On Independence Numbers of the Cartesian Product of Graphs

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Abstract

Let G and H be connected graphs and let $G\Box H$ be the Cartesian product of G by H. A lower and an upper bound for the independence number of the Cartesian product of graphs is proved for the case, where one of the factors is bipartite. Cartesian products with one factor being an odd path or an odd cycle are considered as well. It is proved in particular that if S_1+S_2 is a largest 2-independent set of a graph G, such that $|S_2|$ is as small as possible and if $|S_2| \leq n+2$ then $\alpha(G\Box P_{2n+1}) = (n+1)\,|S_1| + n\,|S_2|$. A similar result is shown for the Cartesian product with an odd cycle. It is finally proved that $\alpha(C_{2k+1}\Box C_{2n+1}) = k\,(2n+1)$, extending a result of Jha and Slutzki.

1 Introduction

The determination of the independence number of a graph is one of the difficult combinatorial problems (cf. [10, p.2, p.480]). Recently it was shown that even approximating clique and independent set is NP-hard, [1, 2, 5]. On the other hand, several polynomial algorithms for decomposing

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a graph with respect to the Cartesian product were proposed [3, 6, 14]. In fact, the decomposition of a graph G can be done in $O(|E(G)| \cdot \log|V(G)|)$ time. Hence it might be possible to determine the independence number of a graph via its factors, because the problem size is much smaller in the factors than in the product. Furthermore, some important classes of graphs are Cartesian products of smaller graphs, as for example hypercubes. We therefore investigate the problem of calculating the independence number of a product graph by the independence numbers of the factors.

A set of vertices (edges) of a graph is *independent* if no two of them are adjacent. The size of a largest independent set of vertices of a graph G is called the *independence number* of G, $\alpha(G)$. A *k–independent set* in G is the union of k disjoint independent sets in G. We denote by $\alpha_k(G)$ the size of a largest k–independent set of a graph G. Clearly, $\alpha_1(G) = \alpha(G)$. Besides α_1 we will be also interested in α_2, the problem which is also (apparently) NP-hard.

An independent set of edges is also called a *matching*. The size of a largest matching of a graph G is called the *matching number* of G, $\tau(G)$. A matching which includes every vertex of a graph is called a *perfect matching*. The *chromatic number* $\chi(G)$ of a graph G is the smallest number of colours needed to colour the vertices of G such that no two adjacent vertices are assigned the same colour. A set of vertices which receive the same colour is called a *colour class* (with respect to a given colouring).

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $(a,x)(b,y) \in E(G \Box H)$ whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. Whenever possible we shall denote the vertices of one factor by $a, b, c, \ldots$ and the vertices of the other factor by x, y, z. For $x \in V(H)$ set $G_x = G \Box \{x\}$ and for $a \in V(G)$ set $H_a = \{a\} \Box H$. We call G_x and H_a a *layer* of G and of H, respectively.

The Cartesian product is commutative, associative and K_1 is a unit. Also, $G \Box H$ is connected if and only if both G and H are connected. We may therefore assume that all the graphs considered in this paper are connected, as well as finite, undirected, simple graphs, i.e. graphs without loops or multiple edges. P_n will denote a path on n vertices. We say that P_n is an odd (resp. even) path if n is odd (resp. even).

In the next section we recall some known results on the independence number of the Cartesian product and point out where difficulties lie. In Section 3 we prove that for a bipartite graph H and any graph G,

$$\frac{|H|}{2}\alpha_2(G) \le \alpha(G \Box H) \le \tau(H)\,\alpha_2(G) + (|H| - 2\tau(H))\,\alpha(G).$$

In the last section we first give two examples showing that the Cartesian product with an odd path gives rise to an interesting problem. Then we prove that if $S_1 + S_2$ is a largest 2–independent set of a graph G, such that $|S_2|$ is as small as possible, and if $|S_2| \leq n+2$ then $\alpha(G \Box P_{2n+1}) = (n+1)\,|S_1| + n\,|S_2|$; otherwise $\alpha(G \Box P_{2n+1}) \leq (n+1)\,(\,|S_1| + |S_2|\,) - n - 2$. We finally show a similar result for the Cartesian product with an odd cycle and prove that for $1 \leq k \leq n$, $\alpha(C_{2k+1} \Box C_{2n+1}) = k\,(2n+1)$.

2 Preliminaries

The first result on independence numbers of the Cartesian product of graphs is due to Vizing [13]:

Theorem 2.1 *For any graphs G and H,*

$$\alpha(G \Box H) \leq \min\{\alpha(G)\,|V(H)|,\ \alpha(H)\,|V(G)|\} \quad \textit{and}$$
$$\alpha(G \Box H) \geq \alpha(G)\,\alpha(H) + \min\{|V(G)| - \alpha(G),\ |V(H)| - \alpha(H)\}.$$

To verify the first inequality we use the fact that layers of the product are isomorphic to factors, hence there is at most $\alpha(G)$ and $\alpha(H)$ independent vertices in any G-layer and H-layer, respectively. To get the second inequality we observe that the product of two independent sets is an independent set. Jha and Slutzki, [9], used this fact to propose a nondeterministic algorithm, which iterates this idea. More precisely, on every step it takes the product of a largest independent set on the remaining vertices in one factor with a largest independent set on the remaining vertices in the second factor. It is not hard to see, that the algorithm produces a maximal independent set, which we will call a *diagonal independent set*. However, as we will see, a diagonal independent set need not be a largest independent set even on products of bipartite graphs.

The following result from [4, p.381] shows a difficulty of calculating the independence number of Cartesian products, since the problem is difficult even on a product with a complete graph.

Theorem 2.2 *For any graph G, $\alpha(G \Box K_k) = \alpha_k(G)$.*

Note that Theorem 2.2 in particular shows that $\chi(G) \leq k$ if and only if $\alpha(G \Box K_k) = |G|$ (see also [11]).

Corollary 2.3 *Let G be a graph on n vertices and let $\chi(G) > k$. Then*

$$\frac{n\,k}{\chi(G)} \leq \alpha(G \Box K_k) \leq k\,\alpha(G).$$

Proof. In a $\chi(G)$–colouring of G we select k largest colour classes. Applying Theorem 2.2 the lower bound follows. The upper bound follows from Theorem 2.1. □

It is clear that the lower bound from Corollary 2.3 can be improved in the case when we have a colouring of G with nonequal sized colour classes. However, the m–partite graph $G_{2,2,\dots,2}$ is an example of a graph for which the upper and the lower bound from Corollary 2.3 coincide.

3 Products with Bipartite Graphs

In this section we consider products where one factor is bipartite. As we will see, the problem is closely related to the determination of a largest 2–independent set.

Proposition 3.1 *If H is a bipartite graph then for any graph G,*

$$\frac{|H|}{2}\,\alpha_2(G) \;\leq\; \alpha(G\Box H) \;\leq\; \tau(H)\,\alpha_2(G) + (|H| - 2\tau(H))\,\alpha(G).$$

Proof. Let $\{a_1, a_2, \dots a_n\} + \{b_1, b_2, \dots, b_m\}$ be a largest 2–independent set of G, $n \geq m$. Let $V(H) = V_1 + V_2$ be the bipartition of H with $|V_1| \geq |V_2|$. Set

$$S = \{(a_i, x) \mid 1 \leq i \leq n; x \in V_1\} \cup \{(b_i, y) \mid 1 \leq i \leq m, y \in V_2\}.$$

Because the product of independent sets is an independent set it follows that S is an independent set. Furthermore,

$$|S| = n\,|V_1| + m\,|V_2| \geq \frac{|V_1| + |V_2|}{2}\,(n+m) = \frac{|H|}{2}\,\alpha_2(G),$$

and the lower bound is proved.

Let S' be a largest independent set of $G\Box H$ and let X be a largest matching of H. Then for each edge $xy \in X$, $|S' \cap (G_x \cup G_y)| \leq \alpha_2(G)$, while for an unmatched vertex $x \in V(H)$ we have $|S' \cap G_x| \leq \alpha(G)$. This implies the upper bound. □

Note that the set S from the previous proof is always a maximal independent set. However, it need not be a largest one. But we have:

Corollary 3.2 *Let G be a graph and let H be a bipartite graph.*
(i) If H has a perfect matching, then $\alpha(G\Box H) = \frac{|V(H)|}{2}\,\alpha_2(G)$.
(ii) If $\alpha_2(G) = 2\alpha(G)$ then $\alpha(G\Box H) = \alpha(G)\,|H|$.

Proof. (i) Since H has a perfect matching, $2\,\tau(H) = |H|$.
(ii) As $\alpha_2(G) = 2\alpha(G)$, we have from Proposition 3.1:

$$|H|\,\alpha(G) \leq \alpha(G\Box H) \leq 2\tau(H)\,\alpha(G) + (|H| - 2\tau(H))\alpha(G) = |H|\,\alpha(G),$$

and the proof is complete. □

It is easy to see that if at least one factor of a product has a perfect matching then the product has a perfect matching as well. It follows the n-cube Q_n has a perfect matching ($Q_1 = K_2$ and for $n \geq 2$, $Q_n = Q_{n-1} \Box K_2$). Hence, let us state:

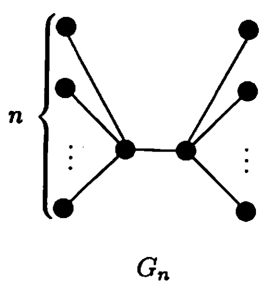
Corollary 3.3 *For any graph G,* $\alpha(G \Box Q_n) = 2^{n-1} \alpha_2(G)$.

4 Products with Odd Paths and Odd Cycles

Let $V(H) = \{x_1, x_2, \ldots, x_n\}$ and let $S \subseteq V(G \Box H)$. Let $X_i = S \cap G_{x_i}$. Then we will write $S = \langle X_1, X_2, \ldots, X_n \rangle$.

Let H be an even path or an even cycle. Then it is easy to see that there is a maximum independent set of $G \Box H$ of the form $\langle A, B, A, \ldots, A, B \rangle$. Moreover, if G is a bipartite graph with the bipartition $V(G) = V_1 + V_2$, then there are exactly two maximum independent sets of $G \Box H$, namely $\langle V_1, V_2, \ldots, V_1, V_2 \rangle$ and $\langle V_2, V_1, \ldots, V_2, V_1 \rangle$. Such solutions will be called *bipartite solutions.*

The situation is more complicated with odd paths and odd cycles. We first show two examples demonstrating this. Consider first the product $G_n \Box P_3$, where G_n is the graph on $2n + 2$ vertices depictured on Fig. 1 (G_n is sometimes called a *double star*). It is easy to verify, that a bipartite solution is of the size $3n + 3$, while the diagonal independent set is of the size $4n + 1$.



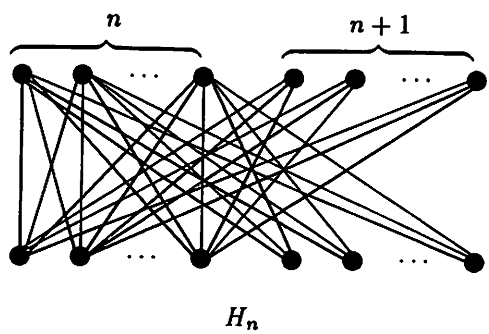


Figure 1: Graphs G_n and H_n

For the second example consider a bipartite graph H_n with the bipartition $V(H_n) = V_1 + V_2$:

$$\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots, b_{n+1}\} + \{a'_1, a'_2, \ldots a'_n, b'_1, b'_2, \ldots, b'_{n+1}\}.$$

In H_n a vertex a_i is adjacent to every vertex in V_2 and a vertex a_i' is adjacent to every vertex in V_1 (see Fig. 1). Consider the product $H_n \Box P_3$. It is again easy to verify, that a bipartite solution is of the size $6n + 3$, while the diagonal independent set is of the size $5n + 4$. Note that these two examples show that the difference between the bipartite solution and a diagonal solution can be arbitrary large.

Lemma 4.1 *For any graph G there exists a maximum independent set of $G \Box P_{2n+1}$ of the form $\langle A, B, A, \dots, B, A\rangle$, where $n \geq 1$.*

Proof. Let $S = \langle X_1, X_2, \dots, X_{2n+1}\rangle$ be a maximum independent set of $G \Box P_{2n+1}$. Choose i, such that $|X_i| + |X_{i+1}| = \max_{1 \leq j \leq 2n}(|X_j| + |X_{j+1}|)$. We may without loos of generality assume that $|X_i| \geq |X_{i+1}|$. Then $S' = \langle X_i, X_{i+1}, \dots, X_{i+1}, X_i\rangle$ is clearly an independent set and by the choice of i, $|S'| \geq |S|$. □

Theorem 4.2 *Let $n \geq 1$ and let $S_1 + S_2$ be a largest 2–independent set of a graph G, such that $|S_2|$ is as small as possible. If $|S_2| \leq n + 2$ then*

$$\alpha(G \Box P_{2n+1}) = (n+1)\,|S_1| + n\,|S_2|,$$

otherwise $\alpha(G \Box P_{2n+1}) \leq (n+1)(\,|S_1| + |S_2|\,) - n - 2$.

Proof. Let $s_1 = |S_1|$ and $s_2 = |S_2|$. Clearly, $S = \langle S_1, S_2, \dots, S_2, S_1\rangle$ is an independent set of $G \Box P_{2n+1}$.

Suppose that $|S| < \alpha(G \Box P_{2n+1})$ and let S' be a largest independent set of $G \Box P_{2n+1}$. Hence $|S'| > |S|$. According to Lemma 4.1 we may assume that $S' = \langle X_1, X_2, \dots, X_2, X_1\rangle$. Let $x_1 = |X_1|$ and $x_2 = |X_2|$. Clearly, $x_1 \geq x_2$.

Note first that $x_1 + x_2 \leq s_1 + s_2$. If $x_1 + x_2 = s_1 + s_2$, then according to the choice of S_2, $s_1 \geq x_1$ and therefore $|S'| \leq |S|$. Hence

$$x_1 + x_2 \leq s_1 + s_2 - 1. \tag{1}$$

It follows from (1) that $x_1 \;\leq\; s_1 + s_2 - 1 - x_2$. Since $x_2 \geq 1$, we obtain

$$x_1 \;\leq\; s_1 + s_2 - 2. \tag{2}$$

From (1) and (2) we conclude:

$$\begin{aligned} |S'| &= (n+1)\,x_1 + n\,x_2 \\ &= n\,(x_1 + x_2) + x_1 \\ &\leq n\,(s_1 + s_2 - 1) + s_1 + s_2 - 2 \\ &= |S| + s_2 - 2 - n. \end{aligned}$$

If $s_2 - 2 - n \leq 0$ we get a contradiction, hence in this case S is a largest independent set. Otherwise the upper bound follows. □

The next theorem shows a similar result for products with odd cycles. Let $(G - S)$ denote the subgraph of G induced by the vertices $V(G) - S$.

Theorem 4.3 *Let $S = S_1 + S_2$ be a largest 2-independent set of a graph G such that $\alpha(G-S)$ is as large as possible. Let S_3 be a maximum independent set of $(G - S)$. Then*

$$n\,|S| + |S_3| \;\leq\; \alpha(G\Box C_{2n+1}) \;\leq\; (n + \frac{1}{2})\,|S|.$$

Furthermore, if G is a bipartite graph, then $\alpha(G\Box C_{2n+1}) = n\,|G|$.

Proof. Clearly, the set $X = \langle S_1, S_2, \ldots, S_2, S_3\rangle$ is an independent set of $G\Box C_{2n+1}$, hence the lower bound.

Suppose now that $X = \langle X_1, X_2, \ldots, X_{2n}, X_{2n+1}\rangle$ is a largest independent set of $G\Box C_{2n+1}$. Because $|S| = \alpha_2(G)$ we have

$$|X_i| + |X_{i+1}| \leq s_1 + s_2 \quad \text{for } 1 \leq i \leq 2n,$$

and also $|X_{2n+1}| + |X_1| \leq s_1 + s_2$. Therefore $2\,|X| \leq (2n + 1)\,(s_1 + s_2)$. This implies the upper bound.

Let now G be a bipartite graph and let $V(G) = V_1 + V_2$ be the bipartition of G. Then $S = \langle V_1, V_2, \ldots, V_1, V_2, \emptyset\rangle$ is an independent set of the claimed size. Since $\alpha(C_{2n+1}) = n$ the equality follows from Theorem 2.1. □

We conclude the paper with the following result, which is proved in [9] for the case $n = k$.

Theorem 4.4 *For $1 \leq k \leq n$, $\alpha(C_{2k+1}\Box C_{2n+1}) = k\,(2n + 1)$.*

Proof. Let $V(C_m) = \{0, 1, \ldots, m-1\}$. By Theorem 2.1, $\alpha(C_{2k+1}\Box C_{2n+1}) \leq k\,(2n + 1)$. To show the lower bound assume first $n = k$. Consider the following set of vertices of $C_{2n+1}\Box C_{2n+1}$:

$$I \;=\; \{\,(\,(i + j) \bmod (2n + 1),\, j\,)\,\},$$

where $i = 0, 2, \ldots, 2n - 2$ and $j = 0, 1, \ldots, 2n$. Obviously, I contains $n(2n+1)$ different vertices. We claim that I is an independent set. Consider different vertices $u = (i+j, j) \in I$ and $v = (i' + j', j') \in I$, where all indices are taken by appropriate modulos. If $j = j'$ then the first components differ by at least two, so u and v are not adjacent. If $j \neq j'$ then u and v could only be adjacent if $j = j' + 1$ or $j = 0$ and $j' = 2n$ (or vice versa).

In both cases the first components of u and v differ by at least one, which implies that u and v are not adjacent. I is therefore an independent set.

Consider now the case $k < n$. Let $I = \langle S_1, S_2, \ldots, S_{2k+1}\rangle$ be the solution for $k = n$ as above. Then

$$I' = \langle S_1, S_2, \ldots, S_{2k+1}, S_1, S_{2k+1}, S_1, \ldots, S_{2k+1}\rangle$$

is an independent set. Furthermore since $|S_i| = k$, $1 \le i \le 2k+1$, $|I'| = k(2n+1)$ and the proof is complete. □

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