The fine structure of $(v, 3)$ directed triple systems: $v \equiv 0$ or $1 \pmod 3$

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ABSTRACT. The fine structure of a directed triple system of index λ is the vector $(c_1, c_2, \ldots, c_\lambda)$, where c_i is the number of directed triples appearing precisely i times in the system. We determine necessary and sufficient conditions for a vector to be the fine structure of a directed triple system of index 3 for $v \equiv 0$ or $1 \pmod 3$.

1 Introduction and definitions

Let a, b and c be three distinct elements. A *transitive* or *directed triple* (a, b, c) is a set of three ordered pairs of the form $\{(a, b), (b, c), (a, c)\}$. A *directed triple system* of order v and *index* λ, or (v, λ) DTS, is a pair $(V, \mathcal{D})$. Here V is a v-set of elements, and $\mathcal{D}$ is a collection of directed triples (called blocks) on V, with the property that every *ordered* pair (x, y) of elements of V appears in precisely λ of the directed triples. Directed triple systems have been studied extensively, often under the name "transitive triple systems". The necessary condition for a (v, λ) DTS to exist is simply that the number of ordered pairs $\lambda v(v-1)$ occurring in blocks be divisible by three. Hence, we require $v \equiv 0, 1 \pmod 3$ for $\lambda \equiv 1, 2 \pmod 3$, and we require only $v \neq 2$ for $\lambda \equiv 0 \pmod 3$. It is well-known that these conditions are also sufficient for the existence of (v, λ) DTSs (see Colbourn and Rosa [6] for a recent survey).

The *fine structure* of a directed triple system of index λ is the vector $(c_1, c_2, \ldots, c_\lambda)$, where c_i is the number of directed triples appearing precisely i times in the system. Colbourn, Mathon, Rosa and Shalaby [4] determined the fine structure of threefold triple systems for $v \equiv 1$ or $3 \pmod 6$, and

Colbourn, Mathon and Shalaby [5] determined the fine structure of threefold triple systems for $v \equiv 5 \pmod 6$. In [7] the author found the fine structure of balanced ternary designs with block size 3, index 3 and $\rho_2 = 3$.

It is easy to determine the fine structure of $(v,2)$ DTSs. Let $(V,\mathcal{D})$ and $(V,\mathcal{D}')$ be two $(v,1)$ DTS having exactly m directed triples in common, where $m \in \{0,1,2,3,\ldots,v(v-1)/3\} \setminus \{(v(v-1)/3)-1\}$; see [9]. Then $(V,\mathcal{D}\cup\mathcal{D}')$ is a $(v,2)$ DTS with exactly m doubly repeated blocks. Moreover there does not exist a $(v,2)$ DTS with exactly $(v(v-1)/3)-1$ doubly repeated blocks. Therefore the vector (c_1,c_2) is a fine structure of a $(v,2)$ DTS if and only if $0 \le c_2 \le (v(v-1)/3)$, $c_2 \ne (v(v-1)/3)-1$, and $c_1 + 2c_2 = 2v(v-1)/3$.

In this paper we study the fine structure of $(v,3)$ DTSs. Indeed, we determine the necessary and sufficient conditions for a vector to be the fine structure of a directed triple system of index 3 for $v \equiv 0$ or $1 \pmod 3$. Since any two of $\{c_1,c_2,c_3\}$ determine the third, we use a more convenient notation for the fine structure: (t,s) is said to be the fine structure of a $(v,3)$ DTS if $c_2 = t$ and $c_3 = v(v-1)/3 - s$ (note that $v(v-1)/3$ is an integer since $v \equiv 0, 1 \pmod 3$). We first need to know the pairs (t,s) which can possibly arise as fine structures. We define $Adm(v) = \{(t,s) | 0 \le t \le s \le v(v-1)/3\} \setminus \{(0,1),(1,1),(1,2),(1,3)\}$, and use the notation $Fine(v)$ for the set of fine structures which actually arise in $(v,3)$ DTSs. Our result is as follows.

Main Theorem:
$Fine(4) = Adm(4) \setminus \{(0,2),(0,3)\}$,
$Fine(6) = Adm(6) \setminus \{(0,2),(0,3),(0,4),(1,4)\}$,
$Fine(7) = Adm(7) \setminus \{(0,3),(1,4),(0,5)\}$ and
$Fine(v) = Adm(v)$ for all $v \equiv 0, 1 \pmod 3$, $v \notin \{4,6,7\}$. □

An argument similar to that used in Lemma 2.1 of [4] or Lemma 1.1 of [7] leads to the following result.

Lemma 1.1. *If $(t,s) \in Fine(v)$ then $0 \le t \le s \le v(v-1)/3$.* □

Lemma 1.2. *$Fine(v) \subseteq Adm(v)$.*

Proof: We must only eliminate the cases $(0,1)$, $(1,1)$, $(1,2)$ and $(1,3)$. Let $(V,\mathcal{D})$ be a $(v,3)$ DTS of type (1,3). Let H be the simple directed graph whose edges are the ordered pairs of elements of V which do not appear in a three times repeated block. Let G be the graph H without direction on edges (the underlying graph). Note that G can have at most two edges between any two distinct vertices. Obviously $|E(G)| = 9$ and all vertices of G have even degree. Since $3H$ can be decomposed into directed triples so that none are repeated three times and precisely one is repeated twice it follows that G cannot have more than one vertex of degree two. Therefore G is isomorphic to one of the following graphs in figure 1.

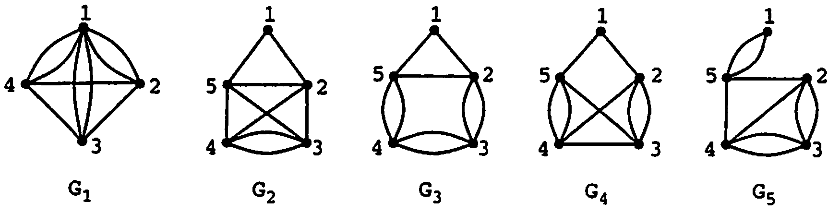


Figure 1

An exhaustive computer search shows that if H is a directed graph whose underlying graph is G_i (above) then $3H$ cannot be decomposed into directed triples so that none are repeated three times and precisely one is repeated twice. So $(1,3) \notin Fine(v)$. A similar method shows that $(0,1)$, $(1,1)$ and $(1,2)$ are not in $Fine(v)$. □

We make use of *pairwise balanced designs* and *group divisible designs* in the next section. A pairwise balanced design ((v, K, λ) PBD) is a collection of subsets of size $k \in K$, called blocks, chosen from a set of v elements in such a way that every pair of elements belongs to λ blocks. A group divisible design, GDD$(K, \lambda, M; v)$, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{m\}$ and $K = \{k\}$ we write GDD$(k, \lambda, m; v)$. Note that if a GDD$(K, 1, M; v)$ exists then a $(v, \{k | k \in K\} \cup \{m | m \in M\}, 1)$ PBD exists.

2 Construction

In this section we show that $Fine(v) = Adm(v)$ for all $v \equiv 0,\ 1 \pmod 3$, $v \geq 15$.

Lemma 2.1. *$(n, n) \in Fine(v)$ for all $0 \leq n \leq v(v-1)/3$ and $n \neq 1$.*

Proof: Let $(V, \mathcal{D})$ and $(V, \mathcal{D}')$ be two $(v, 1)$ DTSs with exactly m blocks in common, where $m \in \{0, 1, 2, \ldots, v(v-1)/3\} \setminus \{(v(v-1)/3) - 1\}$; (see [9]). Then $(V, \mathcal{D} \cup \mathcal{D} \cup \mathcal{D}')$ is a $(v, 3)$ DTS with fine structure $((v(v-1)/3) - m, (v(v-1)/3) - m)$. □

Now we examine $Fine(v)$ for $v = 3$, 4 and 6.

Lemma 2.2. $Fine(3) = \{(0,0), (0,2), (2,2)\}$.

Proof: Let $\mathcal{D} = \{(1,2,3),(3,2,1),(1,3,2),(2,3,1),(2,1,3),(3,1,2)\}$. Then $\mathcal{D}$ yields a $(3,3)$ DTS of type $(0,2)$ on $\{1,2,3\}$. Now the result follows from Lemma 1.2 and Lemma 2.1. □

Lemma 2.3. *$Fine(4) = Adm(4) \setminus \{(0,2),(0,3)\}$.*

Proof: We apply Lemma 2.1 for the types (0,0), (2,2), (3,3) and (4,4). Then we use Designs 1-5 in [8] for the types (0,4), (1,4), (2,4), (3,4) and (2,3). Moreover an exhaustive search shows that $(0,2),(0,3) \notin Fine(4)$. □

Lemma 2.4. *$Fine(6) = Adm(6) \setminus \{(0,2),(0,3),(0,4),(1,4)\}$.*

Proof: First we show that $(0,2)$, $(0,3)$, $(0,4)$ and $(1,4)$ are not in $Fine(6)$. Let $(V,\mathcal{D})$ be a (6,3) DTS of type (t,s), where $(t,s) \in \{(0,2),(0,3),(0,4),(1,4)\}$. Let H be the simple directed graph whose edges are the ordered pairs of elements of V which do not appear in a three times repeated block. Let G be the graph H without direction on edges. Note that G can have at most two edges between any two distinct vertices. Obviously all vertices of G have even degree. Moreover, if $t=0$ then G cannot have a vertex of degree two and if $t=1$ then G can have at most one vertex of degree two. An exhaustive computer search shows that when $(t,s)=(0,2)$ or (0,3), in each case there is exactly one candidate for G; when $(t,s)=(0,4)$ there are exactly 22 candidates for G, and when $(t,s)=(1,4)$ there are exactly 60 candidates for G (up to isomorphism). Checking each of these candidates exhaustively shows that (0,2), (0,3), (0,4) and (1,4) are not in $Fine(6)$.

Secondly we apply Lemma 2.1 and Designs 6-53 in [8]. □

Lemma 2.5. *Let $v \equiv 1$ or $3 \pmod 6$ and $v \geq 15$. Then there exists a $(v,\{3,4,6\},1)$ PBD with exactly one block of size 6, six blocks of size 4 and $(v(v-1)/6)-17$ blocks of size 3.*

Proof: Let $w \in \{15,19,21,25,27,31,33\}$ and let $(W,\mathcal{B}_i)$, $1 \leq i \leq 7$, be Design i in Appendix 1, where $W = \{1,2,3,\ldots,w)\}$. We define $\mathcal{B} = \{\{1,2,3,4,5,6\},\{1,7,8,9\},\{2,7,10,11\},\{3,8,12,13\},\{4,9,14,15\},\{5,10,12,14\},\{6,11,13,15\}\}$. Then $(W,\mathcal{B}\cup\mathcal{B}_i)$ is a $(w,\{3,4,6\},1)$ PBD with the required structure. Now let $v \geq 37$. It is well-known that there exists a $(v,3,1)$ BIBD having a $(w,3,1)$ BIBD as a subdesign, where $w \in \{15,19,21,25,27,31,33\}$. Now if we place a $(w,\{3,4,6\},1)$ PBD (constructed above) on the elements of the subdesign then the resulting design is a $(v,\{3,4,6\},1)$ PBD with the required structure. This completes the proof. □

Lemma 2.6. *There exist a $(v,\{3,4,6\},1)$ PBD with at least one block of size 3, one block of size 4 and one block of size 6 for all $v \equiv 0$ or $4 \pmod 6$, $v \geq 18$.*

Proof: Let $v \equiv 4 \pmod 6$, $v \geq 22$. Then there exists a GDD$(3, 1, \{6, 4^*\}; v)$ (with exactly one group of size 4); see [3]. Therefore there exists a $(v, \{3, 4^*, 6\}, 1)$ PBD with the required structure.

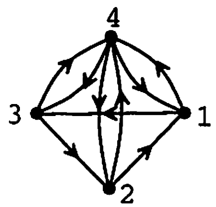
Now let $v \equiv 0 \pmod 6$. We take a GDD$(4, 1, \{2, 5^*\}; 6n + 5)$ which exists for $n \geq 3$ (see [2]). By adding a new point ∞ to the groups of this design we can construct a $(v, \{3, 4, 6^*\}, 1)$ PBD with the required structure, where $v \equiv 0 \pmod 6$, $v \geq 24$. For $v = 18$ we proceed as follows. Let $\mathcal{B}$ be the blocks of Design 8 in Appendix 1 and let $\mathcal{B}' = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10\}, \{11, 12, 13, 14\}, \{15, 16, 17, 18\}\}$. Then $\mathcal{B} \cup \mathcal{B}'$ yields a $(18, \{3, 4, 6^*\}, 1)$ PBD on $\{1, 2, 3, \ldots, 18\}$. This completes the prooof. □

Lemma 2.7. *$Adm(v) \setminus \{(0, 3)\} \subseteq Fine(v)$ for all $v \equiv 1$ or $3 \pmod 6$, $v \geq 15$, and for all $v \equiv 0$ or $4 \pmod 6$, $v \geq 18$.*

Proof: Let $(V, \mathcal{B})$ be a $(v, \{3, 4, 6\}, 1)$ PBD which we obtain from Lemma 2.5 or Lemma 2.6. For each block $B \in \mathcal{B}$ we place the triples of a $(|B|, 3)$ DTS on the elements of B. The result is a $(v, 3)$ DTS. Using DTSs with different types (see Lemmas 2.2, 2.3 and 2.4) on each block of the PBD we find that $Adm(v) \setminus \{(0, 3)\} \subseteq Fine(v)$ □

Lemma 2.8. *$(0, 3) \in Fine(v)$ for $v \equiv 0, 1 \pmod 3$, $v \geq 9$.*

Proof: Let G be the following directed graph.



Then $3G$ can be decomposed into nine directed triples $\mathcal{D} = \{(1, 3, 4), (1, 4, 3), (2, 1, 4), (2, 4, 1), (3, 2, 4), (3, 4, 2), (4, 1, 3), (4, 2, 1), (4, 3, 2)\}$. Now let $\mathcal{D}_i$, $1 \leq i \leq 7$, be the blocks of Designs 1-7 in Appendix 2. Then $(W, \mathcal{D} \cup \mathcal{D}_i \cup \mathcal{D}_i \cup \mathcal{D}_i)$ is a $(w, 3)$ DTS of type $(0, 3)$, where $w \in \{9, 10, 12, 13, 15, 16, 18\}$ and $W = \{1, 2, 3, \ldots, w\}$. Now let $v \geq 19$. It is well-known that there exists a $(v, 1)$ DTS having a $(w, 1)$ DTS as a subdesign, where $w \in \{9, 10, 12, 13, 15, 16, 18\}$; see e.g. [1]. Triplicate each directed triple of the $(v, 1)$ DTS which is not in the subdesign, and then place a $(w, 3)$ DTS of type (0,3) on the elements of the subdesign. The result is a $(v, 3)$ DTS of type (0.3). This completes the proof. □

Lemma 2.9. *$Fine(16) = Adm(16)$.*

Proof: We add a new point ∞ to the groups of a GDD(3,1,5; 15), which exists (see [3]), to obtain a $(16, \{3, 6\}, 1)$ PBD. Now using a method similar to that described in Lemma 2.7 we obtain $Adm(16) \setminus \{(0,3),(1,4)\} \subseteq Fine(16)$. For the type (1,4) we take a $(16,4,1)$ BIBD and place $(4,3)$ DTSs with type $(0,0)$ on the elements of each block of the design except one block which we place a $(4,3)$ DTS with type (1,4). Finally we apply Lemma 2.8 for the type (0,3). □

So far we have proved the following result.

Theorem 2.10. *$Fine(v) = Adm(v)$ for $v \equiv 0, 1 \pmod 3$, $v = 3$ or $v \geq 15$.*

3 Solutions for small orders

In this section we examine $Fine(v)$ for v = 7, 9, 10, 12 and 13. We show that $Fine(7) = Adm(7) \setminus \{(0,3),(1,4),(0,5)\}$ and $Fine(v) = Adm(v)$ for v = 9, 10, 12 and 13.

Lemma 3.1. *If there exists a $(v,3,3)$ BIBD of type (t_1, s_1) and a $(v,3,3)$ BIBD of type (t_2, s_2) then there exists a $(v,3)$ DTS of type (t_1+t_2, s_1+s_2).*

Proof: Let $(\mathbf{Z}_v, \mathcal{B}_i)$ be a $(v,3,3)$ BIBD of type (t_i, s_i), $i = 1,2$ (see [4] and [5] for the fine structure of $(v,3,3)$ BIBDs). Without loss of generality we can assume $a < b < c$ for each block $\{a,b,c\} \in \mathcal{B}_1$ and $d > e > f$ for each block $\{d,e,f\} \in \mathcal{B}_2$. Form $\mathcal{D} = \{(a,b,c) | \{a,b,c\} \in \mathcal{B}_1\} \cup \{(d,e,f) | \{d,e,f\} \in \mathcal{B}_2\}$. Then $(\mathbf{Z}_v, \mathcal{D})$ is a $(v,3)$ DTS of type (t_1+t_2, s_1+s_2). □

Lemma 3.2. *Let $v, w \equiv 0, 1 \pmod 3$. If there exists a $(v,1)$ DTS having a $(w,1)$ DTS as a subdesign then $Fine(w) \subseteq Fine(v)$.*

Proof: Triplicate each directed triple of the $(v,1)$ DTS which is not in the subdesign, and then place a $(w,3)$ DTS on the elements of the subdesign.

Lemma 3.3. *$Fine(7) = Adm(7) \setminus \{(0,3),(1,4),(0,5)\}$.*

Proof: First we show that (0,3), (1,4) and (0,5) are not in $Fine(7)$. Let $(V, \mathcal{D})$ be a $(7,3)$ DTS of type (t,s), where $(t,s) \in \{(0,3),(1,4),(0,5)\}$. Let H and G be the graphs defined in Lemma 2.4. An exhaustive computer search shows that when $(t,s) = (0,3)$ there is exactly one candidate for G, when $(t,s) = (1,4)$ there are exactly 60 candidates for G and when $(t,s) = (0,5)$ there are exactly 180 candidates for G (up to isomorphism). Checking each of these candidates exhaustively shows that (0,3), (1,4) and (0,5) are not in $Fine(7)$. Secondly we apply Lemma 2.1 for the types of the form (n,n), $0 \leq n \leq 14$, $n \neq 1$. Thirdly we take a (7,3,1) BIBD and place (3,3) DTSs with different types on the elements of the blocks of the design to obtain (7,3) DTSs with types of the form $(2n, 2m)$, $0 \leq n, m \leq 7$.

Finally we apply Lemma 3.1 with $v = 7$ (see [4] for the fine structure of a (7,3,3) BIBD). For the remaining types see Designs 54-100 in [8]. □

Lemma 3.4. $Fine(9) = Adm(9)$.

Proof: First we apply Lemma 2.1 for the types of the form (n, n), $0 \leq n \leq 24$, $n \neq 1$. Secondly we take a (9,3,1) BIBD and place (3,3) DTSs with different types on the elements of the blocks of the design to obtain (9,3) DTSs with types of the form $(2n, 2m)$, $0 \leq n$, $m \leq 12$. Thirdly we apply Lemma 3.1 with $v = 9$ (see [4] for the fine structure of a (9,3,3) BIBD). For the type (0,3) we use Lemma 2.8 and for the types (2,3), (1,4) and (3,4) we apply Lemma 3.2 (note that there exists a (9,1) DTS having a (4,1) DTS as a subdesign; see e.g. [1]). The remaining types are settled by Designs 101-159 in [8]. □

Lemma 3.5. $Fine(10) = Adm(10)$.

Proof: We take a $(10, \{3, 4\}, 1)$ PBD and place the triples of a (4,3) DTS or a (3,3) DTS on the elements of each block of size 4 or 3. Using DTSs with different types covers all the types of $Fine(10)$ except the types $(t, s) \in \{(i, 2j + 1) | i = 0, 1 \text{ and } 2 \leq j \leq 14\} \cup \{(0, 3)\}$. The type (0,3) is covered by Lemma 2.8 and the remaining types are settled by Designs 160-185 in [8]. □

Lemma 3.6. $Fine(12) = Adm(12)$.

Proof: Since there exists a GDD(3,1,4; 12) it follows that there exists a $(12, \{3, 4\}, 1)$ PBD. Now a method similar to that described in Lemma 3.5 covers all the types of $Fine(12)$ except the types $(t, s) \in \{(i, 2j+1) | i = 0, 1 \text{ and } 2 \leq j \leq 21\} \cup \{(0, 3)\}$. The type (0,3) is covered by Lemma 2.8 and the remaining types are settled by Designs 186-225 in [8]. □

Lemma 3.7. $Fine(13) = Adm(13)$.

Proof: Since there exists a (13,1) DTS containing a (6,1) DTS as a subdesign (see e.g. [1]) it follows that $Fine(6) \subseteq Fine(13)$ (see Lemma 3.2). We take a (13,3,1) BIBD and place (3,3) DTSs with different types on the elements of each block of the design to obtain (13,3) DTSs with types of the form $(2n, 2m)$, $0 \leq n$, $m \leq 26$. We also apply Lemma 3.1 with $v = 13$ (see [4] for the fine structure of a (13,3,3) BIBD). The remaining types are (0,3), (1,4) and (0,11). For the type (0,3) we apply Lemma 2.8 and for the type (1,4) we take a (13,4,1) BIBD and place (4,3) DTSs with type (0,0) on the elements of each block of the BIBD except one block which we place a (4,3) DTS with type (1,4). Finally we use Design 226 in [8] for the type (0,11). This completes the proof. □

Remark The fine structure for $(v, 3)$ DTSs when $v \equiv 2 \pmod 3$ is more difficult. The difficulty is in determining (and proving) the necessary conditions. However, the author has found some results for this case.

References

[1] E.J. Billington, *The intersection problem for combinatorial designs*, Congressus Numerantium, **92** (1993), 33–54.

[2] A.E. Brouwer, *Optimal packings of K_4's into a K_n*, Journal of Combinatorial Theory Series A **26** (1979), 278–297.

[3] C.J. Colbourn, D.G. Hoffman and R. Rees, *A new class of group divisible designs with block size three*, Journal of Combinatorial Theory, Series A **59**, (1992), 73–89.

[4] C.J. Colbourn, R.A. Mathon, A. Rosa and N. Shalaby, *The fine structure of threefold triple systems: $v \equiv 1$ or 3 (mod 6)*, Discrete Mathematics **92** (1991), 49–64.

[5] C.J. Colbourn, R.A. Mathon and N. Shalaby, *The fine structure of threefold triple systems: $v \equiv 5$ (mod 6)*, Australasian Journal of Combinatorics **3** (1991), 75–92.

[6] C.J. Colbourn and A. Rosa, *Directed and Mendelsohn triple systems*, in Contemporary design theory: a collection of surveys (editors J.H. Dinitz and D.R. Stinson), John Wiley and Sons, New York (1992), 97–136.

[7] A. Khodkar, *The fine structure of balanced ternary designs with block size three*, Utilitas Mathematica **44** (1993), 197–230.

[8] A. Khodkar, *The fine structure of $(v, 3)$ directed triple systems: $v \in \{4, 6, 7, 9, 10, 12, 13\}$*, Research Report, Department of Mathematics, University of Queensland, 1994.

[9] C.C. Lindner and W.D. Wallis, *Embeddings and prescribed intersections of transitive triple systems*, Annals of Discrete Mathematics **15** (1982), 265–272.

Appendix 1

Let $\mathcal{B} = \{\{1,2,3,4,5,6\}, \{1,7,8,9\}, \{2,7,10,11\}, \{3,8,12,13\}, \{4,9,14,15\}, \{5,10,12,14\}, \{6,11,13,15\}\}$ and let $\mathcal{B}_i$ be the blocks of Design i in this appendix, $i \neq 8$. Then $\mathcal{B} \cup \mathcal{B}_i$ yields a $(w, \{3,4,6\}, 1)$ PBD on $\{1,2,3,\ldots,w\}$, where $w \in \{15, 19, 21, 25, 27, 31, 33\}$. Let $\mathcal{B}'$ be the blocks of Design 8 in this appendix and let $\mathcal{B}'' = \{\{1,2,3,4,5,6\}, \{7,8,9,10\}, \{11,12,13,14\}, \{15,16,17,18\}\}$. Then $\mathcal{B}' \cup \mathcal{B}''$ yields a $(18, \{3,4,6\}, 1)$ PBD on $\{1,2,3,\ldots,18\}$.

No.	v	Triples					
1	15	{1,10,13}	{1,11,14}	{1,12,15}	{2,8,15}	{2,9,12}	{2,13,14}
		{3,7,14}	{3,9,11}	{3,10,15}	{4,7,13}	{4,8,10}	{4,11,12}
		{5,7,15}	{5,8,11}	{5,9,13}	{6,7,12}	{6,8,14}	{6,9,10}
2	19	{1,10,17}	{1,11,14}	{1,12,15}	{1,13,19}	{1,16,18}	{2,8,14}
		{2,9,12}	{2,13,16}	{2,15,18}	{2,17,19}	{3,7,18}	{3,9,17}
		{3,10,15}	{3,11,16}	{3,14,19}	{4,7,12}	{4,8,19}	{4,10,16}
		{4,11,17}	{4,13,18}	{5,7,13}	{5,8,11}	{5,9,18}	{5,15,17}
		{5,16,19}	{6,7,17}	{6,8,10}	{6,9,16}	{6,12,19}	{6,14,18}
		{7,14,16}	{7,15,19}	{8,15,16}	{8,17,18}	{9,10,13}	{9,11,19}
		{10,18,19}	{11,12,18}	{12,16,17}	{13,14,17}		
3	21	{1,10,16}	{1,11,12}	{1,13,20}	{1,14,17}	{1,15,18}	{1,19,21}
		{2,8,21}	{2,9,12}	{2,13,14}	{2,15,19}	{2,16,17}	{2,18,20}
		{3,7,20}	{3,9,19}	{3,10,15}	{3,11,21}	{3,14,16}	{3,17,18}
		{4,7,17}	{4,8,19}	{4,10,20}	{4,11,16}	{4,12,18}	{4,13,21}
		{5,7,21}	{5,8,11}	{5,9,16}	{5,13,17}	{5,15,20}	{5,18,19}
		{6,7,16}	{6,8,18}	{6,9,17}	{6,10,19}	{6,12,20}	{6,14,21}
		{7,12,15}	{7,13,19}	{7,14,18}	{8,10,17}	{8,14,20}	{8,15,16}
		{9,10,13}	{9,11,18}	{9,20,21}	{10,18,21}	{11,14,19}	{11,17,20}
		{12,16,21}	{12,17,19}	{13,16,18}	{15,17,21}	{16,19,20}	
4	25	{1,10,13}	{1,11,14}	{1,12,25}	{1,15,16}	{1,17,24}	{1,18,19}
		{1,20,21}	{1,22,23}	{2,8,19}	{2,9,12}	{2,13,23}	{2,14,24}
		{2,15,17}	{2,16,21}	{2,18,25}	{2,20,22}	{3,7,24}	{3,9,10}
		{3,11,19}	{3,14,20}	{3,15,25}	{3,16,23}	{3,17,18}	{3,21,22}
		{4,7,12}	{4,8,22}	{4,10,23}	{4,11,25}	{4,13,24}	{4,16,17}
		{4,18,20}	{4,19,21}	{5,7,15}	{5,8,11}	{5,9,21}	{5,13,20}
		{5,16,18}	{5,17,22}	{5,19,23}	{5,24,25}	{6,7,25}	{6,8,21}
		{6,9,16}	{6,10,19}	{6,12,17}	{6,14,18}	{6,20,23}	{6,22,24}
		{7,13,21}	{7,14,22}	{7,16,19}	{7,17,20}	{7,18,23}	{8,10,17}
		{8,14,23}	{8,15,18}	{8,16,25}	{8,20,24}	{9,11,20}	{9,13,18}
		{9,17,25}	{9,19,22}	{9,23,24}	{10,15,22}	{10,16,24}	{10,18,21}
		{10,20,25}	{11,12,21}	{11,16,22}	{11,17,23}	{11,18,24}	{12,15,23}
		{12,16,20}	{12,18,22}	{12,19,24}	{13,14,16}	{13,17,19}	{13,22,25}
		{14,17,21}	{14,19,25}	{15,19,20}	{15,21,24}	{21,23,25}	
5	27	{1,10,24}	{1,11,12}	{1,13,25}	{1,14,16}	{1,15,18}	{1,17,27}
		{1,19,21}	{1,20,26}	{1,22,23}	{2,8,15}	{2,9,22}	{2,12,23}
		{2,13,16}	{2,14,27}	{2,17,19}	{2,18,21}	{2,20,25}	{2,24,26}
		{3,7,22}	{3,9,27}	{3,10,26}	{3,11,16}	{3,14,19}	{3,15,25}
		{3,17,18}	{3,20,23}	{3,21,24}	{4,7,23}	{4,8,11}	{4,10,17}
		{4,12,16}	{4,13,18}	{4,19,20}	{4,21,22}	{4,24,27}	{4,25,26}
		{5,7,27}	{5,8,24}	{5,9,23}	{5,11,26}	{5,13,17}	{5,15,20}
		{5,16,19}	{5,18,22}	{5,21,25}	{6,7,16}	{6,8,17}	{6,9,18}
		{6,10,19}	{6,12,20}	{6,14,21}	{6,22,24}	{6,23,25}	{6,26,27}
		{7,12,18}	{7,13, 14}	{7,15,19}	{7,17,26}	{7,20,21}	{7,24,25}
		{8,10,21}	{8,14,25}	{8,16,22}	{8,18,20}	{8,19,27}	{8,23,26}
		{9,10,20}	{9,11,19}	{9,12,24}	{9,13,26}	{9,16,21}	{9,17,25}
		{10,13,23}	{10,15,22}	{10,16,25}	{10,18,27}	{11,14,22}	{11,17,21}
		{11,18,25}	{11,20,27}	{11,23,24}	{12,15,27}	{12,17,22}	{12,19,25}
		{12,21,26}	{13,19,24}	{13,20,22}	{13,21,27}	{14,17,23}	{14,18,26}
		{14,20,24}	{15,16,26}	{15,17,24}	{15,21,23}	{16,17,20}	{16,18,24}
		{16,23,27}	{18,19,23}	{19,22,26}	{22,25,27}		

No.	v	Triples					
6	31	{1,10,15}	{1,11,22}	{1,12,31}	{1,13,14}	{1,16,21}	{1,17,20}
		{1,18,28}	{1,19,30}	{1,23,26}	{1,24,25}	{1,27,29}	{2,8,30}
		{2,9,20}	{2,12,27}	{2,13,24}	{2,14,19}	{2,15,21}	{2,16,18}
		{2,17,26}	{2,22,29}	{2,23,28}	{2,25,31}	{3,7,14}	{3,9,23}
		{3,10,30}	{3,11,28}	{3,15,29}	{3,16,17}	{3,18,21}	{3,19,26}
		{3,20,31}	{3,22,24}	{3,25,27}	{4,7,12}	{4,8,25}	{4,10,24}
		{4,11,20}	{4,13,26}	{4,16,19}	{4,17,29}	{4,18,22}	{4,21,23}
		{4,27,30}	{4,28,31}	{5,7,21}	{5,8,31}	{5,9,16}	{5,11,19}
		{5,13,27}	{5,15,28}	{5,17,22}	{5,18,24}	{5,20,23}	{5,25,29}
		{5,26,30}	{6,7,27}	{6,8,23}	{6,9,28}	{6,10,31}	{6,12,22}
		{6,14,20}	{6,16,26}	{6,17,24}	{6,18,25}	{6,19,29}	{6,21,30}
		{7,13,18}	{7,15,22}	{7,16,30}	{7,17,25}	{7,19,23}	{7,20,24}
		{7,26,31}	{7,28,29}	{8,10,16}	{8,11,18}	{8,14,27}	{8,15,17}
		{8,19,20}	{8,21,22}	{8,24,28}	{8,26,29}	{9,10,22}	{9,11,27}
		{9,12,25}	{9,13,21}	{9,17,30}	{9,18,29}	{9,19,31}	{9,24,26}
		{10,13,29}	{10,17,23}	{10,18,26}	{10,19,25}	{10,20,27}	{10,21,28}
		{11,12,16}	{11,14,17}	{11,21,31}	{11,23,30}	{11,24,29}	{11,25,26}
		{12,15,26}	{12,17,28}	{12,18,19}	{12,20,30}	{12,21,29}	{12,23,24}
		{13,16,25}	{13,17,19}	{13,20,28}	{13,22,30}	{13,23,31}	{14,16,24}
		{14,18,30}	{14,21,25}	{14,22,31}	{14,23,29}	{14,26,28}	{15,16,31}
		{15,18,20}	{15,19,27}	{15,23,25}	{15,24,30}	{16,20,29}	{16,22,23}
		{16,27,28}	{17,18,31}	{17,21,27}	{18,23,27}	{19,21,24}	{19,22,28}
		{20,21,26}	{20,22,25}	{22,26,27}	{24,27,31}	{25,28,30}	{29,30,31}
7	33	{1,10,13}	{1,11,12}	{1,14,16}	{1,15,17}	{1,18,27}	{1,19,26}
		{1,20,21}	{1,22,23}	{1,24,25}	{1,28,29}	{1,30,31}	{1,32,33}
		{2,8,29}	{2,9,22}	{2,12,19}	{2,13,16}	{2,14,32}	{2,15,31}
		{2,17,33}	{2,18,28}	{2,20,30}	{2,21,23}	{2,24,26}	{2,25,27}
		{3,7,32}	{3,9,17}	{3,10,24}	{3,11,16}	{3,14,21}	{3,15,33}
		{3,18,19}	{3,20,23}	{3,22,26}	{3,25,29}	{3,27,30}	{3,28,31}
		{4,7,20}	{4,8,10}	{4,11,25}	{4,12,16}	{4,13,18}	{4,17,31}
		{4,19,24}	{4,21,30}	{4,22,28}	{4,23,27}	{4,26,32}	{4,29,33}
		{5,7,13}	{5, 8,15}	{5,9,11}	{5,16,20}	{5,17,21}	{5,18,32}
		{5,19,23}	{5,22,33}	{5,24,30}	{5,25,28}	{5,26,29}	{5,27,31}
		{6,7,23}	{6,8,17}	{6,9,18}	{6,10,19}	{6,12,20}	{6,14,28}
		{6,16,25}	{6,21,22}	{6,24,29}	{6,26,27}	{6,30,32}	{6,31,33}
		{7,12,17}	{7,14,22}	{7,15,19}	{7,16,28}	{7,18,24}	{7,21,25}
		{7,26,31}	{7,27,29}	{7,30,33}	{8,11,33}	{8,14,31}	{8,16,26}
		{8,18,20}	{8,19,21}	{8,22,25}	{8,23,24}	{8,27,32}	{8,28,30}
		{9,10,30}	{9,12,27}	{9,13,19}	{9,16,31}	{9,20,25}	{9,21,24}
		{9,23,26}	{9,28,33}	{9,29,32}	{10,15,22}	{10,16,21}	{10,17,32}
		{10,18,29}	{10,20,26}	{10,23,33}	{10,25,31}	{10,27,28}	{11,14,18}
		{11,17,28}	{11,19,30}	{11,20,27}	{11,21,26}	{11,22,32}	{11,23,29}
		{11,24,31}	{12,15,24}	{12,18,23}	{12,21,28}	{12,22,30}	{12,25,32}
		{12,26,33}	{12,29,31}	{13,14,25}	{13,17,30}	{13,20,33}	{13,21,29}
		{13,22,27}	{13,23,31}	{13,24,32}	{13,26,28}	{14,17,23}	{14,19,27}
		{14,20,29}	{14,24,33}	{14,26,30}	{15,16,27}	{15,18,21}	{15,20,32}
		{15,23,28}	{15,25,26}	{15,29,30}	{16,17,29}	{16,18,30}	{16,19,33}
		{16,22,24}	{16,23,32}	{17,18,26}	{17,19,25}	{17,20,22}	{17,24,27}
		{18,22,31}	{18,25,33}	{19,20,31}	{19,22,29}	{19,28,32}	{20,24,28}
		{21,27,33}	{21,31,32}	{23,25,30}			
8	18	{1,7,14}	{1,8,16}	{1,9,11}	{1,10,15}	{1,12,17}	{1,13,18}
		{10,14,17}	{2,7,17}	{2,8,12}	{2,9,14}	{2,10,18}	{2,11,15}
		{2,13,16}	{3,7,11}	{3,8,14}	{3,9,18}	{3,10,16}	{3,12,15}
		{3,13,17}	{4,7,12}	{4,8,17}	{4,9,15}	{4,10,13}	{4,11,16}
		{4,14,18}	{5,7,18}	{5,8,15}	{5,9,13}	{5,10,12}	{5,11,17}
		{5,14,16}	{6,7,16}	{6,8,13}	{6,9,17}	{6,10,11}	{6,12,18}
		{6,14,15}	{7,13,15}	{8,11,18}	{9,12,16}		

Appendix 2

Let $\mathcal{D}_i$, $1 \leq i \leq 7$, be the blocks of Designs 1-7 in this appendix and let $\mathcal{D} = \{(1,3,4),(1,4,3),(2,1,4),(2,4,1),(3,2,4),(3,4,2),(4,1,3),(4,2,1),(4,3,2)\}$. Then $\mathcal{D}\cup\mathcal{D}_i\cup\mathcal{D}_i\cup\mathcal{D}_i$ yields a $(w,3)$ DTS of type (0,3) on $\{1,2,3,\ldots,w\}$, where $w \in \{9,10,12,13,15,16,18\}$.

No.	v	Transitive triples					
1	9	(1,2,5)	(1,9,7)	(3,8,9)	(2,3,6)	(2,7,8)	(3,7,5)
		(7,4,9)	(4,5,6)	(4,8,7)	(5,2,9)	(5,1,8)	(8,5,4)
		(6,9,4)	(6,7,1)	(7,6,2)	(6,8,3)	(9,6,5)	(9,3,1)
		(9,8,2)	(5,7,3)	(8,1,6)			
2	10	(1,2,5)	(1,6,7)	(8,9,3)	(2,3,6)	(2,7,8)	(9,10,6)
		(3,1,10)	(3,9,7)	(1,9,8)	(4,5,6)	(4,7,9)	(4,8,10)
		(5,10,8)	(5,9,2)	(5,7,3)	(6,10,4)	(8,6,2)	(6,9,1)
		(6,3,8)	(7,10,1)	(8,7,4)	(9,5,4)	(10,7,2)	(10,3,5)
		(2,10,9)	(7,6,5)	(8,5,1)			
3	12	(1,2,5)	(1,6,7)	(1,8,9)	(10,5,11)	(2,3,6)	(2,7,8)
		(2,9,10)	(2,11,12)	(3,1,12)	(3,7,5)	(3,8,10)	(3,9,11)
		(4,5,6)	(7,4,9)	(4,7,11)	(4,10,12)	(8,1,11)	(5,9,2)
		(5,10,3)	(11,5,4)	(6,5,12)	(11,6,8)	(6,1,10)	(10,4,8)
		(6,11,3)	(7,10,1)	(11,7,2)	(12,7,3)	(5,8,7)	(8,12,5)
		(9, 5, 1)	(9,8,3)	(9,6,4)	(10,7,6)	(8,6,2)	(11, 10, 9)
		(12,6,9)	(12,10,2)	(12,11,1)	(9,7,12)	(12,8,4)	
4	13	(1,2,5)	(1,6,7)	(1,11,9)	(1,13,10)	(12,13,5)	(2,3,6)
		(2,7,8)	(2,9,10)	(2,11,12)	(3,5,1)	(3,7,9)	(3,8,10)
		(3,11,13)	(4,5,6)	(4,7,10)	(4,8,11)	(4,9,12)	(5,2,13)
		(5,3,12)	(5,7,4)	(5,9,8)	(5,11,10)	(6,4,13)	(6,8,1)
		(6,9,2)	(6,10,3)	(6,11,5)	(7,6,12)	(7,11,1)	(7,13,2)
		(8,7,3)	(8,12,2)	(13,8,4)	(9,7,5)	(9,11,3)	(9,13,1)
		(10,8,5)	(10,9,4)	(10,12,1)	(10,13,6)	(11,8,6)	(12,9,6)
		(12,10,7)	(12,11,4)	(13,11,7)	(13,12,3)	(10,11,2)	(1,12,8)
		(8,13,9)					
5	15	(1,2,5)	(6,13,7)	(1,14,8)	(10,1,11)	(12,13,3)	(14,1,6)
		(9,3,6)	(2,7,8)	(2,9,15)	(2,11,6)	(2,13,14)	(3,5,1)
		(5,3,7)	(3,8,10)	(3,13,11)	(8,3,14)	(4,5,6)	(2,3,12)
		(4,8,11)	(4,9,12)	(4,15,13)	(10,5,15)	(5,4,14)	(7,6,3)
		(5,8,12)	(8,5,9)	(5,10,13)	(6,12,15)	(6,8,1)	(6,9,2)
		(6,10,4)	(6,5,11)	(12,6,14)	(7,11,2)	(11,1,13)	(12,5,2)
		(7,13,4)	(7,15,5)	(8,13,2)	(14,3,15)	(8,15,4)	(13,9,8)
		(9,13,1)	(14,9,4)	(1,15,9)	(10,8,6)	(10,3,9)	(12,1,7)
		(9,10,7)	(11,8,7)	(11,12,9)	(11,15,3)	(15,10,2)	(1,10,12)
		(12,11,4)	(13,15,6)	(7,9,14)	(14,13,5)	(15,14,11)	(15,12,8)
		(13,12,10)	(15,7,1)	(9,11,5)	(14,2,10)	(4,7,10)	(11,10,14)
		(14,7,12)					

No.	v	Transitive triples					
6	16	(1,2,5)	(1,6,7)	(4,9,8)	(10,1,11)	(1,12,13)	(1,15,14)
		(2,12,6)	(2,16,7)	(10,2,9)	(8,11,1)	(13,2,14)	(16,12,3)
		(3,1,8)	(3,5,7)	(2,11,15)	(9,5,3)	(3,14,6)	(3,13,15)
		(4,5,6)	(7,16,9)	(3,16,11)	(4,12,10)	(7,11,13)	(16,4,14)
		(5,14,8)	(9,14,2)	(13,5,9)	(5,11,2)	(5,12,1)	(5,4,13)
		(6,5,16)	(6,2,8)	(6,1,9)	(6,10,4)	(3,9,10)	(6,13,12)
		(7,6,14)	(7,1,10)	(7,8,2)	(7,3,12)	(12,15,4)	(7,5,15)
		(8,10,6)	(12,8,7)	(8,4,16)	(8,15,13)	(8,14,5)	(15,12,5)
		(9,13,6)	(14,9,7)	(9,4,11)	(13,7,4)	(15,10,8)	(2,10,13)
		(11,12,14)	(10,15,7)	(10,16,5)	(9,15,16)	(11,4,7)	(15,11,6)
		(11,16,8)	(12,11,9)	(12,16,2)	(14,13,11)	(14,10,3)	(10,14,12)
		(14,16,1)	(15,2,3)	(15,9,1)	(16,6,15)	(16,13,10)	(14,4,15)
		(13,1,16)	(13,8,3)	(8,9,12)	(6,11,3)	(11,5,10)	
7	18	(1,2,5)	(1,6,7)	(1,8,9)	(1,10,11)	(12,8,13)	(1,14,15)
		(16,17,2)	(2,3,6)	(5,2,7)	(9,2,10)	(2,11,12)	(2,13,14)
		(15,3,16)	(2,17,18)	(3,1,18)	(3,12,7)	(16,3,8)	(3,17,5)
		(3,9,14)	(18,3,13)	(1,17,16)	(4,5,6)	(4,7,9)	(4,8,11)
		(4,10,12)	(13,10,16)	(4,14,17)	(4,15,13)	(5,8,1)	(14,5,9)
		(10,5,3)	(17,11,13)	(5,12,15)	(5,13,17)	(5,14,16)	(6,5,18)
		(6,2,8)	(6,9,1)	(6,10,4)	(6,11,3)	(12,16,1)	(6,14,13)
		(15,17,12)	(7,10,1)	(7,11,2)	(17,7,3)	(13,7,4)	(7,14,12)
		(7,15,6)	(8,7,18)	(7,8,17)	(8,12,4)	(8,3,10)	(8,14,6)
		(8,15,5)	(8,2,16)	(9,12,17)	(9,13,6)	(12,14,3)	(15,9,4)
		(9,7,16)	(17,9,8)	(10,9,18)	(10,13,2)	(10,14,7)	(10,15,8)
		(16,5,10)	(10,17,6)	(3,15,11)	(14,11,8)	(11,15,7)	(11,16,6)
		(11,5,4)	(11,18,9)	(12,11,10)	(11,17,1)	(12,18,2)	(1,13,12)
		(18,7,5)	(12,9,5)	(14,18,1)	(13,9,3)	(15,14,2)	(6,16,12)
		(16,7,13)	(16,14,4)	(2,9,15)	(13,5,11)	(17,14,10)	(6,17,15)
		(18,12,6)	(13,18,8)	(18,11,14)	(15,18,10)	(18,17,4)	(16,9,11)
		(13,15,1)	(4,16,18)	(18,16,15)			