A few more partitioned balanced tournament designs

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ABSTRACT. A balanced tournament design, $BTD(n)$, defined on a $2n$-set V, is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of the elements of V into an $n \times 2n-1$ array such that (1) every element of V is contained in precisely one cell of each column, and (2) every element of V is contained in at most two cells of each row. If we can partition the columns of a $BTD(n)$ defined on V into three sets C_1, C_2, C_3 of sizes 1, $n-1$, $n-1$ respectively so that the columns in $C_1 \cup C_2$ form a Howell design of side n and order $2n$, an $H(n, 2n)$, and the columns in $C_1 \cup C_3$ form an $H(n, 2n)$, then the $BTD(n)$ is called partitionable. We denote a partitioned balanced tournament design of side n by $PBTD(n)$. The existence of these designs has been determined except for seven possible exceptions. In this note, we describe constructions for four of these designs. This completes the spectrum of $PBTD(n)$ for n even.

1 Introduction

A balanced tournament design, $BTD(n)$, defined on a $2n$-set V, is an arrangement of the $\binom{2n}{2}$ distinct unordered pairs of the elements of V into an $n \times 2n-1$ array such that

(1) every element of V is contained in precisely one cell of each column, and

(2) every element of V is contained in at most two cells of each row.

A $BTD(n)$ is a generalized balanced tournament design with block size $k = 2$, a $GBTD(n, 2)$, [3]. The existence of balanced tournament designs was established in 1977 by Schellenberg, Van Rees and Vanstone.

Theorem 1.1 [12]. *For n a positive integer, $n \neq 2$, there exists a $BTD(n)$.*

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Balanced tournament designs with additional structure and generalized balanced tournament designs have been investigated extensively in the past few years. A survey of results on balanced tournament designs can be found in [9]. More recent results on generalized balanced tournament designs can be found in [3, 4, 5]. Generalized balanced tournament designs are of interest because of their close connections with several other types of combinatorial designs, see for example [3].

In this note, we are interested in partitioned balanced tournament designs. We will use Howell designs to define partitioned balanced tournament designs.

Let V be a set of $2n$ elements. A Howell design of side s and order $2n$, or more briefly an $H(s, 2n)$, is an $s \times s$ array in which each cell is either empty or contains an unordered pair of elements of V such that

(1) each row and each column is Latin (that is, every element of V is in precisely one cell of each row and column) and

(2) every unordered pair of elements of V is in at most one cell of the array.

It follows immediately from the definition of an $H(s, 2n)$ that $n \leq s \leq 2n - 1$.

If we can partition the columns of a $BTD(n)$ defined on V into three sets C_1, C_2, C_3 of sizes 1, $n-1$, $n-1$ respectively so that the columns in $C_1 \cup C_2$ form an $H(n, 2n)$ and the columns in $C_1 \cup C_3$ form an $H(n, 2n)$, then the $BTD(n)$ is called partitionable. We denote the design by $PBTD(n)$. Partitioned balanced tournament designs are related to both Room squares, [14], and the even sided analogue of Room squares, [8]. They can be used to provide schedules of play for round robin tournaments which balance the effects of court and round assignments.

Partitioned balanced tournament designs were introduced by D.R. Stinson in [14], and he conjectured that $PBTD(n)$ exist for all $n \geq 5$. In a series of papers, this conjecture was settled with 7 possible exceptions for n.

Theorem 1.2 [6, 7, 8, 2]. *Let n be a positive integer, $n \geq 5$. There exists a $PBTD(n)$ (or a $PGBTD(n, 2)$) except possibly for $n \in \{9, 11, 15, 26, 28, 34, 44\}$.*

The purpose of this note is to describe constructions for $PBTD(n)$ for $n \in \{26, 28, 34, 44\}$. This will complete the spectrum for $PBTD(n)$ when n is even.

2 Direct Constructions

Intransitive starters and adders can be used to construct $PBTD(\ell)$ for $\ell = 28$ and 34. We define an intransitive starter over Z_{2n} for a $BTD(n+m)$ written on the symbol set $Z_{2n} \cup \{\infty_i \mid i = 1, \ldots, 2m\}$. In order to describe the intransitive starter and adder, we need some additional notation.

Suppose $n > 2m$. Define

$$\begin{array}{rll} B_i = & \{\infty_i, y_i\} & \text{for } i = 1, \ldots, 2m \\ B_{2m+i} = & \{x_{i1}, x_{i2}\} & \text{for } i = 1, \ldots, n-2m \\ R_j = & \{u_{j1}, u_{j2}\} & \text{for } j = 1, \ldots, m \\ C_j = & \{v_{j1}, v_{j2}\} & \text{for } j = 1, \ldots, m-1 \end{array}$$

An intransitive starter for a $BTD(n+m)$ defined on $Z_{2n} \cup \{\infty_1, \infty_2, \ldots, \infty_{2m}\}$ is a triple (S, R, C) where $S = \{B_i \mid i = 1, 2, \ldots, n\}$, $R = \{R_j \mid j = 1, 2, \ldots, m\}$ and $C = \{C_j \mid j = 1, 2, \ldots, m-1\}$ satisfying the following properties.

(1) $\displaystyle\bigcup_{B \in S \cup R} B = Z_{2n} \cup \{\infty_1, \infty_2, \ldots, \infty_{2m}\}$

(2) Let $D_0 = \{0, n\}$. $\{\pm(x_{i1} - x_{i2}) \mid i = 1, \ldots, n-2m\} \cup \{\pm(u_{j1} - u_{j2}) \mid j = 1, \ldots, m\} \cup \{\pm(v_{j1} - v_{j2}) \mid j = 1, \ldots, m-1\} = (Z_{2n} - D_0)$.

(3) $n \equiv 1 \pmod 2$ and $\{\pm(v_{j1} - v_{j2}) \mid j = 1, \ldots, m-1\} \cap \{0, 2, 4, \ldots, 2(n-1)\} = \emptyset$.

Let $A = (a_1, a_2, \ldots, a_n)$ be a complete set of coset representatives of the subgroup $\{0, n\}$ in Z_{2n}. A is called an adder for the intransitive starter (S, R, C) if

$$\bigcup_{\ell=0}^{1} \left(\bigcup_{i=1}^{n} B_i + a_i + \ell n \cup \bigcup_{j=1}^{m-1} C_j + \ell n \right)$$

is equal to the multiset

$$2(Z_{2n} \cup \{\infty_1, \infty_2, \ldots, \infty_{2m}\}) - \{D_{i_1} \cup D_{i_2}\}$$

where $D_j = D_0 + j$, $0 \le j \le n-1$.

Theorem 2.1[?]. *If there is an intransitive starter (S, R, C) over Z_{2n} for a $BTD(n+m)$ and a corresponding adder, then there is a $BTD(n+m)$ which is missing as a subarray a $BTD(m)$. If there exists a $BTD(m)$, then there exists a $BTD(n+m)$.*

We will use the following corollary of Theorem 2.1 for $PBTDs$.

Corollary 2.2 [4]. *Suppose there exists an intransitive starter (S, C, R) and a corresponding adder for a $BTD(n+m)$ defined on $Z_{2n} \cup \{\infty_1, \infty_2, \ldots, \infty_{2m}\}$ with the following properties.*

(i) $\{\pm(u_{j1} - u_{j2}) \mid j = 1, 2, \ldots, m\} \cap \{0, 2, 4, \ldots, 2(n-1)\} = \emptyset$

(ii) $A = \{0, 2, 4, \ldots, 2(n-1)\}$

(iii) $\bigcup_{i=1}^{n}(B_i + a_i) \cup \bigcup_{i=1}^{m-1} C_i = Z_{2n} \cup \{\infty_1, \ldots, \infty_{2m}\} - D_j$ *for some* j

Then there exists a $PBTD(n+m)$ which is missing as a subarray a $BTD(m)$. If there exists a $PBTD(m)$, then there exists a $PBTD(n+m)$.

Lemma 2.3.

(i) There exists a $PBTD(28)$ which contains as a subarray a $PBTD(5)$.

(ii) There exists a $PBTD(34)$ which contains as a subarray a $PBTD(7)$.

Proof:

(i) An intransitive starter and adder for a $PBTD(23{+}5)$ defined on $Z_{46} \cup \{\infty_1, \ldots, \infty_{10}\}$ is listed in Table 1. Since there exists a $PBTD(5)$ (Theorem 1.2), there exists a $PBTD(28)$.

(ii) An intransitive starter and adder for a $PBTD(27{+}7)$ defined on $Z_{54} \cup \{\infty_1, \ldots, \infty_{14}\}$ is listed in Table 2. Since there exists a $PBTD(7)$ (Theorem 1.2 or [11]), there exists a $PBTD(34)$.

□

Table 1
An intransitive starter and adder for a $PBTD(28)$

S	0,2	1,5	7,13	10,18	30,40	3,15	20,34	21,37
A	2	0	4	12	24	34	8	18
S	24,42	6,26	9,31	12,43	22,41	∞_1,16	∞_2,23	∞_3,27
A	14	6	44	22	30	20	36	40
S	∞_4,32	∞_5,33	∞_6,28	∞_7,39	∞_8,17	∞_9,14	∞_{10},45	
A	38	10	32	42	28	26	16	
R	35,36	29,38	8,25	44,19	4,11			
C	41,44	26,31	16,27	20,33				
D_0	0,23							

Table 2
An intransitive starter and adder for a $PBTD(34)$

S	0, 2	1, 5	3, 9	4, 12	30, 40	17, 29	7, 21	22, 38
A	2	4	8	10	0	6	12	14
S	33, 51	8, 28	23, 45	20, 44	6, 32	∞_1, 10	∞_2, 11	∞_3, 25
A	16	52	24	22	38	18	46	28
S	∞_4, 53	∞_5, 24	∞_6, 26	∞_7, 35	∞_8, 27	∞_9, 39	∞_{10}, 52	∞_{11}, 47
A	32	26	20	48	34	36	50	44
S	∞_{12}, 34	∞_{13}, 49	∞_{14}, 50					
A	40	30	42					
R	41, 42	16, 31	46, 37	48, 13	43, 14	36, 19	18, 15	
C	34, 39	1, 8	43, 32	51, 10	45, 24	41, 18		
D_0	0, 27							

3 Basic Frame Construction.

The case $n = 26$ can be done using the basic frame construction ([6]). In order to describe this construction and the constructions in the next section, we need several definitions.

Let V be a set of v elements. Let $G_1, G_2, \ldots, G_m$ be a partition of V into m sets. A $\{G_1, G_2, \ldots, G_m\}$-frame F with block size k, index λ and latinicity μ is a square array of side v which satisfies the properties listed below. We index the rows and columns of F by the elements of V.

(1) Each cell is either empty or contains a k-subset of V.

(2) Let F_i be the subsquare of F indexed by the elements of G_i. F_i is empty for $i = 1, 2, \ldots, m$. (The F_i's are often called the holes of the frame.)

(3) Let $j \in G_i$. Row j of F contains each element of $V - G_i$ μ times and column j of F contains each element of $V - G_i$ μ times.

(4) The collection of blocks obtained from the nonempty cells of F is a $GDD(v; k; G_1, G_2, \ldots, G_m; 0, \lambda)$. (See [16] for the notation for group divisible designs (GDD).)

If there is a $\{G_1, G_2, \ldots, G_m\}$-frame H with block size k, index λ and latinicity μ such that

(1) $H_i = F_i$ for $i = 1, 2, \ldots, m$ and

(2) H can be written in the empty cells of $F - \bigcup_{i=1}^{m} F_i$,

then H is called the complement of F and denoted by $\overline{F}$. If a complement of F exists, we call F a complementary $\{G_1, G_2, \ldots, G_m\}$-frame. A complementary $\{G_1, G_2, \ldots, G_m\}$-frame F is said to be skew if at most one of the cells (i, j) and (j, i) $(i \neq j)$ is nonempty.

We will use the following notation for frames. If $|G_i| = h$ for $i = 1, 2, \ldots, m$, we call F a $(\mu, \lambda; k, m, h)$-frame. The type of a $\{G_1, G_2, \ldots, G_m\}$-frame is the multi-set $\{|G_1|, |G_2|, \ldots, |G_m|\}$. We will say that a frame has type $t_1^{u_1} t_2^{u_2} \ldots t_\ell^{u_\ell}$ if there are u_i G_j's of cardinality t_i, $1 \leq i \leq \ell$. In this note, we will only use frames where $\mu = \lambda = 1$ and $k = 2$. These frames are usually called Room frames. For notational convenience, we will denote a Room frame simply by its type or partitioning $(\{G_1, G_2, \ldots, G_m\})$.

The frame constructions for *BTDs* also use sets of mutually orthogonal partitioned incomplete Latin squares (*OPILs*). Let $P = \{S_1, S_2, \ldots, S_m\}$ be a partition of a set S $(m \geq 2)$. A partitioned incomplete Latin square, having partition P, is an $|S| \times |S|$ array L, indexed by the elements of S, satisfying the following properties.

(1) A cell of L either contains an element of S or is empty.

(2) The subarrays indexed by $S_i \times S_i$ are empty for $1 \leq i \leq m$.

(3) Let $j \in S_i$. Row j of L contains each element of $S - S_i$ precisely once and column j of L contains each element of $S - S_i$ precisely once.

The type of L is the multiset $\{|S_1|, |S_2|, \ldots, |S_m|\}$. If there are u_i S_j's of cardinality t_i, $1 \leq i \leq k$, we say L has type $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$.

Suppose L and M are a pair of partitioned incomplete Latin squares with partition P. L and M are called orthogonal if the array formed by the superposition of L and M, $L \circ M$, contains every ordered pair in $S \times S - \bigcup_{i=1}^{m} (S_i \times S_i)$ precisely once. A set of n partitioned incomplete Latin squares with partition P is called a set of n mutually orthogonal partitioned incomplete Latin squares of type $\{|S_1|, |S_2|, \ldots, |S_m|\}$ if each pair of distinct squares is orthogonal.

We are now in a position to state and apply the basic frame construction for *PGBTDs*.

Theorem 3.1 [6, 4]. *If there exists a complementary* $\{G_1, G_2, \ldots, G_m\}$-*frame* $(m \geq 2)$, *a pair of orthogonal partitioned incomplete Latin squares* *with partition $\{G_1, G_2, \ldots, G_m\}$ and $PBTD(|G_i|+1)$ for $i = 1, 2, \ldots, m$, then there is a $PBTD((\sum_{i=1}^{m} |G_i|)+1)$.*

Lemma 3.2. *There exists a $PBTD(26)$.*

Proof: A starter and adder for a complementary (skew) frame of type 5^5 are:

S	1,2	4,6	8,11	12,16	17,23	21,3	14,22	9,18	13,24	7,19
A	21	13	23	16	1	8	19	3	14	7

Since there is a pair of $OPILS$ of type 5^5 [15] and a $PBTD(6)$ [8], we apply Corollary 3.2 to construct a $PBTD(26)$. □

4 A Frame Product Construction

A frame product construction was used in [2] to take care of several special cases of $PBTD(n)$s. In this section, we show that a much weaker version of this construction can be used for the case $n = 44$. The construction uses a complementary frame and a pair of $OPILS$ with a single shared (holey) ordered transversal.

Let $V = \bigcup_{i=1}^{n} V_i$ and let $W = \bigcup_{i=1}^{n} W_i$. Let F be a complementary $\{V_1, V_2, \ldots, V_n\}$-frame of type t^n defined on V. Let $\overline{F}$ be the complement of F defined on W; $\overline{F}$ is a $\{W_1, W_2, \ldots, W_n\}$-frame of type t^n. Let F' be the array of pairs formed by the superposition of F and $\overline{F}$, $F' = F \circ \overline{F}$. Suppose T is a transversal of F' such that

(i) Every element of $(V - V_i) \cup (W - W_i)$ occurs precisely once in T for some i.

(ii) T contains t empty cells from hole F_i.

Let $L = \{L_1, L_2\}$ be a pair of $OPILS$ of type t^n where L_1 is defined on V with partition $\{V_1, V_2, \ldots, V_n\}$ and L_2 is defined on W with partition $\{W_1, W_2, \ldots, W_n\}$. Let L' be the array of pairs formed by the superposition of L_1 and L_2, $L' = L_1 \circ L_2$. Suppose S is a transversal of L' such that

(i) Every element of $(V - V_i) \cup (W - W_i)$ occurs precisely once in S for some i.

(ii) S contains t empty cells from hole L'_i.

If we can order the pairs in T and S so that every element of $(V - V_i) \cup (W - W_i)$ occurs precisely once as a first coordinate and precisely once as a second coordinate, then we say that the complementary frame F and the pair of $OPILS$, $L = \{L_1, L_2\}$, share a (holey) ordered transversal, $T \cup S$.

To illustrate these definitions, we describe a complementary frame of type 2^7 and a pair of $OPILS$ of type 2^7 with a shared ordered transversal. A skew 2^7 frame ([13]) is displayed in Figure 1 and a pair of $OPILS$ of type 2^7 is $\{L_1, L_2\}$ where L_1 is the holey self orthogonal Latin square displayed in Figure 2 ([10]) and $L_2 = L_1^T$. They share a (holey) ordered transversal, $T \cup S$. The hole F_i is defined on $\{x, y\}$. The ordered pairs in $T \cup S$ are:

$2,4$ $7,11$ $6,9$ $8,10$ $1,5$ $0,3$ $\bar{2},\bar{4}$ $\bar{7},\bar{11}$ $\bar{6},\bar{9}$ $\bar{8},\bar{10}$ $\bar{1},\bar{5}$ $\bar{0},\bar{3}$
$\bar{4},2$ $4,\bar{2}$ $\bar{11},7$ $11,\bar{7}$ $\bar{9},6$ $9,\bar{6}$ $\bar{10},8$ $10,\bar{8}$ $\bar{5},1$ $5,\bar{1}$ $\bar{3},0$ $3,\bar{0}$

				2,5			11, y		4,8	1,3	7, x	9,10	
8, x					3,6			0, y		5,9	2,4		10,11
3,5	9, x					4,7			1, y		6,10	11,0	
7,11	4,6	10, x					5,8			2, y			0,1
	8,0	5,7	11, x					6,9			3, y	1,2	
4, y		9,1	6,8	0, x					7,10				2,3
	5, y		10,2	7,9	1, x					8,11		3,4	
		6, y		11,3	8,10	2, x					9,0		4,5
10,1			7, y		0,4	9,11	3, x					5,6	
	11,2			8, y		1,5	10,0	4, x					6,7
		0,3			9, y		2,6	11,1	5, x			7,8	
			1,4			10, y		3,7	0,2	6, x			8,9
	3,10		5,0		7,2		9,4		11,6		1,8		
2,9		4,11		6,1		8,3		10,5		0,7			

Figure 1
A skew frame of type 2^7, [13]

	5	0	8	10	3	1		9	2	6	7	11	4
3		y	5	9	4	2	0		7	8	10	x	6
8	2		11	y	7	0	10	5		x	9	4	1
5	9	8		6	11	x	3	4	y		1	7	0
1	y	7	9		10	6	5	2	0	11		3	x
10	3	11	x	7		4	1	y	8	5	6		2
0	4	2	6	1	x		8	3	11	9	y	10	
	10	5	1	0	8	3		6	4	7	11	2	9
2		4	y	3	5	9	7		10	0	x	6	8
7	8		0	11	2	y	9	x		4	5	1	10
11	x	9		5	6	8	4	7	1		0	y	3
9	6	1	7		y	10	2	0	x	3		5	11
4	7	10	3	x		11	6	8	5	1	2		y
6	0	x	4	2	1		11	10	9	y	3	8	

Figure 2
L_1, a holey self orthogonal Latin square, [10]

The frame product construction also uses the existence of $IA(n,k,4)s$, [1]. Let V be a finite set of size n. Let K be a subset of V of size k. An incomplete orthogonal array $IA(n,k,s)$ is an $(n^2-k^2)\times s$ array written on the symbol set V such that every ordered pair of $(V\times V)-(K\times K)$ occurs in any ordered pair of columns from the array. An $IA(n,k,s)$ is equivalent to a set of $s-2$ mutually orthogonal Latin squares of order n which are missing a subsquare order k. We need not be able to fill in the $k\times k$ missing subsquares with squares of order k.

Theorem 4.1 [1]. *An $IA(n,k,4)$ exists if and only if $n\geq 3k$ and $(n,k)\neq (6,1)$.*

Theorem 4.2. *Let m be a positive integer, $m\neq 2$ or 6. Suppose there exists*

(1) a complementary frame F of type t^n and a pair of $OPILS$ of type t^n with a shared (holey) ordered transversal,

(2) an $IA(m+k,k,4)$,

(3) a $PBTD(tm+1)$ and

(4) a $PBTD(tm+k+1)$.

Then there exists a $PBTD(tmn+k+1)$.

Proof: Let $V = \bigcup_{i=1}^{n} V_i$ and let $W = \bigcup_{i=1}^{n} W_i$ where $|V_i| = |W_i| = t$ for all i. Let $M = \{1, 2, \ldots, m\}$.

Let F be a complementary $\{V_1, V_2, \ldots, V_n\}$-frame of type t^n and let $\overline{F}$ denote the complement of F defined on W. $\overline{F}$ is a $\{W_1, W_2, \ldots, W_n\}$-frame. Let F' be the array of pairs formed by the superposition of F and $\overline{F}$, $F' = F \circ \overline{F}$. Let $L = \{L_1, L_2\}$ be a pair of $OPILS$ of type t^n. Suppose L_1 has partition $\{V_1, V_2, \ldots, V_n\}$ and suppose that L_2 has partition $\{W_1, W_2, \ldots, W_n\}$. Let L' denote the array of pairs formed by the superposition of L_1 and L_2, $L' = L_1 \circ L_2$. F and L share an ordered transversal $T \cup S$. T is a transversal of F' which contains t empty cells from hole F_n and S is a transversal of L' which contains t empty cells from the last hole defined on $V_n \cup W_n$. We need some additional notation for the pairs in T and S. Let $a(i)$ denote the pair in T which occurs in row i of F' and let $b(i)$ denote the pair in T which occurs in column i of F' for $i = 1, 2, \ldots, w$ where $w = t(n-1)$. Similarly, let $c(i)$ denote the pair in S which occurs in row i of L' and let $d(i)$ denote the pair in S which occurs in column i of L' for $i = 1, 2, \ldots, w$. The pairs in $T \cup S$ are ordered so that each element of $(V - V_n) \cup (W - W_n)$ occurs precisely once as a first coordinate and once as a second coordinate.

Since $m \neq 2$ or 6, there exists a pair of orthogonal Latin squares of side m defined on M, N_1 and N_2. Let N denote the array of pairs formed by the superposition of N_1 and N_2, $N = N_1 \circ N_2$. N_{xy} is the array formed by replacing each pair (a, b) in N with the pair $((a, x), (b, y))$.

We use an $IA(m+k, k, 4)$ to construct a pair of orthogonal Latin squares of side $m + k$ which is missing a pair of orthogonal Latin squares of side k. (The smaller Latin squares need not exist.) Let I denote the $m + k$ square array of pairs formed by superimposing the pair of Latin squares. Let $\beta = \{\beta_i \mid i = 1, 2, \ldots, k\}$ and let $\alpha = \{\alpha_i \mid i = 1, 2, \ldots, k\}$ where $U = \alpha \cup \beta$. I_{xy} will denote I defined on the symbols $M \times \{x, y\} \cup U$ where the missing subarray is defined on U. More precisely, if (x, y) is an ordered pair in T or S, then the pair of Latin squares used to construct I_{xy} will be defined on $(M \times x) \cup \alpha$ and $(M \times y) \cup \beta$ respectively, where the missing subarrays are defined on α and β. I_{xy} can be written in the following form:

$$I_{xy} = \begin{array}{|c|c|} \hline A_{xy} & C_{xy} \\ \hline R_{xy} & O \\ \hline \end{array}$$

where O is an empty square array of side k.

Let $B_1 = [F'L']$. B_1 is a $tn \times 2tn$ array. We construct an $(mtn + k) \times (2mtn + 2k)$ array as follows. Replace each pair (x, y) in $F' - T$ and $L' - S$

with the $m \times m$ array N_{xy}. Replace each ordered pair in $T \cup S$ with the $m \times m$ array A_{xy}. The resulting array B_2 has size $mtn \times 2mtn$. We add k new rows and $2k$ new columns to B_2.

We define B_3 to be the following $k \times (2tmn + 2k)$ array. B_3 contains the k new rows to be added to B_2. The subarrays labeled E in B_3 are empty arrays of size $k \times tm$ and the subarray labeled E' is an empty array of size $k \times 2k$.

$B_3 =$

$R_{b(1)}$	$\cdots$	$R_{b(w)}$	E	$R_{d(1)}$	$\cdots$	$R_{d(w)}$	E	E'

We define B_4 to be the following $(tmn + k) \times 2k$ array. B_4 contains the $2k$ new columns to be added to B_2. The subarrays labeled E in B_4 are empty arrays of size $tm + k \times k$.

$B_4 =$

$C_{a(1)}$	$C_{c(1)}$
$\vdots$	$\vdots$
$C_{a(w)}$	$C_{c(w)}$
E	E

We use B_2, B_3 and B_4 to construct an array B' of size $mtn+k \times 2(mtn+k)$.

$B' =$

B_2	B_4
B_3	$\mathcal{E}$

B' has the following structure. The arrays labeled E are empty square arrays of order mt. E_1 is an empty $k \times 2k$ array, E_2 is an empty $mt \times 2k$ array and the arrays labeled E_3 are empty arrays of size $k \times mt$.

$B' =$

E					E					
	E					E				
		$\ddots$					$\ddots$			
			E					E		
				E					E	E_2
				E_3					E_3	E_1

We fill in the empty arrays of B' with $PBTDs$. Let D_i be a $PBTD(mt+1)$ defined on $M \times (V_i \cup W_i) \cup \{\infty_1, \infty_2\}$ for $i = 1, 2, \ldots, n-1$. D_i can be written in the following form where D_i^1 and D_i^2 are square arrays of order mt.

$D_i=$

D_i^1	D_i^2	D_i^3
D_i^4	D_i^5	∞_1, ∞_2

The partitioning of D_i is the first mt columns of the array together with the last column and the second mt columns of the array with the last column.

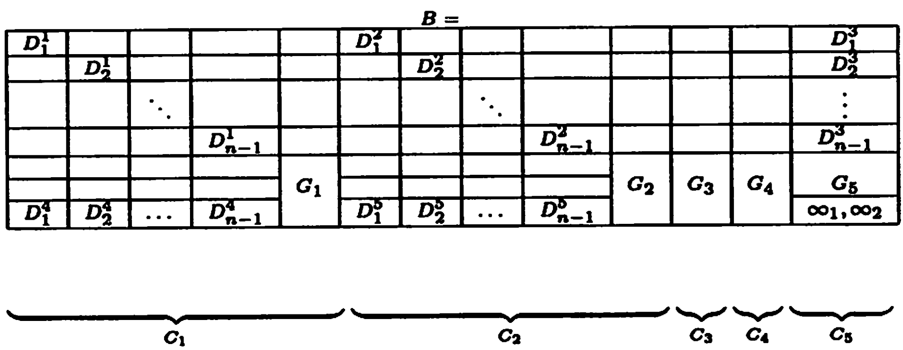
Let G be a $PBTD(mt+k+1)$ defined on $M \times (V_n \cup W_n) \cup U \cup \{\infty_1, \infty_2\}$. G can be written in the following form. G_1 and G_2 are $(mt+k) \times mt$ arrays and G_3 and G_4 are $(mt+k) \times k$ arrays.

$G=$

G_1	G_2	G_3	G_4	G_5
				∞_1, ∞_2

The partitioning of G is $G_1 \cup G_3$ together with the last column and $G_2 \cup G_4$ together with the last column.

We place these arrays in B' as follows.



The resulting array B is an $(mnt+k+1) \times (2mnt+2k+1)$ array defined on $(M \times (V \cup W)) \cup U \cup \{\infty_1, \infty_2\}$. It is straightforward to verify that B is a $PBTD(mnt+k+1)$. The partitioning of B is $C_1 \cup C_3 \cup C_5$ and $C_2 \cup C_4 \cup C_5$. □

Corollary 4.3. *There exists a* $PBTD(44)$.

Proof: Let $n = 7$, $t = 2$, $m = 3$, $k = 1$ in Theorem 4.2. A complementary frame of type 2^7 and a pair of *OPILS* of type 2^7 with a shared (holey) ordered transversal are described above. The $IA(4,1,4)$ exists by Theorem 4.1 and there exist $PBTD(7)$ and $PBTD(8)$ by Theorem 1.2. □

5 Conclusions

We have shown that there exist $PBTD(n)$ for $n = 26$, 28, 34 and 44. This completes the spectrum for $PBTD(n)$ for $n \equiv 0 \pmod 2$. The existence question for $PBTDs$ has now been settled with 3 possible exceptions.

Theorem 5.1. *There exist $PBTD(n)$ for $n \geq 5$ except possibly for $n \in \{9, 11, 15\}$.*

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