The Ascending Star Subgraph Decomposition of Some Star Forests

Huaitang Chen
Mathematics Department
Linyi Teachers' College
Linyi, Shandong
P.R. of China

Kejie Ma
Institute of Operations Reseasrch
Qufu Normal University
Qufu, Shandong
P.R. of China

Huishan Zhou
Department of Mathematics and Computer Sciences
Georgia State University
University Plaza
Atlanta, Georgia, 30303

ABSTRACT. Let G be a graph of size $\binom{n+1}{2}$ for some integer $n \geq 2$. Then G is said to have an ascending star subgraph decomposition if G can be decomposed into n subgraphs $G_1, G_2, \ldots, G_n$ such that each G_i is a star of size i with $1 \leq i \leq n$. We shall prove in this paper that a star forest with size $\binom{n+1}{2}$ possesses an ascending star subgraph decomposition under some conditions on the number of components or the size of components.

1 Introduction

For definitions and notations not presented here, we follow [2]. Let G be a graph of size q, and let n be the positive integer with $\binom{n+1}{2} \leq q < \binom{n+2}{2}$. Then G is said to have an ascending subgraph decomposition (ASD) if G can be decomposed into n subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$. Furthermore, if each G_i is a star (matching, path, ... etc.,), then G is said to have an ascending star (matching, path, ... etc., respectively) subgraph decomposition or simply a star (matching, path, ... etc., respectively) ASD.

Alavi, Boals, Chartrand, Erdös and Oellermann proposed the following conjecture [1]: Every graph of positive size has an ASD. In the same paper they have reduced the verification of the conjecture to the following equivalent version: Every graph of size $\binom{n+1}{2}$ for $n = 1, 2, \ldots$ has an ASD. In order to obtain some insight of the eventual proof of this conjecture, many authors have considered variations by restricting either the requirement on the decomposed graph or the conditions on the factor subgraphs. Those variations have their own significance too. Faudree, Gyarfas and Schelp showed [4] that a star forest of size $\binom{n+1}{2}$ has an ASD, from which Erdös suggested two problems: a star forest of size $\binom{n+1}{2}$ with each component having more than n edges has a star ASD; a graph of size $\binom{n+1}{2}$ has a star forest ASD. And in his joint paper with others as mentioned above [1], a slightly different version of the first problem is formulated: a star forest of size $\binom{n+1}{2}$ with each component having size between n and $2n-2$ (n and $2n-2$ are included) has a star ASD.

It was mentioned in [4] that the complete graph K_{n+1} with $n+1$ vertices could easily be proved to have a star ASD and a path ASD. They further proved that any graph obtained from K_{n+1} by deleting any n edges has a star ASD. Chen and Ma [3] and Fu [6] proved that a star forest of size $\binom{n+1}{2}$, where each component has size greater than n, and where two components differ in size by at most one has a star ASD. Two further restrictions on the size of components of a star forest that guarantee the star ASD can be found in [3]. Zhao obtained the star ASD for a star forest of size $\binom{n+1}{2}$ by restricting either the number of components to two or the number of components to three while keeping the size of each components at least n [9].

There are five results about the matching ASD of a graph. Three of them put restrictions on the maximum degree of a graph. Two were proved in [1]: a graph of size $\binom{n+1}{2}$, $n \geq 4$, with maximum degree at most 2, has a matching ASD; a forest of size $\binom{n+1}{2}$, with maximum degree d ($2 \leq 2d-2 \leq n$), has a matching ASD. One was proved in [4]: a graph of size $\binom{n+1}{2}$, with maximum degree d ($n \geq 4d^2 + 6d + 3$), has a matching ASD. The other two can be found in [6]: Let G be a graph of size $\binom{n+1}{2}$. If G can be decomposed into n edge disjoint subgraphs G_i, $i = 1, 2, \ldots, n$, such that the size of G_i is i and for each $k \in \{2, 3, \ldots, n\}$, there is at most one edge of G_k which is incident with some vertex of the edge induced subgraph induced by the union of $G_1, G_2, \ldots, G_{k-1}$, then G has a matching ASD. If G is a disconnected graph with n components, which have sizes $1, 2, \ldots, n-1$ and n, then G has a matching ASD. In [5] Fu also obtained a result on the ASD of a graph with restrictions on the maximum degree: a graph of size $\binom{n+1}{2}$ with maximum degree at most $(n-1)/2$ has an ASD. In [6] he obtained a result on the ASD of complete bipartite graphs of size $\binom{n+1}{2}$.

In an ASD of G, a member of the decomposition is isomorphic to a subgraph of all other members of the decomposition with larger size. A closely related packing problem was considered in [7] where this property was not demanded. In particular, the auther conjectured that the complete graph K_{n+1} can be decomposed into n edge disjoint trees of sizes $1, 2, \ldots, n$.

We shall now prove the following four results about the star ASD for a star forest:

Let G be a star forest with components $G_1, G_2, \ldots, G_k$ such that G_i has size a_i $(i = 1, 2, \ldots, k)$, where $\sum_{i=1}^{k} a_i = \binom{n+1}{2}$ for a natural number n. Then G has a star ASD under one of the following four conditions:

1. $k = 3$, and $(a_i, a_j) \neq (1, 1)$, $(a_i, a_j) \neq (2, 2)$ for any pair i and j with $i, j \in \{1, 2, 3\}$ and $i \neq j$. (Note: This condition implies that $n \geq 3$.)
2. $a_i \geq n$ $(i = 1, 2, \ldots, k)$ and at least $(k-2)$ a_i's among $a_1, a_2, \ldots, a_k$ are equal;
3. $a_i \geq n$ $(i = 1, 2, \ldots, k)$ and $\max\{a_i \mid i = 1, 2, \ldots, k-1\} - \min\{a_i \mid i = 1, 2, \ldots, k-1\} \leq 1$.
4. $a_i \geq n$ $(i = 1, 2, \ldots, k)$ and $\max\{a_i \mid i = 1, 2, \ldots, k\} - \min\{a_i \mid i = 1, 2, \ldots, k\} \leq 2$.

All the above four results are proved before 1990 and they supposed to be published in 1993. Because of some incidents in the publishing house, it was delayed and three results became out of dated. The last three results (presented in 2, 3 and 4) became special cases of a general result published in [8]. We eliminate the proofs in this paper. Note that their methods of proof are totally different.

2 Main Results

In connection with the problem mentioned in Section 1, we present and prove an equivalent number-theoretic problem following the presentations mentioned in [1]. We first introduce some new terms. Let $a, b_1, b_2, \ldots, b_t$ be natural numbers. If a $= \sum_{i=1}^{t} b_i$, then a is said to be *decomposed into* $b_1, b_2, \ldots, b_t$, denoted by $a = [b_1, b_2, \ldots, b_t]$. If we let $N_a = \{b_1, b_2, \ldots, b_t\}$, then we simply write $a = [N_a]$ to mean $a = [b_1, b_2, \ldots, b_t]$. Furtheomore, if $b_1, b_2, \ldots, b_t$ are pairwise distinct, then we have a *distinct decomposition* of a. For natural numbers $a_1, a_2, \ldots, a_k$ (not necessarily distinct), if each $a_i = [N_{a_i}] = [b_1^i, b_2^i, \ldots, b_{s_i}^i]$ $(i = 1, 2, \ldots, k)$ is a distinct decomposition and $N_{a_i} \cap N_{a_j} = \emptyset$ for $i \neq j$, then the decomposition of $a_1, a_2, \ldots, a_k$ is called a *distinct decomposition*. If, furthermore, $\cup_{i=1}^{k} N_{a_i} = N$, i.e.,

$N_{a_1}, N_{a_2}, \ldots, N_{a_k}$ is a partition of N, where N is a finite subset of natural numbers, then $N_{a_1}, N_{a_2}, \ldots, N_{a_k}$ is said to be a whole decomposition of $\{a_1, a_2, \ldots, a_k\}$ by N. We shall also say that $\{a_1, a_2, \ldots, a_k\}$ can be *wholly decomposed* by N, or simply $a_1, a_2, \ldots, a_k$ can be *wholly decomposed* by N.

Now the result stated in graph-theoretic terms in the end of last section can be formulated equivalently as follows.

Theorem. *Let a_1, a_2 and a_3 be natural numbers with $a_1+a_2+a_3 = \binom{n+1}{2}$, $(a_i, a_j) \neq (1,1)$ and $(a_i, a_j) \neq (2,2)$ for any pair i and j with $i, j \in \{1,2,3\}$ and $i \neq j$. Then a_1, a_2, a_3 can be wholly decomposed by $1, 2, \ldots, n$.*

The following lemma, which will be frequently used in our proof, can be found in [9].

Lemma. *Let a_1 and a_2 be natural numbers with $a_1 + a_2 = \binom{n+1}{2}$ for a natural number n. Then a_1, a_2 can be wholly decomposed by $1, 2, \ldots, n$.*

Proof of The Theorem. Consider the value of $a_1 + a_2$. If there exists an integer $r \leq n$ such that $a_1 + a_2 = \binom{r+1}{2}$, then by the lemma there exists a partition of the set $\{1, 2, \ldots, r\}$ into two subsets N_1 and N_2 such that for $i = 1, 2$, $a_i = [N_i]$. Let $N_3 = \{r+1, r+2, \ldots, n\}$. Then $a_3 = [N_3]$. Therefore, N_1, N_2, N_3 is a whole decomposition of a_1, a_2, a_3 by $1, 2, \ldots, n$.

Assume now that there exist integers $r < n$ and σ: $1 \leq \sigma \leq r$ such that $a_1 + a_2 = \binom{r+1}{2} + \sigma$. Set $\sigma' = r + 1 - \sigma$. Then $1 \leq \sigma' \leq r$. Set $M_3 = \{\sigma', r+2, r+3, \ldots, n\}$. Then $a_3 = [M_3]$.

Claim: At least one of a_1 and a_2 will be at least σ.

Assume, otherwise, that $a_1 < \sigma$, $a_2 < \sigma$. Then $a_1 + a_2 = \binom{r+1}{2} + \sigma < 2\sigma$, which implies that $0 < r < 1$, a contradiction.

Without loss of generality, suppose $\sigma \leq a_2$. Let $b_2 = a_2 - \sigma$. Then $a_1 + b_2 = \binom{r+1}{2}$.

If there exists $r_1 \geq 1$ such that $a_1 = \binom{r_1+1}{2}$, then we have a partition of the set $\{1, 2, \ldots, r\}$ into two subsets M_1 and M_2:

$$M_1 = \{1, 2, \ldots, r_1\} \text{ and } M_2 = \{r_1+1, r_1+2, \ldots, r\} \qquad (*)$$

with $a_1 = [M_1]$ and $b_2 = [M_2]$.

Otherwise there exists $r_1 \geq 1$ and δ $(1 \leq \delta \leq r_1)$ such that $a_1 = \binom{r_1+1}{2} + \delta$. Let $\delta' = r_1 + 1 - \delta$. Then we have a partition of the set $\{1, 2, \ldots, r\}$ into two subsets M_1 and M_2:

$$M_1 = \{1, 2, \ldots, \delta'-1, \delta'+1, \ldots, r_1, r_1+1\} \text{ and } M_2 = \{\delta', r_1+2, \ldots, r\} \qquad (**)$$

with $a_1 = [M_1]$ and $b_2 = [M_2]$.

Since $1 \leq \sigma' \leq r$, it follows that $\sigma' \in M_1 \cup M_2$. If $\sigma' \in M_2$, then we can set $N_1 = M_1$, $N_2 = (M_2 \backslash \{\sigma'\}) \cup \{r+1\}$ where $r+1 = \sigma + \sigma'$ and $N_3 = \{\sigma', r+2, \ldots, n\}$. Then N_1, N_2, N_3 is a whole decomposition of a_1, a_2, a_3 by $1, 2, \ldots, n$.

Now we assume that $\sigma' \in M_1$. According to the two different cases of $a_1 = \binom{r_1+1}{2}$ and $a_1 = \binom{r_1+1}{2} + \delta$ $(1 \leq \delta \leq r_1)$, M_1 and M_2 will have two different forms in (*) and (**).

Case 1: $a_1 = \binom{r_1+1}{2}$. We discuss according to the following two subcases $r = r_1$ and $r > r_1$.

Case 1.1: $r = r_1$. In this case $M_2 = \emptyset$.

If $\sigma \neq \sigma'$, then we can find the two distinct σ and σ' in M_1 such that $\sigma + \sigma' = r + 1$; thus we can set

$$N_1 = (M_1 \backslash \{\sigma, \sigma'\}) \cup \{r+1\}, \ N_2 = \{\sigma\}, \text{ and } N_3 = M_3,$$

which is a whole decomposition of a_1, a_2, a_3 by $1, 2, \ldots, n$. (From now on we shall omit the words after "which" for simplicity. So when we set N_1, N_2 and N_3, they should always be a whole decomposition of a_1, a_2, a_3 by $1, 2, \ldots, n$.)

If $\sigma = \sigma'$, then r is an odd number. If $r \neq 1, 3$, then we can find three distinct numbers 1, $\sigma - 1$ and σ in M_1; thus we can set

$$N_1 = (M_1 \backslash \{1, \sigma - 1, \sigma\}) \cup \{r+1\}, \ N_2 = \{1, \sigma - 1\}, \text{ and} N_3 = M_3.$$

If $\sigma = \sigma'$, $r = 3$ and $n > r + 1 = 4$, then $\sigma = \sigma' = 2$; hence we can set

$$N_1 = \{1, 5\}, \ N_2 = \{2\}, \text{ and } N_3 = \{3, 4, r+3, \ldots, n\}.$$

If $\sigma = \sigma'$, $r = 3$, and $n = r + 1 = 4$, then $a_2 = a_3 = \sigma = 2$, which is not allowed.

If $\sigma = \sigma'$ and $r = 1$, then $a_1 = a_2 = \sigma = 1$, which is not allowed.

Case 1.2: $r_1 < r$.

If r_1 is an even number or r_1 is an odd number but $\sigma' \neq (r_1+1)/2$, then $\tau \equiv r_1 + 1 - \sigma' \neq \sigma'$. Thus we can set

$$N_1 = (M_1 \backslash \{\sigma', \tau\}) \cup \{r_1+1\}, \ N_2 = (M_2 \backslash \{r_1+1\}) \cup \{\tau, r+1\}, \text{ and } N_3 = M_3.$$

If r_1 is an odd number but $r_1 \neq 1, 3$ and $\sigma' = (r_1+1)/2$, then we can set

$$N_1 = (M_1 \backslash \{1, \sigma' - 1, \sigma'\}) \cup \{r_1 + 1\},$$
$$N_2 = (M_2 \backslash \{r_1 + 1\}) \cup \{1, \sigma' - 1, r + 1\}, \text{ and } N_3 = M_3.$$

If $r_1 = 3$ and $\sigma' = (r_1+1)/2 = 2$, then if $r = r_1 + 1 = 4$, we can set $N_1 = \{1, 5\}$, $N_2 = \{4, 3\}$, and $N_3 = \{2, 6, \ldots, n\}$; and if $r > r_1 + 1 = 4$, we can set $N_1 = \{1, 5\}$, $N2 = (M_2 \backslash \{5\}) \cup \{3, r+1\}$, and $N_3 = M_3$.

If $r_1 = 1$, then $\sigma' = 1$ and $\sigma = r$. Therefore, $n \geq r+2$, for otherwise $a_1 = a_3 = 1$, which is not allowed. We can set $N_1 = M_1$, $N_2 = (M_2\backslash\{2\})\cup\{r+2\}$, and $N_3 = (M_3\backslash\{1, r+2\}) \cup \{2, r+1\}$. (If $r = 2$, then we have $N_1 = \{1\}$, $N_2 = \{4\}$, and $N_3 = \{2, 3, 5, \dots, n\}$; if $r > 2$, then we have $N_1 = \{1\}$, $N_2 = \{3, 4, \dots, r, r+2\}$, and $N_3 = \{2, r+1, r+3, \dots, n\}$.)

Case 2: $a_1 = \binom{r_1+1}{2} + \delta$ $(1 \leq \delta \leq r_1)$. In this case $r_1 < r < n$.

Recall that M_1 and M_2 are given by (**), and $\sigma' \neq \delta'$ since $\sigma' \in M_1$. We consider two cases.

Case 2.1: $\sigma' < \delta'$. In this case $\delta' > 1$.

If δ' is an odd number or δ' is an even number with $\sigma' \neq \delta'/2$, then there exists τ in M_1 different than σ' such that $\delta' = \tau + \sigma'$. Hence we can set

$$N_1 = (M_1\backslash\{\tau, \sigma'\}) \cup \{\delta'\},\ N_2 = (M_2\backslash\{\delta'\}) \cup \{\tau, r+1\},\ \text{and } N_3 = M_3.$$

If δ' is an even number with $\sigma' = \delta'/2$ and $\sigma' \neq 1, 2$, then there exist two different numbers 1 and $\sigma' - 1$ in M_1. Hence we can set

$$N_1 = (M_1\backslash\{1, \sigma' - 1, \sigma'\}) \cup \{\delta'\},$$
$$N_2 = (M_2\backslash\{\delta'\}) \cup \{1, \sigma' - 1, r+1\},\ \text{and } N_3 = M_3.$$

If δ' is an even number with $\sigma' = \delta'/2$ and $\sigma' = 2$, then $\delta' = 4$. We shall have the following two subcases depending on whether $r = r_1 + 1$.

If $r = r_1 + 1$, then $\sigma = r + 1 - 2 = r_1$ and $M_2 = \{4\}$. We can set

$$N_1 = (M_1\backslash\{2, 3, r_1+1\}) \cup \{4, r_1+2\},$$
$$N_2 = \{3, r_1+1\},\ \text{and } N_3 = M_3 = \{2, r_1+3, \dots, n\}.$$

If $r > r_1 + 1$, then $\sigma = r + 1 - 2 = r - 1$. We can set

$$N_1 = (M_1\backslash\{2, 3, r_1+1\}) \cup \{4, r_1+2\},$$
$$N_2 = (M_2\backslash\{4, r_1+2\}) \cup \{3, r_1+1, r+1\},$$

and

$$N_3 = M_3 = \{2, r+2, \dots, n\}.$$

If δ' is an even number with $\sigma' = \delta'/2$ and $\sigma' = 1$, then $\delta' = 2$ and $\sigma = r$. We shall have the following two subcases depending on whether $r = r_1 + 1$.

If $r = r_1 + 1$, then we can set

$$N_1 = (M_1\backslash\{1, r_1+1\}) \cup \{r_1+2\},\ N_2 = \{2, r_1+1\},\ \text{and } N_3 = M_3.$$

If $r > r_1 + 1$, then we can set

$$N_1 = (M_1\backslash\{1, r_1+1\}) \cup \{r_1+2\},$$
$$N_2 = (M_2\backslash\{r_1+2\}) \cup \{r_1+1, r+1\},\ \text{and } N_3 = M_3.$$

Case 2.2: $\sigma' > \delta'$. Since $\sigma' \in M_1$, it follows that $\sigma' \leq r_1 + 1$. We consider following two subcases.

Case 2.2.1: $r \geq r_1 + 2$. Let $\tau = r_1 + 2 - \sigma'$.

If $\tau \neq \sigma'$ and $\tau \neq \delta'$, then we can set

$$N_1 = (M_1 \backslash \{\tau, \sigma'\}) \cup \{r_1+2\},\ N_2 = (M_2 \backslash \{r_1+2\}) \cup \{\tau, r+1\}, \text{ and } N_3 = M_3.$$

If $\tau = \delta'$ where $\delta' \neq 1, 2$, then we can set

$$N_1 = (M_1 \backslash \{1, \delta' - 1, \sigma'\}) \cup \{r_1 + 2\},$$
$$N_2 = (M_2 \backslash \{r_1 + 2\}) \cup \{1, \delta' - 1, r + 1\}, \text{ and } N_3 = M_3.$$

If $\tau = \delta' = 2$, then $\sigma' = r_1$. We can set

$$N_1 = (M_1 \backslash \{1, 3, r_1\}) \cup \{2, r_1 + 2\},$$
$$N_2 = (M_2 \backslash \{2, r_1 + 2\}) \cup \{1, 3, r + 1\}, \text{ and } N_3 = M_3.$$

If $\tau = \delta' = 1$, then $\sigma' = r_1 + 1$. We can set

$$N_1 = (M_1 \backslash \{2, r_1 + 1\}) \cup \{1, r_1 + 2\},$$
$$N_2 = (M_2 \backslash \{1, r_1 + 2\}) \cup \{2, r + 1\}, \text{ and } N_3 = \{r_1 + 1, r + 2, \ldots, n\}.$$

If $\tau = \sigma'$, then $\sigma' < \sigma' + \delta' < 2\sigma' = r_1 + 2$. Hence $\alpha \equiv \sigma' + \delta'$ is a number in M_1 between σ' and $r_1 + 2$. We can set

$$N_1 = (M_1 \backslash \{\sigma', \alpha\}) \cup \{\delta', r_1 + 2\},$$
$$N_2 = (M_2 \backslash \{\delta', r_1 + 2\}) \cup \{\alpha, r + 1\}, \text{ and } N_3 = M_3.$$

Case 2.2.2: $r = r_1 + 1$. In this case, $M_2 = \{\delta'\}$. Since $\sigma' > \delta'$ and $\sigma' + \sigma = r_1 + 2$, it follows that $\sigma + \delta' < r_1 + 2$.

First, we assume that $\sigma + \delta' = r_1 + 1$. Then $\sigma' = \delta' + 1$. Recall that $\delta' \leq r_1$.

If $\delta' < r_1$, then we can set

$$N_1 = (M_1 \backslash \{\delta' + 1, r_1 + 1\}) \cup \{\delta', r_1 + 2\}, N_2 = \{r_1 + 1\}, \text{ and } N_3 = M_3.$$

If $\delta' = r_1 \neq 1$, then $\sigma' = r_1 + 1$. We can set

$$N_1 = (M_1 \backslash \{1, r_1 + 1\}) \cup \{r_1 + 2\},\ N_2 = \{r_1 + 1\},$$

and

$$N_3 = (M_3 \backslash \{r_1 + 1\}) \cup \{1, r_1\} = \{1, r_1, r_1 + 3, \ldots, n\}.$$

If $\delta' = r_1 = 1$, then $a_1 = a_2 = 2$, which is not allowed.

Second, we assume that $\sigma+\delta' < r_1+1$. Recall that $\sigma+\sigma' = r+1 = r_1+2$. Let $a \equiv \delta' + \sigma > \delta'$.

If $a \neq \sigma'$, then we can set

$$N_1 = (M_1 \backslash \{\sigma', a\}) \cup \{\delta', r_1 + 2\}, N_2 = \{a\}, \text{ and } N_3 = M_3.$$

If $a = \sigma' \neq \delta' + 1$, where $\delta' \neq 1$, then we can set

$$N_1 = (M_1 \backslash \{1, \sigma' - 1, \sigma'\}) \cup \{\delta', r_1 + 2\}, N_2 = \{1, \sigma' - 1\}, \text{ and } N_3 = M_3.$$

If $a = \sigma' \neq \delta' + 1$, where $\delta' = 1$, then $\sigma = \sigma' - 1$. We can set

$$N_1 = (M_1 \backslash \{\sigma' - 1, \sigma'\}) \cup \{r_1 + 2\}, N_2 = \{1, \sigma' - 1\}, \text{ and } N_3 = M_3.$$

If $a = \sigma' = \delta' + 1$, where $\delta' \neq 1$, then $\sigma = 1$, $\sigma' = r_1 + 1 = a$, and $\delta' = r_1$. We can set

$$N_1 = (M_1 \backslash \{1, r_1+1\}) \cup \{r_1+2\}, N_2 = \{1, r_1\}, \text{ and } N_3 = \{r_1+1, r_1+3, \ldots, n\}.$$

If $a = \sigma' = \delta' + 1$, where $\delta' = 1$, then $a = \sigma' = 2$, $\sigma = 1$, and $r_1 = 1$, hence, $a_1 = a_2 = 2$, which is not allowed. □

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