On The Sizes Of Least Common Multiples Of Several Pairs Of Graphs

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ABSTRACT. For nonempty graphs G and H, H is said to be G-decomposable (written $G|H$) if $E(H)$ can be partitioned into sets $E_1, \cdots, E_n$ such that the subgraph induced by each E_i is isomorphic to G. If H is a graph of minimum size such that $F \mid H$ and $G \mid H$, then H is called a least common multiple of F and G. The size of such a least common multiple is denoted by $\ell\mathrm{cm}(F,G)$. We show that if F and G are bipartite, then $\ell\mathrm{cm}(F,G) \leq q(F) \cdot q(G)$, where equality holds if $(q(F), q(G)) = 1$. We also determine $\ell\mathrm{cm}(F,G)$ exactly if F and G are cycles or if $F = P_m$, $G = K_n$ where n is odd and $\left(m-1, \frac{1}{2}(n-1)\right) = 1$, in the latter case extending a result in [8].

1 Introduction

A nonempty graph H is *decomposable* into the subgraphs $G_1, \cdots, G_n$ of H if G_i is isolate-free (*i.e.*, has no isolated vertices) for each $i = 1, \cdots, n$ and $E(H)$ can be partitioned into $E(G_1) \cup \cdots \cup E(G_n)$. If $G_i \cong G$ for each i, then H is *G-decomposable*, in which case G *divides* H or, equivalently, is a *divisor* of H, denoted by $G \mid H$. If H is G-decomposable into at least two copies of G and $G \ncong K_2$, then H is *non-trivially* G-decomposable and G is a *proper divisor* of H. Clearly, $K_2 \mid H$ and $H \mid H$; thus K_2 and H are called the *trivial divisors* of H. Obviously, if $G \mid H$ then $q(G) \mid q(H)$. That the converse is not true can be seen by, for example, taking $H \cong K_{1,3} + e$ and $G \cong 2K_2$.

We henceforth consider only isolate-free graphs. Following [3], H is called a *least common multiple* of F and G if H is a graph of minimum size such that $F \mid H$ and $G \mid H$. The set of all least common multiples of F and G is denoted by $\mathrm{LCM}(F, G)$ and the size of any such graph by $\ell\mathrm{cm}(F, G)$. Least common multiples of any nonempty finite set of graphs are defined similarly. That every such set of graphs has a least common multiple follows directly from the following result of Wilson [10].

Theorem A. [10]. *Let F be a graph of size q. Then $F \mid K_p$ provided p is sufficiently large, $q \mid \binom{p}{2}$ and $d \mid (p-1)$, where d is the greatest common divisor of the degrees of the vertices of F.*

The sizes of least common multiples of several classes of graphs were determined in [3,8,9]. For other results on least common multiples and the related concept of greatest common divisors (defined in [3]), see [1,2,4,6,7].

While it is obvious that $\ell\mathrm{cm}(F, G) = k\,\ell\mathrm{cm}(q(F), q(G))$ for some integer $k \geq 1$, Theorem A does not provide a good upper bound for k. A lower bound is given in Theorem B. In Section 2 we show that for F and G bipartite, $\ell\mathrm{cm}(F, G) \leq q(F) \cdot q(G)$. Equality obviously holds if $(q(F), q(G)) = 1$.

Theorem B. [8]. *For any connected graphs F and G,*

$$\ell\mathrm{cm}(F, G) \geq \begin{cases} q(G) & \text{if } p(F) \leq p(G) \text{ and } q(F) \mid q(G) \\ ML & \text{otherwise} \end{cases}$$

where

$$L = \ell cm\,(q(F), q(G))$$

and

$$M = \max\left\{ \left\lceil \frac{2\delta(G)q(F)}{\Delta(F)L} \right\rceil, \left\lceil \frac{(p(F)-1)q(G)}{(p(G)-1)L} \right\rceil \right\}.$$

In Section 3 we determine $\ell\mathrm{cm}(C_m, C_n)$ for all $m, n \geq 3$, thus improving the result for bipartite graphs when m and n are even.

Least common multiples of paths versus complete graphs were investigated in [8], where $\ell\mathrm{cm}(P_m, K_3)$, $\ell\mathrm{cm}(P_m, K_4)$ and $\ell\mathrm{cm}(P_m, K_n)$, for all $m \geq 2$ and n odd, $\left(m-1, \binom{n}{2}\right) = 1$, were determined. In Section 4 we determine $\ell\mathrm{cm}(P_m, K_n)$ where n is odd and $\left(m-1, \frac{1}{2}(n-1)\right) = 1$.

2 Bipartite Graphs

In general, upper bounds obtained by using Theorem A would be extremely large and no good general upper bound for $\ell\mathrm{cm}(F, G)$ is known. We begin by improving this situation for bipartite graphs and obtain an upper bound in this case which is often equal to the lower bound in Theorem B.

Proposition 1. *For any bipartite graphs F and G,*

$$\ell cm(F, G) \leq q(F) \cdot q(G).$$

Proof: Informally, we construct a bipartite graph H with $F \mid H$, $G \mid H$ by replacing each edge e of F with a copy of G, where the partite sets of G replace the endvertices of e. Let F (G, respectively) have bipartition (A, B) ((C, D), respectively), where $A = \{a_1, \cdots, a_q\}$, $B = \{b_1, \cdots, b_r\}$, $C = \{c_1, \cdots, c_s\}$ and $D = \{d_1, \cdots, d_t\}$. Let $V(H) = V \cup W$, where

$$V = \{v_{11}, \cdots, v_{1s}, v_{21}, \cdots, v_{2s}, \cdots, v_{qs}\}$$

and

$$W = \{w_{11}, \cdots, w_{1t}, w_{21}, \cdots, w_{2t}, \cdots, w_{rt}\}.$$

Join v_{ij} to w_{xy} if and only if $a_i b_x \in E(F)$ and $c_j d_y \in E(G)$. Consider any fixed edge $c_j d_y$ of G. Then the edges $\{v_{ij} w_{xy} \mid a_i b_x \in E(F)\}$ induce a copy of F in H and such copies of F corresponding to distinct edges of G are edge disjoint, hence $F \mid H$. Similarly, $G \mid H$ and the result follows since $q(H) = q(F) \cdot q(G)$. □

Since the graph H constructed above is bipartite, repeated applications of Proposition 1 immediately give

Corollary 2. *For any bipartite graphs $G_1, \cdots, G_n$,*

$$\ell cm(G_1, \cdots, G_n) \leq \prod_{i=1}^{n} q(G_i).$$

Further, if the graphs G_i have coprime sizes, then equality holds.

3 Cycles

By determining $\ell\text{cm}(C_m, C_n)$ for all pairs of cycles, we now improve Proposition 1 in the case where m and n are even. For any path P in a graph G, let $-P$ be the path obtained by reversing the direction of P.

Theorem 3.

(i) $\ell cm(C_m, C_{km}) = 2km \qquad (k \geq 2)$

(ii) $\ell cm(C_m, C_n) = \ell cm(m, n)$ *otherwise.*

Proof: (i) Clearly, $\ell\text{cm}(C_m, C_{km}) \neq km$, hence $\ell\text{cm}(C_m, C_{km}) \geq 2km$. We construct $G \in \text{LCM}(C_m, C_{km})$ with $q(G) = 2km$ as follows: Partition km

into $2k$ parts of size at most $m-1$ each; say $km = n_1 + \cdots + n_{2k}$, where $n_i \le m-1$ for each i, and note that $km = (m-n_1)+\cdots+(m-n_{2k})$. Consider two disjoint cycles $R \cong S \cong C_{km}$. Partition R (S, respectively) into $2k$ consecutive internally disjoint paths R_i (S_i, respectively) with length n_i ($m-n_i$ respectively) and endvertices u_{i-1} and u_i (w_{i-1} and w_i), where $u_0 = u_{2k}$ ($w_0 = w_{2k}$), $i = 1, \cdots, 2k$. Let G be the graph obtained by identifying u_i and w_i for each $i = 1, \cdots, 2k$. Clearly, $C_{km} \mid G$. Also, the $2k$ copies of C_m in G are the cycles formed by R_i followed by $-S_i$, $i = 1, \cdots, 2k$. Therefore $G \in \mathrm{LCM}(C_m, C_{km})$ and $\ell\mathrm{cm}(C_m, C_{km}) = 2km$.

(ii) Let $n > m$ and $k = \gcd(m,n)$; say $m = rk$ and $n = sk$ so that $\ell\mathrm{cm}(m,n) = rsk$. If $r = 2$, let G be the graph of size $rsk = 2n$ obtained from two copies of C_n with vertex sequences $v_0, \cdots, v_{n-1}$ and $w_0, \cdots, w_{n-1}$, respectively, by identifying the vertices v_{ik} and w_{ik} for each $i = 0, \cdots, s-1$. As in the proof of (i), $C_m \mid G$ and $C_n \mid G$. Thus we may henceforth assume that $r \ge 3$.

Partition n into $s-1$ parts of size at most $m-1$ each, say $n = n_1 + \cdots + n_{s-1}$ where $0 < n_j \le m-1$ for each j. (Elementary arithmetic shows that this is always possible.) Similarly, partition m into $r-1$ parts of size less than $n-1$ each, say $m = m_1 + \cdots + m_{r-1}$, $0 < m_i < n-1$ for each i. Note that

$$\sum_{i=1}^{r-1}(n-m_i) = (r-1)n - \sum_{i=1}^{r-1} m_i = rsk - n - m = \sum_{j=1}^{s-1}(m-n_j). \quad (1)$$

Consider three disjoint cycles $R \cong C_m$, $S \cong C_n$ and $T \cong C_{rsk-m-n}$. Partition R (S, respectively) into $r-1$ ($s-1$, respectively) consecutive internally disjoint paths R_i (S_j) with length m_i (n_j) and endvertices u_{i-1} and u_i (v_{j-1} and v_j), where $u_0 = u_{r-1}$ ($v_0 = v_{s-1}$), for $i = 1, \cdots, r-1$ ($j = 1, \cdots, s-1$). Partition T into $r-1$ ($s-1$ respectively) consecutive internally disjoint paths T_i (T'_j) with length $n-m_i$ ($m-n_j$) and endvertices x_{i-1} and x_i (y_{j-1} and y_j), for $i = 1, \cdots, r-1$ ($j = 1, \cdots, s-1$), where $x_0 = x_{r-1}$ ($y_0 = y_{s-1}$). Note that by (1), these partitions of T are always possible. Also note that possibly $x_i = y_j$ for some i and j.

Now let G be the graph obtained by identifying u_i and x_i (v_j and y_j) for each $i = 0, \cdots, r-2$ ($j = 0, \cdots, s-2$). Then $C_m \mid G$, the copies of C_m being R and the cycles formed by S_j followed by $-T'_j$, $j = 1, \cdots, s-1$; similarly, the copies of C_n in G are S and the cycles formed by R_i followed by $-T_i$, $i = 1, \cdots, r-1$. Since $q(G) = rsk = \ell\mathrm{cm}(m,n)$ the result follows. □

4 Paths versus Complete Graphs

If we restrict Theorem B to the case $F \cong P_m$ and $G \cong K_n$, we obtain

Theorem C. [8]. *For all integers $m \geq 2$ and $n \geq 3$,*

$$\ell cm(P_m, K_n) \geq \begin{cases} \binom{n}{2} & \text{if } m \leq n \text{ and } m-1 \mid \binom{n}{2} \\ ML & \text{otherwise,} \end{cases}$$

where $L = \ell cm\left(m-1, \binom{n}{2}\right)$ and $M = \lceil (m-1)(n-1)/L \rceil$.

In the case where n is odd, the following upper bound was also obtained in [8].

Theorem D. [8]. *For all $m \geq 2$ and $n \geq 3$, where n is odd,*

$$\ell cm(P_m, K_n) \leq (m-1)\binom{n}{2}.$$

An immediate corollary of the above two results is

Theorem E. [8]. *For all $m \geq 2$ and $n \geq 3$, where n is odd and $\left(m-1, \binom{n}{2}\right) = 1$,*

$$\ell cm(P_m, K_n) = (m-1)\binom{n}{2} = ML.$$

We now extend Theorem E to show that the lower bound given in Theorem C is also exact if $\left(m-1, \frac{1}{2}(n-1)\right) = 1$. We need the following result, which can easily be obtained as a consequence of the standard proof of the fact that every complete graph of odd order n is the edge sum of $\frac{1}{2}(n-1)$ hamilton cycles (*cf.* [5, p. 237]).

Theorem F. *Let u and v be two vertices of K_n, where n is odd. There exists an edge-decomposition of K_n into $n-1$ (u,v)-paths $Q_1, \cdots, Q_{n-1}$ such that Q_i has length i and $Q_i \cup Q_{n-i}$ is a hamilton cycle of K_n.*

(Graphs of size $\binom{n}{2}$ (for some integer $n \geq 2$) which are decomposable into $n-1$ paths $Q_1, \cdots, Q_{n-1}$ with $|E(Q_i)| = i$ $(1 \leq i \leq n-1)$ are called *path perfect* graphs.)

Theorem G. *If n is even, then K_n is decomposable into $n-1$ paths of length $\frac{1}{2}n$, all of which originate at the same vertex of K_n.*

Proof: Let $V(K_n) = \{0, \cdots, n-1\}$ and consider the representation of K_n obtained by arranging the vertices $1, \cdots, n-1$ in a regular $(n-1)$-gon with vertex 0 in the centre. Let Q_i be the path with vertex sequence $0, i, i+1, i-1, i+2, i-2, \cdots, r$, where $r = i + \frac{1}{4}n$ if $n \equiv 0 \pmod 4$ and $r = i - \left\lfloor \frac{1}{4}n \right\rfloor$ if $n \equiv 2 \pmod 4$. Then Q_i has length $\frac{1}{2}n$ and it is easy to see from the representation that $E(Q_1) \cup \cdots \cup E(Q_n)$ partitions $E(K_n)$. □

The following two types of graphs will be used in the construction of least common multiples of P_m and K_n. Let $G_1 \cong \cdots \cong G_q \cong K_n$. Denote by

$F_{n,q}$ the chain of graphs $G_1, \cdots, G_q$ such that $V(G_i) \cap V(G_{i+1}) = \{a_{i+1}\}$ and $V(G_i) \cap V(G_j) = \emptyset$ if $j \notin \{i-1, i, i, +1\}$, and by $H_{n,q}$ the necklace consisting of $G_1, \cdots, G_q$ with the same conditions as for $F_{n,q}$, but with arithmetic modulo q. In $F_{n,q}$, let a_1 (a_{q+1}, respectively) be any vertex of G_1 (G_q, respectively) distinct from a_2 (a_q, respectively).

We use the notation of Theorem C in the following theorems.

Theorem 4. *If $n \mid m-1$ and n is odd, then*

$$\ell cm(P_m, K_n) = (m-1)(n-1) = ML.$$

Proof: Let $m - 1 = \mu n$ for some integer $\mu \geq 1$. By Theorem C,

$$\ell \mathrm{cm}(P_m, K_n) \geq \lceil (m-1)(n-1)/L \rceil \times L,$$

where $L = \ell \mathrm{cm}\left(m-1, \ \frac{1}{2}n(n-1)\right)$. Now, $(m-1)(n-1)$ is a multiple of $m-1$ and of $\frac{1}{2}n(n-1) = (m-1)(n-1)/2\mu$. Thus $(m-1)(n-1)/L$ is integral and $\ell \mathrm{cm}(P_m, K_n) \geq (m-1)(n-1) = ML$.

For the reverse inequality, let $q = 2\mu$ and consider the graph $F_{n,q}$ with $\frac{1}{2}qn(n-1)$ edges as well as the $(n-1)$ (a_1, a_{q+1})-paths P_j, passing through $a_2, \cdots, a_q$, and respectively obtained by the concatenation of q paths guaranteed by Theorem F: Let $Q_{r,i}$ denote the relevant path of length i in the subgraph G_r of $F_{n,q}$. Then

$$\begin{aligned}
P_1 &= Q_{1,n-1} \cup Q_{2,1} \cup Q_{3,n-1} \cup Q_{4,1} \cup \cdots \cup Q_{q,1} \\
P_2 &= Q_{1,n-2} \cup Q_{2,2} \cup Q_{3,n-2} \cup Q_{4,2} \cup \cdots \cup Q_{q,2} \\
&\vdots \\
P_{n-1} &= Q_{1,1} \cup Q_{2,n-1} \cup Q_{3,1} \cup Q_{4,n-1} \cup \cdots \cup Q_{q,n-1}.
\end{aligned}$$

Each P_j has length $\mu n = m-1$ and these $(n-1)$ paths partition $E(F_{n,q})$. Consequently, $\ell \mathrm{cm}(P_m, K_n) = ML$. □

Theorem 5. *If $m - 1 \mid n$, then $\ell cm(P_m, K_n) = ML$.*

Proof: Let $n = \nu(m-1)$ for some integer $\nu \geq 1$. If $m \leq n$ and $m - 1 \mid \binom{n}{2}$, *i.e.* if $\nu > 1$ and n is odd or ν is even, then by Theorem C, $\ell \mathrm{cm}(P_m, K_n) \geq \binom{n}{2}$. But in this case, $L = \frac{1}{2}\nu(m-1)(n-1) \geq (m-1)(n-1)$ and so $ML = \binom{n}{2}$. In all other cases $ML = n(n-1)$. The lower bound is given by Theorem C. We show that the upper bound also holds.

If n is odd and $\nu = 1$, the result is proved in Theorem 4. If n is odd and $\nu > 1$, then K_n is decomposable into $\frac{1}{2}(n-1)$ hamilton cycles, each of which is decomposable into ν paths of length $m-1$. If n and ν are both even, then by Theorem G, K_n is decomposable into $n-1$ paths of length $\frac{1}{2}n = \frac{1}{2}\nu(m-1)$, each of which is decomposable into $\frac{1}{2}\nu$ paths of length

$m-1$. If n is even and ν is odd, consider the graph $F_{n,2}$. For $i = 1, 2$, the subgraph $G_i \cong K_n$ of $F_{n,2}$ is decomposable into $n-1$ paths of length $\frac{1}{2}n$, each path originating at a_2. Hence $F_{n,2}$ is decomposable into $n-1$ paths of length n, each of which is decomposable into ν paths of length $m-1$. □

Now let $m-1 = k\mu$ and $n = k\nu$, where n is odd, $\mu, \nu \geq 2$, $(\mu, \nu) = 1$, $k > 1$ *i.e.* $(m-1, n) \neq 1$ and $(\mu, \frac{1}{2}(n-1)) = 1$. (Note that since $(k, n-1) = 1$, $(\mu, \frac{1}{2}(n-1)) = 1$ if and only if $(m-1, \frac{1}{2}(n-1)) = 1$.) Then, using the notation of Theorem C,

$$L = \ell\text{cm}\left(k\mu, \frac{1}{2}k\nu(n-1)\right) = \frac{1}{2}k\mu\nu(n-1) = \frac{1}{2}\mu n(n-1)$$
$$= \frac{1}{2}(m-1)\nu(n-1),$$
$$M = \left\lceil \frac{2}{\nu} \right\rceil = 1,$$

and

$$ML = \frac{1}{2}\mu n(n-1).$$

Theorem 6. *If n is odd and $(m-1, \frac{1}{2}(n-1)) = 1$, then*

$$\ell cm(P_m, K_n) = \frac{1}{2}\mu n(n-1) = ML.$$

Proof: If $\mu = 1$, the result is proved in Theorem 5 and if $\nu = 1$, the result is proved in Theorem 4. Hence we may assume $\mu > 1$ and $\nu \geq 3$ (note that ν is odd). Then $m-1 \nmid \binom{n}{2}$ and by Theorem C, $\ell\text{cm}(P_m, K_n) \geq ML$. For the reverse inequality we exhibit a graph $G \in \text{LCM}(P_m, K_n)$ of size $\frac{1}{2}\mu n(n-1)$. We consider three cases.

Case 1. $\mu = 2$. In the graph $G = F_{n,2}$ each G_i, $i = 1, 2$, decomposes into $\frac{1}{2}(n-1)$ hamilton cycles, each of which decomposes into 2 paths originating at a_2 (where $\{a_2\} = V(G_1) \cap V(G_2)$) and of respective length $m-1 = 2k$ and $n-(m-1) = (\nu-2)k$. Each of the $\frac{1}{2}(n-1)$ paths of length $(\nu-2)k$ of G_1 can be concatenated with one of the $\frac{1}{2}(n-1)$ paths of length $(\nu-2)k$ of G_2 to form a path of length $2k(\nu-2)$, which decomposes into $\nu-2$ paths of length $m-1$. Therefore $F_{n,2} \in \text{LCM}(P_m, K_n)$.

Case 2. $\mu \geq 3$ is odd. Consider the eulerian graph $G = H_{n,\mu}$ of size $\frac{1}{2}\mu n(n-1)$. Let S' be the sequence of integers

$$n-1, 1, n-2, 2, n-3, 3, \cdots, \frac{1}{2}(n+1), \frac{1}{2}(n-1)$$

and let $S = \{s_r\}$ be the sequence of length $\mu(n-1)$ obtained by the concatenation of μ copies of S'. Let R be the circuit of G obtained by traversing the necklace clockwise $n-1$ times, beginning at a_1, and by using the (a_i, a_{i+1})-path Q_{s_r} of length s_r in G_i if G_i is the r'th copy of K_n encountered on the circuit. Using arithmetic modulo μ, the path Q_{n-1} of length $n-1$ is used in the subgraphs G_i of $H_{n,\mu}$ for

$$i \in \{1, (n-1)+1, 2(n-1)+1, \cdots, (\mu-1)(n-1)+1\}.$$

These subgraphs are all distinct, for suppose $\alpha(n-1)+1 \equiv \beta(n-1)+1 \pmod{\mu}$ for $0 \le \alpha < \beta \le \mu - 1$. Then $(\beta-\alpha)(n-1) \equiv 0 \pmod{\mu}$, contradicting $(\mu, n-1) = 1$. Therefore the path Q_{n-1} is used exactly once in each G_i, $i = 1, \cdots, \mu$. Similarly, each Q_j, $j = 1, \cdots, n-1$, is used exactly once in each G_i. Thus R is an eulerian circuit of G. Now consider any vertex u of G, say $u \in V(G_i)$, and let C be the cycle of length x (say) consisting of the subtrail of R between two successive passages through u. If $E(C) \cap E(G_i) = \{a_i a_{i+1}\}$, then C contains n edges in any pair of successive copies of K_n, beginning with G_{i+1}, and thus $x = 1 + \frac{1}{2}(\mu-1)n$. Otherwise, if C contains at least two edges in G_i, then C contains at least $n-1$ edges in any two successive copies of K_n beginning with G_{i+1}, and therefore $x \ge 2 + \frac{1}{2}(\mu-1)(n-1)$. Thus

$$x > 1 + \frac{1}{2}(\mu-1)(n-1) \ge 1 + \frac{1}{2}(\mu-1)(3k-1)$$

and

$$\begin{aligned} x-(m-1) &> 1 - k\mu + \frac{1}{2}(\mu-1)(3k-1) \\ &= \frac{1}{2}(\mu-3)(k-1) \\ &\ge 0. \end{aligned}$$

Therefore R can be decomposed into paths of length $k\mu = m-1$ and thus $H_{n,\mu} \in \mathrm{LCM}(P_m, K_n)$.

Case 3. $\mu \ge 4$ is even. Let $G = H_{n,\mu}$ and for each $i = 1, \cdots, n-1$, let S_i be the sequence $n-i, i, n-i, i, \cdots, n-i, i$ of length μ. Let S^* be the sequence $S_1 S_2 \cdots S_{n-1}$ of length $\mu(n-1)$ and define the eulerian circuit R^* of G similar to R in Case 2, using the sequence S^* instead of S. For any vertex u of G, the number of edges of R^* between two consecutive passages through u is at least

$$\begin{aligned} &\frac{1}{2}(\mu-2)n + 3 > \frac{1}{2}(\mu-2)\,k\nu \\ &\ge \mu k = m-1 \quad \text{if} \quad \mu \ge 6 \quad \text{or if} \quad \nu \ge 4 \quad \text{and} \quad \mu \ge 4. \end{aligned}$$

Thus in these cases, R^* can be decomposed into paths of length $m-1$. In the case $\nu = 3$ and $\mu = 4$, *i.e.*, $n = 3k$ and $m-1 = 4k$, let T_i, $i = 1, \cdots, \lfloor 3k/2 \rfloor$, be the sequence $i, i, 3k - i, 3k - i, 3k - i, 3k - i, i, i$. By using Theorem F to ensure that an edge-decomposition of each K_{3k} is obtained, let X_i be the closed trail of length $12k$ in $G = H_{3k,4}$ corresponding to T_i. Each X_i can be decomposed into three subtrails of length $4k$ and since each subtrail contains at most three of the vertices a_1, a_2, a_3 and a_4, a decomposition of X_i into three paths P_{4k+1} results. But the concatenation of the X_i forms an eulerian circuit of G and we thus have the required path decomposition of G. □

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