To the paper of H.L. Abbott and B. Zhou on 4-critical planar graphs

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In [1] it is proved that each 4-critical plane graph contains either a 4- or a 5-cycle or otherwise a face of size between 6 and 11.

This may be strengthened as follows:

Theorem. *If a plane graph has the minimum degree at least 3 and no triangles sharing an edge, then there is a face of size between 4 and 10; the bound 10 is the best possible.*

Proof: To obtain an extremal construction, we just cut off the corners of the icosahedron.

Now let G be a counterexample to the main statement of our Theorem. The Euler formula

$$|V| - |E| + |F| = 2$$

for G may be rewritten in the symmetrical form

$$\sum_{x \in V \cup F} (r(x) - 4) = \sum_{x \in V \cup F} g(x) = -8, \tag{1}$$

where $r(x)$ is the number of edges incident with x. We redistribute the contribution $g(x)$ of vertices and faces of C as follows. Each face f transfers to each incident vertex v: $1/3$ if $r(f) = 3$; $2/3$ if $r(f) > 3$, $r(v) = 3$, and v is incident with a triangle; and $1/3$ otherwise. The modified contributions are denoted by g^*. Next, we prove that $g^*(x) \geq 0$ for each $x \in V \cup F$.

Let f be a face in G. If $r(f) = 3$, then $g^*(f) = g(f) - 3(-1/3) = 3 - 4 + 3 \cdot 1/3 = 0$. If $r(f) \geq 12$, then $g^*(f) \geq r(f) - 4 - r(f) \cdot 2/3 = (r(f) - 12)/3 \geq 0$. Due to the properties of G, it remains to assume that $r(f) = 11$. Observe that f can not be incident to eleven 3-vertices, each of which is incident with a triangle. Therefore $g^*(f) \geq 7 - 10 \cdot 2/3 - 1/3 = 0$.

Now take $v \in V$. Remind that since no triangles share an edge, v is incident with at most $\lfloor r(v)/2 \rfloor$ triangles. If $r(v) = 3$, we have two subcases:

If v is incident with a triangle, then $g^*(v) \geq -1-1/3+2\cdot 2/3 = 0$. Otherwise $g^*(v) \geq -1 + 3 \cdot 1/3 = 0$. For $r(v) \geq 4$, it holds $g^*(v) \geq 0$. Therefore it remains to note that the number of non-triangular faces incident to v, each of which contribute at least $1/3$ to $g^*(v)$, is not less than the number of triangles incident to v, contributing $-1/3$ each.

Thus we have from (1)

$$0 \leq \sum_{x \in V \cup F} g^*(x) = \sum_{x \in V \cup F} g(x) = -8.$$

This contradiction completes the proof.

References

[1] H.L. Abbott and B. Zhou, On small faces in 4-critical planar graphs. *Ars Combinatoria* 32(1991) 203–207.