Vertex-Neighbor-Integrity of Trees

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Abstract. Let G be a graph. A vertex subversion strategy of G, S, is a set of vertices in G whose closed neighborhood is deleted from G. The survival-subgraph is denoted by G/S. The vertex-neighbor-integrity of G, VNI(G), is defined to be $\mathrm{VNI(G)} = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\}$, where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S. In this paper, we show the minimum and the maximum vertex-neighbor-integrity among all trees with any fixed order, and also show that for any integer l between the extreme values there is a tree with the vertex-neighbor-integrity l.

I. Introduction

In 1987 Barefoot, Entringer, and Swart introduced the integrity of a graph to measure the "vulnerability" of the graph. [1] [2] In 1990 [5] we modeled a spy network by a graph whose vertices represent the stations and whose edges represent the lines of communication, and considered the question of not only removing some vertices but also of removing all of their adjacent vertices. Now we connect this idea and the concept of integrity to create a new graph parameter, called "vertex-neighbor-integrity".

To make the development of this new parameter clear, we give the definitions step by step. Let G be a graph and u be any vertex in G. $N(u) = \{v \in V(G) | v \neq u,\ v \text{ and } u \text{ are adjacent}\}$ is the *open neighborhood* of u, and $N[u] = \{u\} \cup N(u)$ denotes the *closed neighborhood* of u. A vertex u in G is said to be *subverted* if the closed neighborhood of u, $N[u]$, is deleted from G. A set of vertices $S = \{u_1, u_2, ..., u_m\}$ is called a *vertex subversion strategy* of G if each of the vertices in S has been subverted from

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G. Let G/S be the *survival-subgraph* left after each vertex of S has been subverted from G. The *vertex-neighbor-integrity* of a graph G, VNI(G), is defined to be

$$\mathrm{VNI}(G) = \min_{S \subseteq V(G)} \{|S| + \omega(G/S)\},$$

where S is any vertex subversion strategy of G, and $\omega(G/S)$ is the maximum order of the components of G/S.

Example 1: K_n is a complete graph of oder n. $\mathrm{VNI}(K_n) = 1$.
Let v be a vertex in K_n. Then $K_n/\{v\} = \emptyset$, so $|\{v\}|+\omega(K_n/\{v\}) = 1+0 = 1$. Therefore,

$$\mathrm{VNI}(K_n) = \min_{S \subseteq V(K_n)} \{|S| + \omega(K_n/S)\} = |\{v\}| + \omega(K_n/\{v\}) = 1.$$

Example 2: $K_{n,m}$, where $n > 1$ and $m > 1$, is a complete bipartite graph with a bipartition (X,Y), where $|X| = n$ and $|Y| = m$. $\mathrm{VNI}(K_{n,m}) = 2$.
Let v be a vertex in $K_{n,m}$. Then $K_{n,m}/\{v\}$ contains $n-1$ (or $m-1$) isolated vertices, so

$$\begin{aligned}\mathrm{VNI}(K_{n,m}) &= \min_{S \subseteq V(K_{n,m})} \{|S| + \omega(K_{n,m}/S)\} \\ &= |\{v\}| + \omega(K_{n,m}/\{v\}) \\ &= 1 + 1 = 2.\end{aligned}$$

In this paper, we provide the minimum and maximum vertex-neighbor-integrity among all trees with any fixed order, and also that for any integer l between the extreme values there is a tree whose vertex-neighbor-integrity is l. $\lceil x \rceil$ is the smallest integer greater than or equal to x. $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

II. The Vertex-Neighbor-Integrity of Trees

Since $\mathrm{VNI}(K_{1,n-1})$=1 for $n \geq 1$, and $\mathrm{VNI}(G) \geq 1$ for any connected graph G, it follows that $\mathrm{VNI}(T) \geq \mathrm{VNI}(K_{1,n-1})$ for all trees T of order $n \geq 1$. If a tree T ($\neq K_{1,n-1}$) has order at least 4, then $\mathrm{VNI}(T) \geq 2$, since there is no vertex v in T such that $T/\{v\} = \emptyset$. Therefore $K_{1,n-1}$ is a unique tree that has the minimum vertex-neighbor-integrity among all trees of order $n \geq 1$.

Next, we show that the path P_n has the maximum vertex-neighbor-integrity among all trees of order $n \geq 1$. First, we evaluate the vertex-neighbor-integrity of P_n.

Lemma 1 [2]: For positive integers, n and m, if n is fixed, then the function $g(m) = m + \lceil n/m \rceil$ has the minimum value $\lceil 2\sqrt{n} \rceil$ at $m = \lceil \sqrt{n} \rceil$.

Theorem 2:

$$\mathrm{VNI}(P_n) = \begin{cases} \lceil 2\sqrt{n+3} \rceil - 4, & \text{if } n \geq 2; \\ 1, & \text{if } n = 1. \end{cases}$$

Proof: If $n = 1$, it is clear that VNI(P_n)=1. Hence assume that $n \geq 2$. Let $V(P_n)=\{v_1, v_2, v_3, ..., v_n\}$, and $S=\{v_{i_1}, v_{i_2}, v_{i_3}, ..., v_{i_m}\}$ be any subset of $V(P_n)$. If $v_i \in S$, then P_n/S does not contain v_{i-1}, v_i, and v_{i+1}. Hence

$$\omega(P_n/S) \geq \left\lceil \frac{n - 3|S|}{|S| + 1} \right\rceil = \left\lceil \frac{n - 3m}{m + 1} \right\rceil.$$

$$\mathrm{VNI}(P_n) = \min_{S \subseteq V(P_n)} \{|S| + \omega(P_n/S)\}$$

$$\geq \min_{m \geq 0} \left\{ m + \left\lceil \frac{n - 3m}{m + 1} \right\rceil \right\} \qquad (1)$$

$$= -4 + \min_{m \geq 0} \left\{ m + 1 + \left\lceil \frac{n + 3}{m + 1} \right\rceil \right\}$$

$$= -4 + \lceil 2\sqrt{n+3} \rceil. \qquad \text{(By Lemma 1.)}$$

$|S|=m=\lceil \sqrt{n+3} \rceil - 1$ achieves the minimum value of $\{m + \lceil (n-3m)/(m+1) \rceil\}$ and the equality of (1) holds by taking S to be a set of m equally spaced vertices of P_n with the distance of two consecutive members equal to $3 + \lceil (n-3m)/(m+1) \rceil$. $m = \lceil \sqrt{n+3} \rceil - 1$ and $n - 3m \geq 0$ if and only if $n \geq 9$ or $n = 6$. If $n \geq 9$ or $n = 6$, then the set of S is taken as described above. If $n = 1$ or 2, S is chosen the set of any single vertex. If $3 \leq n \leq 5$, S is chosen the set of a single central vertex. If $n = 7$ or 8, let $V(P_n) = \{v_i | 1 \leq i \leq n\}$ and $E(P_n) = \{[v_i, v_{i+1}] | 1 \leq i \leq n-1\}$. Then S is chosen the set of the vertices v_3 and v_6. QED.

Using the following lemma, we now show that the path P_n has the maximum vertex-neighbor-integrity among all trees of order n.

Lemma 3: If T is a tree of order n and $0 \leq m \leq n$, then there is a subset $S \subseteq V(T)$ such that $|S| = m$ and $\omega(T/S) \leq \lceil (n-3m)/(m+1) \rceil$.

Proof: Assume that the result is not true for some n, and let T be a tree of order n with largest diameter, say d, satisfying

$$\omega(T/S) > \left\lceil \frac{n-3|S|}{|S|+1} \right\rceil,$$

for any subset $S \subseteq V(T)$. From the proof of Theorem 2, we know that $T \not\cong P_n$, i.e., $d \leq n-2$. Let $P=(v_1, v_2, ..., v_{d+1})$ be a longest path in T. Then there is a vertex v in the path P such that the degree of v is greater than 2; let the least index of such vertices be k. Then $1 < k < d+1$. Now construct the tree T′ which is $T - [v_k, v_{k+1}] + [v_1, v_{k+1}]$ with order n and diameter $d' > d$ (as shown in Figure 1). By the assumption on T, there is a subset $S' \subseteq V(T') = V(T)$ such that $|S'| = m$ and

$$\omega(T'/S') \leq \left\lceil \frac{n-3m}{m+1} \right\rceil.$$

The subversion of v_{k+1} from T produces the path component $(v_1, v_2, ..., v_{k-1})$ and the component containing v_{k+3}, called C_{k+3}. (Note: C_{k+3} may be $\emptyset$.) The subversion of v_{k+1} from T′ produces the component containing the path $(v_2, v_3, ..., v_k)$, called C'_2, and the component containing v_{k+3}, called C'_{k+3}. (Note: C'_{k+3} may be $\emptyset$.) C'_2 contains a $(k-1)$-path and $C_{k+3} \cong C'_{k+3}$, so $T/\{v_{k+1}\} \subseteq T'/\{v_{k+1}\}$. It is clear that $T/\{v_k\} \subseteq T'/\{v_k\}$. $T/\{v\} \cong T'/\{v\}$, where v is adjacent to v_{k+1} in T′ and $v \neq v_1$. $T/\{u\} \subseteq T'/\{u\}$, where u is adjacent to v_k in T′. Hence v_{k+1}, v_k, v, u $\notin S'$, for all v, $v \neq v_1$ and v is adjacent to v_{k+1} in T′, and for all u, u is adjacent to v_k in T′.

Next we show that $v_1 \notin S'$:

Assume that $v_1 \in S'$.

(1) If there is no vertex v_{t_i}, where $2 \leq t_i \leq k-2$, in S$'$, then let S be S$'$ with v_1 replaced by v_k. T/S and T$'$/S$'$ have only one different component as follows: T/S has the path component $\mathcal{P} = (v_1, v_2, ..., v_{k-2})$, and T$'$/S$'$ has the component $\mathcal{C}'$ containing the path $(v_3, v_4, ..., v_k)$. Besides that, T/S and T$'$/S$'$ have the same components. Since the order of $\mathcal{P} <$ the order of $\mathcal{C}'$, all of the components of T/S have sizes smaller than or equal to $\omega(\mathrm{T}'/\mathrm{S}')$, and $\omega(\mathrm{T/S}) \leq \omega(\mathrm{T}'/\mathrm{S}') \leq \lceil (n-3m)/(m+1) \rceil$, a contradiction.

(2) If there are vertices v_{t_1}, v_{t_2}, ..., v_{t_r}, where $r \geq 1$, in S$'$ with $2 \leq t_1 < t_2 < ... < t_r \leq k-2$, then we consider the following cases.

(i) If $t_1 = 2$ and $t_2 \geq 4$, then let S be S$'$ with v_{t_1} replaced by v_k, and v_{t_j} replaced by v_{t_j-2}, where $2 \leq j \leq r$.

(ii) If $t_1 = 2$ and $t_2 = 3$, then let S be S$'$ with v_{t_1} replaced by v_k, v_{t_2} replaced by any vertex u adjacent to v_k in T$'$, and v_{t_j} replaced by v_{t_j-2}, where $3 \leq j \leq r$.

(iii) If $t_1 \geq 3$, then let S be S$'$ with v_{t_j} replaced by v_{t_j-2}, where $1 \leq j \leq r$, and v_1 replaced by v_k.

Now we consider the sizes of all components of T/S and T$'$/S$'$. T/S and T$'$/S$'$ have the following different components:

T/S has the components ——

path $\mathcal{P}_1 = (v_3, ..., v_{t_2-4})$, if $t_1 = 2$ and only if $t_2 \geq 7$, or
$(v_{t_1}, ..., v_{t_2-4})$, if $t_1 \geq 3$,
path $\mathcal{P}_j = (v_{t_j}, ..., v_{t_{j+1}-4})$, where $2 \leq j \leq r-1$, and only if $r \geq 3$,
path $\mathcal{P}_r = (v_{t_r}, ..., v_{k-2})$,
$\mathcal{C}_u$: the component containing u_2, u_3, ..., (note: $\mathcal{C}_u$ may be $\emptyset$), as shown in Figure 1,
path $\mathcal{P}_0 = (v_1, ..., v_{t_1-4})$, only if $t_1 \geq 5$.

T$'$/S$'$ has the components ——

path $\mathcal{P}_1' = (v_4, ..., v_{t_2-2})$, if $t_1 = 2$ and only if $t_2 \geq 6$, or
$(v_{t_1+2}, ..., v_{t_2-2})$, if $t_1 \geq 3$,
path $\mathcal{P}_j' = (v_{t_j+2}, ..., v_{t_{j+1}-2})$, where $2 \leq j \leq r-1$, and only if $r \geq 3$,
$\mathcal{C}_r'$: the component containing the path $(v_{t_r+2}, ..., v_k)$, u_1, and $\mathcal{C}_u$ (containing u_2, u_3, ...), (note: $\mathcal{C}_u$ may be $\emptyset$), as shown in Figure 1,
path $\mathcal{P}_0' = (v_3, ..., v_{t_1-2})$, only if $t_1 \geq 5$.

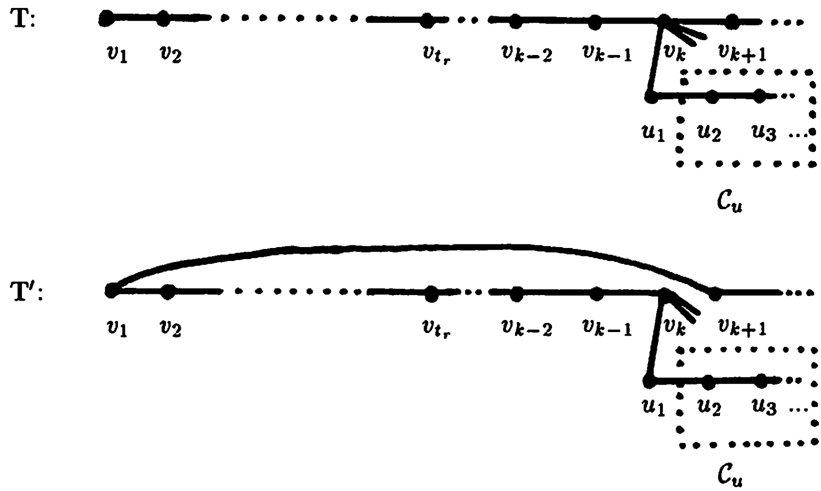

Figure 1

Besides above, T/S and T′/S′ have the same components. Since the order of $\mathcal{P}_1 \leq$ the order of $\mathcal{P}'_1$, the order of $\mathcal{P}_j =$ the order of $\mathcal{P}'_j$, for all $2 \leq j \leq r-1$, the order of $\mathcal{P}_r <$ the order of $\mathcal{C}'_r$, the order of $\mathcal{C}_u <$ the order of $\mathcal{C}'_r$, and the order of $\mathcal{P}_0 =$ the order of $\mathcal{P}'_0$, all of the components of T/S have the sizes smaller than or equal to $\omega(\mathrm{T}'/\mathrm{S}')$, and $\omega(\mathrm{T}/\mathrm{S}) \leq \omega(\mathrm{T}'/\mathrm{S}') \leq \lceil (n-3m)/(m+1) \rceil$, a contradiction.

Therefore $v_1 \notin \mathrm{S}'$.

Since it has been shown that $v_k, v_{k+1}, v, u \notin \mathrm{S}'$, for all v adjacent to v_{k+1} in T′, and for all u adjacent to v_k in T′, we know v_k and v_{k+1} must be in T′/S′. It follows that there must exist $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$, $r \geq 1$, in S′ with $2 \leq i_1 < i_2 < i_3 < \ldots < i_r \leq k-2$, since otherwise v_k and v_{k+1} are in the same component of T′/S′. Thus, taking S=S′, $\omega(\mathrm{T}/\mathrm{S}) = \omega(\mathrm{T}'/\mathrm{S}') \leq \lceil (n-3m)/(m+1) \rceil$, a contradiction.

Let S* be S′ with v_{i_j} replaced by $v_{i_j+k-i_r}$, where $1 \leq j \leq r$. Then $\omega(\mathrm{T}/\mathrm{S}^*) > \lceil (n-3m)/(m+1) \rceil$, and this can hold only if the path component $(v_1, v_2, \ldots, v_{i_1+k-i_r-2})$ of T/S* has order greater than $\lceil (n-3m)/(m+1) \rceil$. Hence

$$i_1 + k - i_r - 2 \geq \left\lceil \frac{n-3m}{m+1} \right\rceil + 1.$$

Let C'_k and C'_{k+1} be two different components in T'/S' containing v_k and v_{k+1} respectively, and h be the number of vertices in C'_{k+1} that are not in the set $\{v_1, v_2, ..., v_{i_1-2}\}$, hence

$$1 \leq h \leq \left\lceil \frac{n-3m}{m+1} \right\rceil - (i_1 - 2) \leq k - i_r - 1.$$

Now, let S be the set S′ with v_{i_j} replaced by v_{i_j+h}, $1 \leq j \leq r$, and consider the sizes of the components of T/S. By the constructions of S′ and S, all components of T/S, except those containing v_1 and v_k, have at most $\lceil (n-3m)/(m+1) \rceil$ vertices. The vertex set of the component of T/S containing v_1 is obtained from the vertex set of C'_{k+1} by deleting the h vertices $C'_{k+1} - \{v_1, v_2, ..., v_{i_1-2}\}$ and appending vertices $v_{i_1-1}, v_{i_1}, v_{i_1+1}, ..., v_{i_1+h-2}$ with no change in number of vertices. Similarly, the vertex set of the component of T/S containing v_k is obtained from the vertex set of C'_k by deleting the h vertices $v_{i_r+2}, v_{i_r+3}, ..., v_{i_r+h}, v_{i_r+h+1}$ and appending the h vertices of $C'_{k+1} - \{v_1, v_2, ..., v_{i_1-2}\}$ with no change in number of vertices. Hence $\omega(T/S) \leq \lceil (n-3m)/(m+1) \rceil$, a contradiction.

Therefore we obtain the result of the lemma. QED.

Theorem 4: The path P_n has the maximum vertex-neighbor-integrity among all trees of order $n \geq 1$.

Proof: It is trivial for $n = 1$.

Let T be any tree of order $n \geq 2$. By Lemma 3, we have

$$\mathrm{VNI}(T) = \min_{S \subseteq V(T)} \{|S| + \omega(T/S)\}$$

$$\leq \min_{m \geq 0} \left\{ m + \left\lceil \frac{n-3m}{m+1} \right\rceil \right\}, \qquad \text{if } |S| = m.$$

By the proof of Theorem 2, $\mathrm{VNI}(P_n) = m + \lceil (n-3m)/(m+1) \rceil$ with $m = \lceil \sqrt{n+3} \rceil - 1$. Therefore the path P_n has the maximum vertex-neighbor-integrity among all trees of order $n \geq 1$. QED.

From the proof of Theorem 2, it is easy to obtain the following result, which will be used later on.

Corollary 5: There are only paths $P_n = P_2$, P_3, or P_6 satisfying the following condition (A) —— for any subset S of $V(P_n)$, if $\mathrm{VNI}(P_n) = |S| + \omega(P_n/S)$, where $n \geq 2$, then $\omega(P_n/S) = 0$.

Proof: If P_n, where $2 \leq n \leq 5$, satisfies the condition (A), then by the proof of Theorem 2, $n = 2$ or $n = 3$.

If P_n, where $n \geq 9$ or $n = 6$, satisfies the condition (A), let S^* be a subset of $V(P_n)$ satisfying $VNI(P_n) = |S^*| + \omega(P_n/S^*)$, then by the proof of Theorem 2,

$$\omega(P_n/S^*) = \left\lceil \frac{n - 3|S^*|}{|S^*| + 1} \right\rceil,$$

where

$$|S^*| = \lceil \sqrt{n+3} \rceil - 1.$$

Since P_n satisfies the condition (A) and $VNI(P_n) = |S^*| + \omega(P_n/S^*)$, we have $\omega(P_n/S^*) = 0$, and hence $VNI(P_n) = |S^*|$. By Theorem 2, $VNI(P_n) = \lceil 2\sqrt{n+3} \rceil - 4$. Therefore $\lceil 2\sqrt{n+3} \rceil - 4 = \lceil \sqrt{n+3} \rceil - 1$, and $\lceil 2\sqrt{n+3} \rceil - \lceil \sqrt{n+3} \rceil = 3$. This implies that $n = 6$ or 9 (since $n = 7$ and 8 have been excluded).

It is easy to verify that P_6 satisfies the condition (A) but P_7, P_8, and P_9 do not. Hence there are only paths $P_n = P_2$, P_3, or P_6 satisfying the condition (A) —— for any subset S of $V(P_n)$, if $VNI(P_n) = |S| + \omega(P_n/S)$, where $n \geq 2$, then $\omega(P_n/S) = 0$. QED.

Lemma 6 [2]: For any positive integer n, $\lceil 2\sqrt{n+1} \rceil - \lceil 2\sqrt{n} \rceil \leq 1$, and the equality holds if and only if $n = m^2$ or $n = m^2 + m$ for some integer m.

We have shown that the path P_n has the maximum vertex-neighbor-integrity among all trees of order n. However, P_n is not the only tree which has the maximum vertex-neighbor-integrity. We evaluate the vertex-neighbor-integrity of $T_{n,k}$ (as shown in Figure 2), where $1 \leq k \leq n-2$, in the next theorem, stating also that there are at least $\lfloor \sqrt{n+3} - (13/4) \rfloor$ nonisomorphic trees of order n having the same vertex-neighbor-integrity as P_n.

Theorem 7:

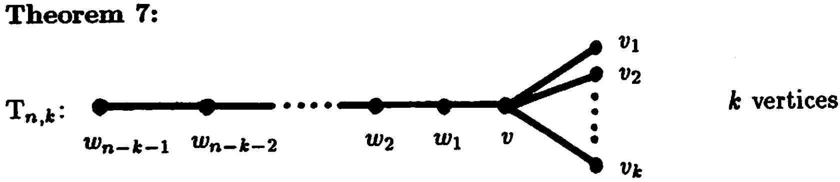

Figure 2

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \begin{cases} \lceil 2\sqrt{n+3}\rceil - 4, & \text{if } 1 \le k \le \sqrt{n+3} - \frac{13}{4}; \\ \lceil 2\sqrt{n-k}\rceil - 3, & \text{if } \sqrt{n+3} - \frac{13}{4} \le k \le n-7 \text{ and } k \ne n-9; \\ \lceil 2\sqrt{n-k+1}\rceil - 3, & \text{if } k = n-5, n-6, n-9; \\ 2, & \text{if } k = n-4, n-3; \\ 1, & \text{if } k = n-2. \end{cases}$$

Proof: If $k = n-2$, $\mathrm{T}_{n,k}$ is a star, and $\mathrm{VNI}(\mathrm{T}_{n,k}) = 1$.

Now we consider the case of $k \le n-3$. Let S^* be a subset of $\mathrm{V}(\mathrm{T}_{n,k})$ satisfying $\mathrm{VNI}(\mathrm{T}_{n,k}) = |\mathrm{S}^*| + \omega(\mathrm{T}_{n,k}/\mathrm{S}^*)$.

If $v_i \in \mathrm{S}^*$, for some end-vertex v_i, we may let $\mathrm{S}' = (\mathrm{S}^* - \{v_i\}) \cup \{v\}$, then

$$|\mathrm{S}'| + \omega(\mathrm{T}_{n,k}/\mathrm{S}') \le |\mathrm{S}^*| + \omega(\mathrm{T}_{n,k}/\mathrm{S}^*).$$

Since

$$\begin{aligned} |\mathrm{S}^*| + \omega(\mathrm{T}_{n,k}/\mathrm{S}^*) &= \mathrm{VNI}(\mathrm{T}_{n,k}) \\ &= \min_{\mathrm{S} \subseteq \mathrm{V}(\mathrm{T}_{n,k})} \left\{ |\mathrm{S}| + \omega(\mathrm{T}_{n,k}/\mathrm{S}) \right\}, \end{aligned}$$

we have

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = |\mathrm{S}'| + \omega(\mathrm{T}_{n,k}/\mathrm{S}').$$

Therefore without loss of generality, we may assume that the end-vertices $v_1, v_2, \ldots, v_k \notin \mathrm{S}^*$.

Now we discuss two possibilities:

Case 1. If $v \in \mathrm{S}^*$, then

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \mathrm{VNI}(\mathrm{P}_{n-(k+2)}) + 1$$

$$= \begin{cases} (\lceil 2\sqrt{(n-k-2)+3}\rceil - 4) + 1, & \text{if } n-(k+2) \geq 2; \\ 1+1, & \text{if } n-(k+2) = 1. \end{cases}$$

(By Theorem 2.)

$$= \begin{cases} \lceil 2\sqrt{n-k+1}\rceil - 3, & \text{if } k \leq n-4; \\ 2, & \text{if } k = n-3. \end{cases}$$

Case 2. If $v \notin S^*$, then let $P = (w_1, w_2, w_3, ..., w_{n-k-1})$ be a path (as shown in Figure 2), and r be the least index for which $w_r \in S^*$.

Subcase 1. If $r = 1$, then

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \begin{cases} 2, & \text{if } n-(k+3) = 0, \\ \mathrm{VNI}(\mathrm{P}_{n-(k+3)}) + 2, & \text{if } n-(k+3) = 2, 3, 6, \\ \mathrm{VNI}(\mathrm{P}_{n-(k+3)}) + 1, & \text{if } n-(k+3) \neq 0, 2, 3, 6. \end{cases}$$

(By Corollary 5.)

$$= \begin{cases} 2, & \text{if } k = n-3, \\ \lceil 2\sqrt{n-k}\rceil - 2, & \text{if } k = n-5, n-6, n-9, \\ 2, & \text{if } k = n-4, \\ \lceil 2\sqrt{n-k}\rceil - 3, & \text{if } k \neq n-3, n-4, n-5, n-6, n-9. \end{cases}$$

(By Theorem 2.)

Subcase 2. If $r > 1$, then $v, v_1, v_2, ...,$ and v_k are in the same component of $T_{n,k}/S^*$. So

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \mathrm{VNI}(\mathrm{P}_n) = \lceil 2\sqrt{n+3}\rceil - 4.$$

If $k = n-4$ and $k \geq 1$, then $\lceil 2\sqrt{n-k+1}\rceil - 3 = 2$, and $\lceil 2\sqrt{n+3}\rceil - 4 \geq 2$. If $k = n-3$ and $k \geq 1$, then $\lceil 2\sqrt{n+3}\rceil - 4 \geq 2$.

Hence,

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = 2, \quad \text{if } \ k = n-3 \ \text{ or } \ k = n-4,$$

and

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \min_{1 \le k \le n-5} (\lceil 2\sqrt{n-k+1} \rceil - 3, \lceil 2\sqrt{n-k} \rceil - h, \lceil 2\sqrt{n+3} \rceil - 4),$$

where $h = 2$ (when $k = n-5, n-6, n-9$) or 3 (when $k \neq n-5, n-6, n-9$). By Lemma 6,

$$\lceil 2\sqrt{n-k+1} \rceil - 3 \le \lceil 2\sqrt{n-k} \rceil - 2,$$

where $k = n-5, \ n-6, \ \text{or} \ n-9$.

$$\lceil 2\sqrt{n+3} \rceil - 4 \le \lceil 2\sqrt{n-k} \rceil - 3 \quad \text{if } \ k \le \sqrt{n+3} - \frac{13}{4}$$

and

$$\lceil 2\sqrt{n+3} \rceil - 4 \ge \lceil 2\sqrt{n-k} \rceil - 3 \quad \text{if } \ k \ge \sqrt{n+3} - \frac{13}{4}.$$

Therefore,

$$\mathrm{VNI}(\mathrm{T}_{n,k}) = \begin{cases} \lceil 2\sqrt{n+3} \rceil - 4, & \text{if } 1 \le k \le \sqrt{n+3} - \frac{13}{4}; \\ \lceil 2\sqrt{n-k} \rceil - 3, & \text{if } \sqrt{n+3} - \frac{13}{4} \le k \le n-7 \ \text{ and } \\ & \quad k \neq n-9; \\ \lceil 2\sqrt{n-k+1} \rceil - 3, & \text{if } k = n-5, n-6, n-9; \\ 2, & \text{if } k = n-4, n-3; \\ 1, & \text{if } k = n-2. \end{cases}$$

QED.

Among all trees of order n, the maximum vertex-neighbor-integrity is $\lceil 2\sqrt{n+3} \rceil - 4$, and the minimum is 1. For any integer l between the extreme values, we also can find a tree whose vertex-neighbor-integrity is l, as shown below.

Theorem 8: If l is any integer, where $1 \leq l \leq \lceil 2\sqrt{n+3} \rceil - 4$, then there is a tree T of order n such that VNI(T) $= l$.

Proof: First we show that $\text{VNI}(T_{n,k+1}) \leq \text{VNI}(T_{n,k}) \leq \text{VNI}(T_{n,k+1}) + 1$, for all k, $1 \leq k \leq n-3$. (Compare $T_{n,k+1}$ in Figure 3 with $T_{n,k}$ in Figure 2.)

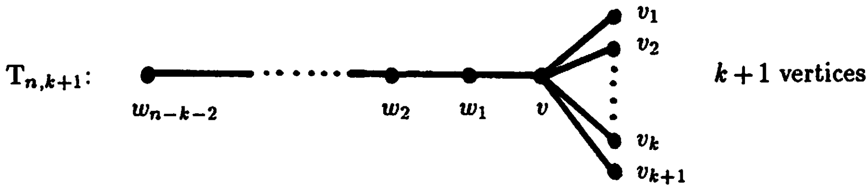

Figure 3

By Theorem 7, it is easy to obtain the first inequality. To show the second inequality, we let S^* be a subset of $V(T_{n,k+1})$ satisfying $\text{VNI}(T_{n,k+1}) = |S^*| + \omega(T_{n,k+1}/S^*)$. It is clear that $v_{k+1} \notin S^*$. So S^* is also a subset of $V(T_{n,k})$. Then $\omega(T_{n,k}/S^*) \leq \omega(T_{n,k+1}/S^*) + 1$ and $\text{VNI}(T_{n,k}) \leq |S^*| + \omega(T_{n,k}/S^*) \leq |S^*| + \omega(T_{n,k+1}/S^*) + 1 = \text{VNI}(T_{n,k+1}) + 1$.

Since path $P_n = T_{n,1}$ and star $K_{1,n-1} = T_{n,n-2}$ are extreme cases of $T_{n,k}$, it follows that $T_{n,k}$, where $1 \leq k \leq n-2$, achieve all values between $\text{VNI}(K_{1,n-1}) = 1$ and $\text{VNI}(P_n) = \lceil 2\sqrt{n+3} \rceil - 4$. QED.

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