$, then

$$y_x = \begin{cases} p^e q^f - i & \text{if } (i,p) = 1, \text{ and} \\ c_1 - i & \text{otherwise.} \end{cases}$$

For any $x = iq \in X_2'$, we have $1 \leq i \leq (c_2 - 1)/2$. We determine $y_x \in I_2'$ as follows. If $q \equiv 1 \pmod 4$, then

$$y_x = \begin{cases} ((q-1)/2)c_2 + i & \text{if } (((q-1)/2)c_2 + i, q) = 1, \text{ and} \\ ((q+1)/2)c_2 + i & \text{otherwise.} \end{cases}$$

If $q \equiv 3 \pmod 4$, then

$$y_x = \begin{cases} ((q-1)/2)c_2 - i & \text{if } (((q-1)/2)c_2 - i, q) = 1, \text{ and} \\ ((q+1)/2)c_2 - i & \text{otherwise.} \end{cases}$$

For any $x = im_1 \in X(m_1)$ with $m_1 \neq 1, m, p, q$, there exists y_x satisfying conditions 5.1, 5.2 and 5.3 in any given interval $[a,b)(\subseteq [0,(m-1)/2])$ with length $2m_2$, where $m_2 = m/m_1$, and a and b are integers. In fact, from Lemma 8.1, for all such x, there exists an integer $y \in [a,b)$ such that

$$y \equiv \begin{cases} i \pmod{m_2} & \text{if } m_1 \equiv 1 \pmod 4, \text{ and} \\ -i \pmod{m_2} & \text{if } m_1 \equiv 3 \pmod 4, \end{cases}$$

and $(y, m_1) = 1$.

Therefore to choose y_x for all $x \in X_3'$ it is sufficient to use an interval of length $\sum_{m_2} 2m_2$, where m_2 runs all divisors of m except $m, 1, p^{e-1}q^f$ and $p^e q^{f-1}$ (because m_1 runs all divesors of m except $1, m, p$ and q). Lemma 8.2 says that $\sum_{m_2} 2m_2 \leq |I_3'|$, so we can determine y_x for all $x \in X_3'$ satisfying $Y_3' \subseteq I_3'$, where $Y_3' = \{y_x \mid x \in X_3'\}$.

It is clear that $X \cap Y = \phi$ and $X \cup Y$ can be extended a half-set modulo m. This and Theorem 5.1 enables us to prove the following theorem.

8.3 Theorem *Let p and q be distinct odd primes satisfying $p < q, p \geq 5$ and $q \geq 11$, and let e and f be natural numbers. Put $n = p^e q^f + 1$. Then there exists a Dudeney set of Hamilton cycles in K_n.*

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