Hamiltonian cycles in the square of a two-connected graph with given circumference

Antoni Marczyk
Instytut Matematyki AGH
Kraków, Al. Mickiewicza 30
Poland

ABSTRACT. Let p denote the circumference of a two-connected graph G. We construct a hamiltonian cycle in G^2 which contains more than $p/2$ edges of G. Using this construction we prove some properties of hamiltonian cycles in the square of G.

1 Introduction

At the Graph Theory Conference in Niedzica, Poland, in 1990, Günter Schaar asked if it is true that the square of every two-connected minimal graph G with circumference $\geq p$ contains a hamiltonian cycle which passes through p or more edges of G.

According to the well-known result of Fleischner [4] the square of every two-connected graph is hamiltonian. Recently Schaar [8] using a result of Riha [6] proved the following conjecture of Traczyk [9].

Theorem 1. *For every two-connected graph G of order $n \geq 4$ there exists a hamiltonian cycle in G^2 that contains at least four edges of G.* □

Let $c(G)$ denote the circumference of G, i.e. the length of a longest cycle of G. In the present article we show that in G^2 we can find a hamiltonian cycle containing at least $\min(c(G), \lceil c(G)/2 \rceil + 4)$ edges of G, provided G is two-connected graph of order ≥ 3. So we give an answer to Schaar's question for $p = 3, \ldots, 9$. Applying this result we obtain some properties of hamiltonian cycles in the square of a two-connected graph.

2 Definitions

All graphs considered are finite graphs having no loops or multiple edges. For a graph G, we denote by $V(G)$ the set of vertices of G, the set of edges by $E(G)$.

Let X be a set. The symbols $\mathcal{P}_1(X)$, $\mathcal{P}_2(X)$ and $\mathcal{P}_{1,2}(X)$ stand for the set of one-element, two-element, and one- or two-element subsets of X, respectively.

Let G be a graph and X and Y two elements of $\mathcal{P}_{1,2}(V(G))$. We say that X and Y are matched in G iff every vertex of X is joined to some vertex of Y and every vertex of Y is joined to a vertex of X.

A path-system of a graph G is a subgraph of G whose each component is a path. A path which goes from x to y will be called an $x-y$ path. By $l(h)$ we denote the length of a path h.

Let $h_i = y_{11}, y_{12}, \dots, y_{1r_i}$ $(i = 1, \dots, n)$ and
$h = x_1, y_{11}, y_{12}, \dots, y_{1r_1}, x_2, y_{21}, \dots, y_{2r_2}, x_3, \dots, y_{n1}, \dots, y_{nr_n}, x_n$
be paths of a graph H. We shall use the following notation for h:
$h = x_1, h_1, x_2, h_2, x_3, \dots, h_n, x_n$.

Two graphs G_1 and G_2 are homeomorphic iff there are graphs G_1', G_2' such that G_i' is obtained from G_i by substituting some of the edges G_i by paths, $i = 1, 2$, and G_1', G_2', are isomorphic.

A two-connected graph G is said to be minimal if for any edge e of G the graph $G - e$ is not two-connected.

A graph G is 1-hamiltonian if it is hamiltonian and every graph obtained by removing some vertex of G is also hamiltonian.
All concepts not defined in this paper can be found in [1].

3 Preliminaries

Let G be a $4t$-vertex graph consisting of a cycle C_{2t} of vertices $y_o, y_1, \dots, y_{2t-1}$ $(t \geq 2)$ and t vertex-disjoint paths $h_1, h_2, \dots, h_t$ such that for each $i = 1, \dots, t$, $l(h_i) = 3$, and both end-vertices of h_i belong to C_{2t}. Thus, $d_G(x) = 3$, for x belonging to C_{2t}, and $d_G(x) = 2$ otherwise. Denote by $\mathcal{G}(4t)$ the class of all graphs G defined above.

Suppose that $G \in \mathcal{G}(4t)$. Let $M_o = \{y_o y_1, y_2 y_3, \dots, y_{2t-2} y_{2t-1}\}$ and $M_1 = \{y_1 y_2, y_3 y_4, \dots, y_{2t-1} y_o\}$ be two perfect matchings of C_{2t} (M_o and M_1 are subgraphs of G) and let z_i denote the neighbour of y_i in G that does not belong to C_{2t} $(i = 0, \dots, 2t-1)$. We shall write that a graph $\hat{G}$ belongs to the class $\hat{\mathcal{G}}(4t)$ if $\hat{G}$ can be obtained from a graph $G \in \mathcal{G}(4t)$ by replacing each edge $y_{2k-1} y_{2k}$ $(k = 1, \dots, t-1)$ by two independent edges e_k and f_k connecting the set $\{y_{2k-1}, z_{2k-1}\}$ with the set $\{y_{2k}, z_{2k}\}$.

Lemma 1. *Every graph $\hat{G}$ which belongs to the class $\hat{\mathcal{G}}(4t)$ contains a hamiltonian cycle which will be called hamiltonian cycle generated by M_o.*

Proof: The paths $h_1, \dots, h_t$ together with all edges of M_o form a two-factor D of $\hat{G}$. Let A be the connected component of D that contains the edge $y_o y_1$. If $A = D$ then A is a hamiltonian cycle of $\hat{G}$ generated by M_o;

if $A \neq D$, define $r = \min\{k \mid y_{2k}y_{2k+1}$ does not belong to $A\}$ and let B be the cycle of D containing $y_{2r}y_{2r+1}$. Construct a new cycle A_1 by removing the edges $y_{2r-1}z_{2r-1}$ and $y_{2r}z_{2r}$ from A and B and adding the edges e_r and f_r. Clearly A_1 contains all vertices of A and B. Then replace in D the cycles A and B by A_1 and proceed in the same manner. This process can be repeated until a hamiltonian cycle has been constructed. □

A tree T in which the vertices of degree > 1 form the empty set or induce a path $x_1, x_2, \ldots, x_n$ $(n \geq 1)$ will be called a caterpillar. It is known (see [5]) that the square of a tree T of order ≥ 3 is hamiltonian if and only if T is a caterpillar. A caterpillar with $n \geq 1$ is denoted also by $T(P_1, P_2, \ldots, P_n)$, where $P_i = \{p_{i1}, p_{i2}, \ldots, p_{ir_i}\}$ $(i = 1, \ldots, n)$ is the set of one-degree vertices adjacent to x_i.

Lemma 2. *Let G be a graph and $T = T(P_1, \ldots, P_n)$ $(n \geq 2;\ (P_1, P_n \neq \emptyset)$ a subgraph of G which is a caterpillar. Suppose that there is a partition $(S_i)_{i \in \{1,\ldots,n\}}$ of the set $V(G) - V(T)$ such that for each i, $S_i \cup P_i$ is the set of vertices of a path h_i in G^2 which goes from p_{i1} to p_{ir_i}. Then*

(i) *there is a hamiltonian cycle C in G^2 that contains the edges x_1p_{11} and x_np_{n1};*

(ii) *there exists in G^2 a path h with end-vertices x_1 and x_n that contains $n-1$ edges of G and such that $V(h) = V(G) - (P_1 \cup S_1 \cup P_n \cup S_n)$;*

(iii) *for each $i = 1, \ldots, n-1$ there exist in G^2 a $x_1 - p_{11}$ path h and a $x_n - p_{n1}$ path h' such that h and h' are vertex-disjoint, $V(h) \cup V(h') = V(G)$, h contains an edge of G which is incident to x_i and h' contains an edge of G which is incident to x_{i+1}.*

Proof: It is easy to check that the following cycles $x_1, h_1, x_2, h_3, x_4, \ldots, h_{n-1}, x_n, h_n, x_{n-1}, h_{n-2}, \ldots, h_2, x_1$ for n even ≥ 2, and $x_1, h_1, x_2, h_3, x_4, \ldots, x_{n-1}, h_n, x_n, h_{n-1}, \ldots, h_2, x_1$ for odd, are hamiltonian in G^2 and contain the edges x_1p_{11} and x_np_{n1} of G. Note that this statement is also true for $n = 1$ and $|P_1| \geq 2$.

Since the edges x_1x_2 and x_ip_{i1} $(i = 2, \ldots, n-1)$ belong to G, the path $h = x_1, x_2, h_2, x_3, h_3, \ldots, x_{n-1}, h_{n-1}, x_n$ fulfills the condition (ii).

Now suppose $P_i \neq \emptyset$ and $P_{i+1} \neq \emptyset$. By (i) there are hamiltonian cycles C_1 and C_2 in the subgraphs of G^2 induced by the sets $\sum_{j=1}^{i} \{x_j\} \cup P_j \cup S_j$ and $\sum_{j=i+1}^{n} \{x_j\} \cup P_j \cup S_j$ that contain x_1p_{11} and x_np_{n1}. It is clear that the paths $h = C_1 - x_1p_{11}$ and $h' = C_2 - x_np_{n1}$ satisfy (iii). If $P_i = \emptyset$, then $i > 1$ and we can substitute the caterpillar $T(P_1, \ldots, P_i)$ by $T(P_1, \ldots, P_{i-2}, P_{i-1} \cup \{x_i\})$ and the path h_i by h_i, x_i, and apply the above method. Clearly, the case $P_{i+1} = \emptyset$ can be treated in the same manner. □

Observe that for each $i = 2, \dots, n-2$ we can find a path in G^2 verifying (ii) and containing the edge $x_i x_{i+1}$ and two edges of G incident to x_i and x_{i+1}. Moreover, there is a path g [resp. g'] of G^2 satisfying (ii) which contains $x_1 x_2$ [resp. $x_{n-1} x_n$] and an edge of G incident to x_2 [resp. x_{n-1}].

The following result of Schaar [8] resembles but is different from a theorem of Fleischner [3].

Lemma 3. *For every two-connected graph G with $n \geq 4$ vertices and any vertex $x \in V(G)$ there are vertices $y, z \in V(G)$ being adjacent to x in G and a Hamiltonian path h in $G^2 - x$ joining y and z and containing at least two edges of $E(G - x)$.* □

The following result of Chartrand, Hobbs, Jung, Kapoor and Nash-Williams (see [2]) follows immediately from Lemma 3.

Corollary. *The square of every two-connected graph G of order ≥ 4 is 1-hamiltonian.*

Lemma 4. *Let G be a 2-connected graph and C a cycle in G of length ≥ 4. Then there exist a partition $\mathcal{B} = (B_s)_{s \in I}$ of the set $V(G) - V(C)$ and a function $f : \mathcal{B} \to \mathcal{P}_{1,2}(V(C))$ which satisfy the following conditions:*

(1) each B_s is the set of vertices of a path h_s in G^2,

(2) $| f(h_s) | \leq | B_s |$ and the set of end-vertices of h_s and $f(h_s)$ are matched in G,

(3) if $f(h_s)$ and $f(h_{s'})$ $(s \neq s')$ are both of cardinality 2 then they are disjoint.

Proof: We shall construct a path-system $\mathcal{B}$ and a function f, which have the required properties, in four steps (compare the algorithm (A) in [7]).

Step 1. Denote by $U_3(C)$ the set of all connected components of $G - C$ containing at least 3 vertices and let $S \in U_3(C)$. Let G_S be the graph obtained from S by replacing all the neighbours of S on C by a new vertex O_S, that we join to a vertex x of S iff x is adjacent in G to a vertex of C. Obviously, G_S is a two-connected graph with at least four vertices. By Lemma 3 there exist a Hamiltonian path g_S in $G_S^2 - O_S$ joining two vertices x_o and y_o which are both adjacent to O_S in G_S.

Let F be the set of edges of $G_S^2 - O_S$ that do not belong to G^2, i.e. F is the set of edges xy such that x and y are adjacent to O_S in G_S and the distance in G between x and y is greater than two. The graph $g_S - F$ consists of a set D_S of vertex-disjoint paths (possibly trivial) of the subgraph of G^2 induced by $V(S)$ and such that each end-vertex of a path belonging to D_S is adjacent in G to some vertex of C. Moreover, every vertex of S belongs to a path of D_S.

Suppose now that S is a connected component of $G - C$ which possesses less than three vertices. For S having exactly one vertex, we denote by D_S the one-element set which consists of the trivial path containing the only element of S, and for S such that $| V(S) |= 2$, D_S will stand for the set of two trivial paths each of them containing one vertex of $V(S)$. Observe that in the last case every element of D_S is adjacent to a vertex of C because G is two-connected.

Define $\mathcal{B}' = \bigcup D_S$ where S runs through the set of all connected components of $G - C$ (we shall identify a path belonging to D_S with its set of vertices) and let f' denote an arbitrarily chosen function verifying the condition (1) and (2) (by the construction of $\mathcal{B}'$ such a function exists).

Step 2. We shall construct a spanning path-system $\mathcal{B}_1$ of $G^2 - C$ and a function $f_1 : \mathcal{B}_1 \to \mathcal{P}_{1,2}(C)$ verifying the conditions (1), (2) and the following one:

(4) $f_1(h_\alpha) = f_1(h_\beta)$ and $| f_1(h_\alpha |= 2$ implies $h_\alpha = h_\beta$.

Consider the set $\{h_{\alpha_1}, h_{\alpha_2}, \dots, h_{\alpha_r}\} = (f')^{-1}(\{a,b\}) \neq \emptyset$, where a and b are two distinct vertices of C. For r even, replace all the paths h_{α_i} by a path h of G^2 such that $V(h) = \bigcup_i V(h_{\alpha_i})$ and each end-vertex of h is joined in G to $a = a(h)$. For r odd, replace in the same way the paths $h_{\alpha_2}, h_{\alpha_3}, \dots, h_{\alpha_r}$ by a path h of G^2. Proceeding similarly with every pair a, b of distinct vertices of C with $(f')^{-1}(\{a,b\}) \neq \emptyset$ we obtain a collection $\mathcal{B}_1$ of paths of $G^2 - C$.

Now put $f_1(h) = \{a(h)\}$ for every new path h, and $f_1(h) = f'(h)$ otherwise. Thus the required partition and function have been found.

Step 3. Consider a graph H_o with $V(H_o) = V(C)$ and $E(H_o) = \{xy \mid$ there is a path h of $\mathcal{B}_1$ such that $f_1(h) = \{x,y\}\}$. Let $C' = a_1, a_2, \dots, a_r, a_1$ be a cycle in H_o, and let for $i = 1,..,r$, $e_i = a_i a_{i+1}$ (indices are taken modulo r).

Replace all the paths g_{e_i} in $\mathcal{B}_1$ satisfying $f_1(g_{e_i}) = \{a_i, a_{i+1}\}$ ($i = 1, \dots, r$) by a new one, named g, which is formed by all vertices belonging to paths g_{e_i}, and whose end-points are adjacent to a_1 by two edges of G. Let $\mathcal{B}_2$ be this modified set of paths and H_1 the graph obtained by deleting from H_o all the edges of C'. Define $f_2(g) = \{a_1\}$, and $f_2(h) = f_1(h)$ for h belonging to $\mathcal{B}_2$ and $h \neq g$. Obviously, the set $\mathcal{B}_2$ and the function f_2 satisfy the conditions (1), (2) and (4).

Repeating this operation as many time as needed we obtain a subgraph H of H_o being a forest, a spanning path-system $\mathcal{B}_3$ in $G^2 - C$ and a function f_3 verifying the conditions (1), (2) and (4).

Step 4. Observe that each tree and forest can be decomposed into edge-disjoint paths in such a way that no two paths have the same end-point. Let D be such a decomposition of the graph H, $h = a_1, a_2, \dots, a_r$ a path of D and $g_{a_1a_2}, g_{a_2a_3}, \dots, g_{a_{r-1}a_r}$ the paths of $\mathcal{B}_3$ which correspond to the edges $a_1a_2, \dots, a_{r-1}a_r$ of h. It is clear that there exists a path $g(h)$ in G^2, formed by vertices of all paths $g_{a_ia_{i+1}}$, such that one of its end-vertices is joined in G to a_1 the other one to a_r.

For each $h \in D$ replace the paths $g_{a_1a_2}, g_{a_2a_3}, \dots, g_{a_{r-1}a_r}$ by $g(h)$ and define $f(g(h)) = \{a_1, a_r\}$ for every new path $g(h)$, and $f(g) = f_3(g)$ otherwise. Let $\mathcal{B}$ stands for the obtained set of paths. By the above construction $\mathcal{B}$ and f are the set and function required in Lemma 4. □

4 Results

Before we prove the main theorem we give a short sketch of the proof of this result which improve its readability. Suppose C is a longest cycle in a two-connected graph G and let p denote its length. We shall show that there is a hamiltonian cycle C_2 in G^2 which passes through $\min(\lceil p/2 \rceil + 4, p)$ or more edges of G.

At the beginning we construct a family $\mathcal{B}$ of paths of the graph $V(G) - C$ having the properties (1), (2) and (3) of Lemma 4. Next, we choose these paths of $\mathcal{B}$ whose endvertices are joined in G to two different vertices on C. This path-system and the cycle C form a subgraph G_1 of G^2 homeomorphic to a graph G' belonging to $\mathcal{G}(4t)$ for some t. Now we modify the graph G' in order to obtain a graph $\hat{G}$ of $\hat{\mathcal{G}}(4t)$. It follows from Lemma 1 that $\hat{G}$ possesses a hamiltonian cycle $\hat{C}$. Applying Lemma 2 we can find a subgraph G_2 of G^2 which is homeomorphic to a subgraph of $\hat{G}$ which contains $\hat{C}$. Therefore, there exists a cycle C_1 of G_2 corresponding to $\hat{C}$. Finally, we construct a hamiltonian cycle C_2 by replacing some portions of C_1 by other paths of different lengths. Each of these paths contains a large number of edges of G and we can easily estimate the cardinality of $E(C_2) \cap E(G)$.

Now we are going to prove the following main result.

Theorem 2. *For every two-connected graph G with $c(G) = p \geq 3$ there exists a hamiltonian cycle in G^2 that contains at least* $\min(\lceil p/2 \rceil + 4, p)$ *edges of G.*

Proof: Because the only two-connected graph with $c(G) = 3$ is a triangle, we can assume that $c(G) \geq 4$. Let $C = x_1, x_2, \dots, x_p, x_1$ $(p \geq 4)$ be a longest cycle of G with a fixed orientation. For $u \in V(C)$ we use u^+ to denote the successor of u on C and u^- to denote its predecessor. Consider a partition $\mathcal{B}$ of the set $V(G) - V(C)$ with a function f which satisfy the conditions (1) - (3) of Lemma 4. For a path h of $\mathcal{B}$ with end-points x and y and such that $| f(h) | > 1$ we denote by $\phi(h)$ a path of G^2 obtained by adding an edge $e_x \in E(G)$ joining x to a vertex of $f(h)$ and an edge $e_y \in E(G)$ which joins y to the other vertex of $f(h)$. The graph G_1 formed by the cycle C and all the paths $\phi(h)$ of $\mathcal{B}$ with $| f(h) | = 2$ is a subgraph of G^2 homeomorphic to a graph $G' \in \mathcal{G}(4t)$ for some $t \leq p/2$.

Let $y_o, \dots, y_{2t-1}$ be the vertices of degree 3 in G_1 occurring on C in consecutive order and let z_i $(i = 1 \dots, 2t-1)$ stand for the neighbour of y_i in G_1 that does not belong to C. Denote by $\mu(y_i, y_{i+1})$ the portion of C between y_i and y_{i+1} (indices modulo $2t$) indicated by the order of C and by $[y_i, y_{i+1}]$ the set $\mathcal{P}_1(V(\mu(y_i, y_{i+1})))$. The portions $\mu(y_i^+, y_{i+1}^-)$ and $\mu(y_i^+, y_{i+1})$ of C (for $y_i^+ \neq y_{i+1}$) are analogously defined.

Without any loss of generality we can assume that

$$w = \sum_{i=0}^{t-1} l(\mu(y_{2i}, y_{2i+1})) \geq \sum_{i=1}^{t} l(\mu(y_{2i-1}, y_{2i})).$$

Hence $w \geq \lceil p/2 \rceil$.

By Lemma 2 for every pair y_{2k-1}, y_{2k} there are two vertex-disjoint paths h_k and h'_k in G^2 joining the set $\{y_{2k-1}, z_{2k-1}\}$ to the set $\{y_{2k}, z_{2k}\}$ and such that $V(h_k) \cup V(h'_k)$ is equal to the set of vertices of the path-system $f^{-1}([y_{2k-1}, y_{2k}]) \cup \{\mu(y_{2k-1}, y_{2k})\} \cup \{z_{2k-1}, z_{2k}\}$. Construct a graph $\hat{G} \in \hat{\mathcal{G}}(4t)$ by deleting from G' every edge $y_{2k-1}y_{2k}$ $(k = 1, \dots, t-1)$ and by adding to G' new edges $y_{2k-1}z_{2k}$ and $z_{2k-1}y_{2k}$ [resp. $y_{2k-1}y_{2k}$ and $z_{2k-1}z_{2k}$] if the end-vertices of h_k and h'_k are y_{2k-1}, z_{2k} and z_{2k-1}, y_{2k} [resp. y_{2k-1}, y_{2k} and z_{2k-1}, z_{2k}]. By Lemma 1 there exists in $\hat{G}$ a hamiltonian cycle $\hat{C}$ generated by M_o. Let C_1 be the corresponding cycle in the subgraph G_2 of G^2 consisting of G_1 and paths h_k and h'_k for $k < t$ (G_2 is homeomorphic to $\hat{G} \cup \{e \mid e = y_{2k-1}, y_{2k}, k = 1, \dots, t-1\}$).

Now we shall construct a hamiltonian cycle C_2 in G^2 by replacing some portions of C_1 by other paths of different lengths.

Firstly, for every $k = 1, \dots, t-1$ such that $y_{2k}^+ \neq y_{2k+1}$, replace the portion $\mu(y_{2k}, y_{2k+1})$ of C_1 by a $y_{2k} - y_{2k+1}$ path of G^2 whose vertex set consists of all the vertices of the path-system $f^{-1}([y_{2k}^+, y_{2k+1}^-]) \cup \{\mu(y_{2k}, y_{2k+1})\}$ and which contains at least $l(\mu(y_{2k}, y_{2k+1}))$ edges of G. Secondly, for every $k \in \{1, \dots, t\}$ such that $\hat{C}$ does not contain any edge joining the set $\{y_{2k-1}, z_{2k-1}\}$ to the set $\{y_{2k}, z_{2k}\}$ in $\hat{G}$, replace the edge $y_{2k}z_{2k}$ by a $y_{2k} - z_{2k}$ path whose vertex-set coincides with set of vertices of the path-system $f^{-1}([y_{2k-1}^+, y_{2k}]) \cup \{\mu(y_{2k-1}^+, y_{2k}]\} \cup \{z_{2k}\}$, and the edge $y_{2k-1}z_{2k-1}$ by a $y_{2k-1} - z_{2k-1}$ path whose vertex-set is equal to the set of vertices of $f^{-1}(\{y_{2k-1}\}) \cup \{y_{2k-1}, z_{2k-1}\}$. The existence of such paths is guaranteed by Lemma 2. Moreover, in the second case, each of these paths can have at least one edge of G.

If $\hat{C}$ contains an edge joining $\{y_{2k-1}, z_{2k-1}\}$ and $\{y_{2k}, z_{2k}\}$ then $\hat{C}$ must contain two such edges being independent (because of the construction of $\hat{C}$); now, it is easily seen that we have obtained a hamiltonian cycle C_2 in G^2 which possesses at least $\lceil p/2 \rceil$ edges of G.

We can assume that $\hat{C}$ has no edge which goes from the set $\{y_{2t-1}, z_{2t-1}\}$ to the set $\{y_o, z_o\}$. Therefore, C_2 contains at least $\lceil p/2 \rceil$ +2 edges of G. Moreover, if $U_3(C)$ is the empty set, then, by Lemma 2, C_2 possesses at least p edges of G.

So suppose there is a connected component S of $G - C$ with $\mid V(S) \mid \geq 3$. By Lemma 3 the path-system D_S (see the proof of Lemma 4) contains at least two edges of G. Therefore the cycle C_2 passes through at least $\lceil p/2 \rceil + 4$ edges of G. This completes the proof of the theorem. □

The following two theorems were found by Fleischner and Schaar (see [7]).

Theorem 3. *Let G be a two-connected graph G of order ≥ 4 and let x, y ($x \neq y$) be any two vertices of G. Then there exist three distinct edges e, f and g of G, and a hamiltonian cycle C in G^2 such that*

(a) e, f are incident to x,

(b) g is incident to y,

(c) C contains e, f and g.

Proof: Assume that G is a two-connected graph of order ≥ 4 and let x and y be two distinct vertices of G. By Menger's theorem there exists in G a cycle $C = x_1, x_2, \ldots, x_p, x_1$ with $p \geq 4$ containing x and y. Construct graphs $\hat{G}$, G_2, $\hat{C}$, C_1 in the same way as in the proof of Theorem 2. Now consider an integer k such that $\hat{C}$ contains no edge joining the set $\{y_{2k-1} z_{2k-1}\}$ with the set $\{y_{2k} z_{2k}\}$. We may assume that $k = t$.

Suppose that $y \in \mu(y_j, y_{j+1})$ and consider the following cases.

Case 1. $x \in \mu(y_i^+, y_{i+1}^-)$, $i = j \pmod 2$, $y \neq y_j$ and $y \neq y_{j+1}$. Without any loss of generality we may assume that $i = 0$. Thus the edges $y_i y_{i+1}$ and $y_j y_{j+1}$ belong to the matching M_o. Using Lemma 2 and the construction in the proof of Theorem 2 we obtain a hamiltonian cycle in G^2 which passes through two edges of G incident to x and one or two edges incident to y.

Case 2. $x \in \mu(y_i^+, y_{i+1}^-)$ and $i = j+1 \pmod 2$. We can label the vertices of G_1 in such a way that $j + 1 = 0$. Now $y_i y_{i+1}$ belongs to M_o. We may apply Lemma 2 and the construction of Theorem 2 in order to obtain the required hamiltonian cycle.

Case 3. For some i, $x = y_i$, $i = j \pmod 2$, $y \neq y_j$ and $y \neq y_{j+1}$. Assume $i = 0$. Hence $y_i y_{i+1}$ and $y_j y_{j+1}$ belong to M_o and $y_{i-1} y_i = y_{2t-1} y_o$. We use Lemma 2 and Theorem 2 in order to construct the required hamiltonian cycle.

Case 4. For some i, $x = y_i$, $i = j+1 \pmod 2$, and ($y \neq y_i^-$ or $y \neq y_{i-1}$). We set $i = 2t - 1$ and we obtain the desired cycle in the same way as in Case 3.

Case 5. For some i, $x = y_i$ and $y = y_i^- = y_{i-1}$. Set $i = 0$. We construct a hamiltonian cycle in G^2 in the same manner as in Case 3.

Because all cases have been examined the proof is complete. □

The following results can be obtained by the same method as in the proof of Theorem 3.

Theorem 4. *Let G be a two-connected graph. Then given an edge e of G we can find another edge f of G which is adjacent to e, and a hamiltonian cycle of G^2 containing e and f.* □

5 Concluding remarks

It remains an open problem to prove the following conjecture.

Conjecture: *There exists a hamiltonian cycle in the square two-connected graph G of order ≥ 3 that contains at least $c(G)$ edges of G.*

Note that the bound $c(G)$ cannot be increased. For example, consider a graph G obtained from the complete bipartite graph $K_{2,n}$ by substituting an edge by a path of length $p-3$. It is easy to check that the order of G may be made arbitrarily large, $c(G) = p$ and G^2 does not contain a hamiltonian cycle having more than p edges of G.

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