Some Constructions of Semiframes

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ABSTRACT. A (k, λ)-semiframe of type g^u is a group divisible design of type g^u $(\mathcal{X}, \mathcal{G}, \mathcal{B})$, in which $\mathcal{B}$ is written as a disjoint union $\mathcal{B} = \mathcal{P} \cup \mathcal{Q}$ where $\mathcal{P}$ is partitioned into partial parallel classes of $\mathcal{X}$ (with respect to some $G \in \mathcal{G}$) and $\mathcal{Q}$ is partitioned into parallel classes of $\mathcal{X}$. In this paper, new constructions for these designs are provided with some series of designs with $k = 3$. Cyclic semiframes are discussed. Finally an application of semiframes is also mentioned.

1 Introduction

The notion of a semiframe was introduced by Rees [10] who conducted further research in [11]. These designs are natural generalizations of frames and resolvable group divisible designs, and have some applications for the construction of other types of designs, such as incomplete group divisible designs, resolvable designs with spanning sets of resolvable subdesigns and complete graphs K_{gu} admitting a one-factorization with an orthogonal set of u disjoint sub-one-factorizations of K_g (see [10, 11]).

A *group divisible design*, (k, λ)-GDD of type g^u, is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ where $\mathcal{X}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{X}$ into u *groups* and $\mathcal{B}$ is a collection of subsets, called *blocks*, of $\mathcal{X}$ such that

(i) for each group $G \in \mathcal{G}$ and each $B \in \mathcal{B}$, $|G \cap B| \leq 1$;

(ii) any pair of points from distinct groups occurs in exactly λ blocks;

(iii) $|B| = k$ for all blocks $B \in \mathcal{B}$, $|G| = g$ for all groups $G \in \mathcal{G}$ and $|\mathcal{X}| = gu$.

A *parallel class* of blocks in a GDD is a subset $\mathcal{B}' \subseteq \mathcal{B}$ which partitions the point set $\mathcal{X}$, while a *partial parallel class* of blocks with respect to $\mathcal{X}'$ ($\subseteq \mathcal{X}$) is a subset $\mathcal{B}'' \subseteq \mathcal{B}$ which partitions $\mathcal{X} - \mathcal{X}'$.

A GDD is said to be *resolvable*, denoted by RGDD, if $\mathcal{B}$ admits a partition into parallel classes; a *frame* is a GDD in which $\mathcal{B}$ admits a partition into partial parallel classes with respect to some group $G \in \mathcal{G}$.

A (k, λ)-*semiframe of type* g^u is a (k, λ)-GDD of type g^u in which the collection of blocks $\mathcal{B}$ can be written as a disjoint union $\mathcal{B} = \mathcal{P} \cup \mathcal{Q}$ where $\mathcal{P}$ can be partitioned into partial parallel classes with respect to some group and $\mathcal{Q}$ can be partitioned into parallel classes. Note that when $\mathcal{P} = \phi$ such a design is an RGDD, while if $\mathcal{Q} = \phi$ the design is a frame.

We will assume here that there are both parallel classes and partial parallel classes present (such a semiframe is said to be *proper*). The extreme cases were well studied in the literature (cf. [1, 4, 7, 12, 13, 14, 15, 16, 17]) and hence we do not pay attention to these cases here.

Rees [10, 11] discussed some constructions for proper semiframes. Stinson [18] presented a list of open problems, the third of which states "find more examples of semiframes". In this paper, we will provide more methods of constructing proper semiframes and illustrate these constructions with some examples. Additionally, a new type of semiframe, which we call a cyclic semiframe, is discussed. Finally, one more application of semi-frames for the construction of RGDDs is also described.

2 Preliminary results

In this section we discuss the necessary conditions for the existence of a proper semiframe.

Lemma 2.1. *In a proper (k, λ)-semiframe $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ of type g^u, there exists an integer d such that for each group $G_j \in \mathcal{G}$, there are d partial parallel classes with respect to G_j.*

Proof: Let $x \in G_j$ and $\mathcal{B}(x) = \{B \in \mathcal{B} \colon x \in B\}$. Consider the number of blocks containing x in two ways:

(1) $|\mathcal{B}(x)| = \lambda(|\mathcal{X}| - |G_j|)/(k-1) = \lambda g(u-1)/(k-1)$;

(2) Every block of $\mathcal{B}(x)$ belongs either to a parallel class or to a partial parallel class with respect to $G_i \neq G_j$. If p and d_i denote the number of parallel classes and partial parallel classes with respect to G_i, respectively, then

$$|\mathcal{B}(x)| = p + \sum_{1 \leq i \leq u} d_i - d_j.$$

Hence $d_j = p + \sum_{1 \le i \le u} d_i - \lambda g(u-1)/(k-1) = d_i$ for all $i \ne j$, $i, j \in \{1, 2, \ldots, u\}$. Thus we can denote d_j by d for all $j \in \{1, 2, \ldots, u\}$. □

Henceforth we use (k, λ)-SF$(p, d; g^u)$ to denote a proper (k, λ)-semiframe of type g^u in which there are p parallel classes and d partial parallel classes with respect to $G_j \in \mathcal{G}$.

Lemma 2.2. *Suppose that there exists a (k, λ)-SF$(p, d; g^u)$. If r is the least residue of g modulo $k-1$, then*

(1) $g \equiv 0 \pmod{k}$ *and* $\lambda r(u-1) \equiv 0 \pmod{k-1}$;

(2) $p \equiv \lambda r(u-1)/(k-1) \pmod{u-1}$ *and* $d = \lambda g/(k-1) - p/(u-1)$.

Proof: From the hypotheses, it follows that $g \equiv 0 \pmod{k}$. We also have

$$\frac{\lambda g(u-1)}{k-1} = p + (u-1)d$$

which can be written as $d = \lambda g/(k-1) - p/(u-1)$. Thus if $g \equiv r \pmod{k-1}$ for $0 \le r < k-1$, then $\lambda r(u-1) \equiv 0 \pmod{k-1}$ and $p \equiv \lambda r(u-1)/(k-1)$ $\pmod{u-1}$. □

3 Constructions

First we present four constructions without proofs. These are slight generalizations of those given in [10, 11]. Note that a *transversal design* TD(k, v) is a $(k, 1)$-GDD of type v^k.

Construction 3.1. The existence of a (k, λ)-SF$(p, d; g^u)$ and a resolvable TD(k, n) implies the existence of a (k, λ)-SF$(np, nd; (ng)^u)$.

Construction 3.2. Suppose that $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ is a (k, λ)-frame with t_i groups of size s_i for $i = 1, 2, \ldots, n$. Further suppose that, for each $i = 1, 2, \ldots, n$, there exists a (k, λ)-SF$(p_i, d; g^{1+s_i/g})$. Then there exists a (k, λ)-SF$(\sum_i p_i t_i, d; g^{1+|\mathcal{X}|/g})$.

Let $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ be a GDD of type g^u. A subset $\mathcal{B}' \subseteq \mathcal{B}$ is said to be *μ-balanced* if

(i) $\mathcal{B}'$ can be partitioned into $(u-1)/\mu$ parallel classes, and

(ii) $\mu\mathcal{B}'$ can be partitioned into u partial parallel classes, each of which is with respect to some group.

Construction 3.3. Let $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ be a (k, λ)-SF$(p, d; g^u)$ and μ be a positive integer, and suppose that there are $(u-1)/\mu$ parallel classes of blocks whose union forms a μ-balanced set. Further let $n \ge \mu$ be a positive integer for which there exists a resolvable TD(k, n). Then there exists a (k, λ)-SF$(np - i(u-1), nd + i; (ng)^u)$ for each $i = 0, 1, \ldots, \lfloor n/\mu \rfloor$.

Construction 3.4. Suppose that there is a $TD(u+1,n)$ and that, for each $i = 1,2,\ldots,n$, there is a (k,λ)-SF$(p_i, d_i; g^u)$. Then there exists a (k,λ)-SF$(\sum_i p_i, \sum_i d_i; (ng)^u)$.

Some new methods of constructions are now provided. First we use resolvable GDDs.

Construction 3.5. The existence of a $(k,1)$-RGDD of type g^u and a (k',λ)-SF$(p,d;n^k)$ implies the existence of a (k',λ)-SF$(p\cdot\frac{g(u-1)}{k-1}, gd; (ng)^u)$.

Proof: Let $(\mathcal{X},\mathcal{G},\mathcal{B})$ be a $(k,1)$-RGDD of type g^u and $S(x)=\{x_1,x_2,\ldots,x_n\}$. Then we will construct a (k',λ)-SF$(p\cdot\frac{g(u-1)}{k-1}, gd; (ng)^u)$) $(\mathcal{X}',\mathcal{G}',\mathcal{B}')$ where

$$\mathcal{X}' = \bigcup_{x\in\mathcal{X}} S(x),\ \mathcal{G}' = \{\bigcup_{x\in G} S(x) : G\in\mathcal{G}\},$$

and $\mathcal{B}'$ will be described later. It is easy to see that in $(\mathcal{X},\mathcal{G},\mathcal{B})$ there are exactly $r = g(u-1)/(k-1)$ parallel classes, $P_1, P_2, \ldots, P_r$. On each parallel class P_i, we can construct p parallel classes P_{ij}, $j = 1,2,\ldots,p$, by taking the union of the jth parallel classes of semiframes. Hence there are exactly pr parallel classes.

Let $Q^B_{x,i}$ denote the ith partial parallel class with respect to $S(x)$ in a (k',λ)-SF$(p,d;n^k)$, where $x\in B\in\mathcal{B}$. Then the required partial parallel classes with respect to $\cup_{y\in G}S(y)$, $x\in G$, are

$$\bigcup_{x\in B\in\mathcal{B}} Q^B_{x,i},\ i=1,2,\ldots,d,$$

and

$$\mathcal{B}' = (\bigcup_{1\le i\le d}\bigcup_{x\in\mathcal{X}}\bigcup_{x\in B\in\mathcal{B}} Q^B_{x,i}) \bigcup (\bigcup_{1\le i\le r}\bigcup_{1\le j\le p} P_{ij}).$$

□

Now we use almost resolvable designs to construct semiframes, where an *almost resolvable design* AR(k,v) is a $(k,k-1)$-frame of type 1^v.

Construction 3.6. Suppose that there exist an AR(k,v) and a resolvable TD(k,u) where $u\equiv 0 \pmod k$. Then there exists a $(k,k-1)$-SF$(v-1,u-1;u^v)$ in which the $v-1$ parallel classes form a 1-balanced set.

Proof: Let $(\mathcal{X},\mathcal{G},\mathcal{A})$ be an AR(k,v) with $\mathcal{X}=\{x_1,x_2,\ldots,x_v\}$. Further let $\mathcal{Z}=\mathcal{X}\times\mathcal{Y}$ with $\mathcal{Y}=\{y_1,y_2,\ldots,y_u\}$.

For each $A\in\mathcal{A}$, we can construct a resolvable TD(k,u) on $A\times\mathcal{Y}$ with groups $\{a\}\times\mathcal{Y}$, $a\in A$, and parallel classes $C_i(A)$, $1\le i\le u$. Let $C_i(\mathcal{A}_j) = \cup_{A\in\mathcal{A}_j} C_i(A)$, $1\le i\le u$, $1\le j\le v$, where $\mathcal{A}_j$ is the partial parallel class of $(\mathcal{X},\mathcal{G},\mathcal{A})$ with respect to $\{x_j\}$. Then let

$$C(\mathcal{A}) = \bigcup_{1\le i\le u, 1\le j\le v} C_i(\mathcal{A}_j).$$

It is readily checked that $(\mathcal{Z}, \{\{x\} \times \mathcal{Y} : x \in \mathcal{X}\}; C(\mathcal{A}))$ is a $(k, k-1)$-frame of type u^v, where $C_i(\mathcal{A}_j)$, $1 \le i \le u$, $1 \le j \le v$, is the partial parallel class with respect to $\{x_j\} \times \mathcal{Y}$. Now we will prove that this frame is in fact a semiframe with the desired 1-balanced set.

Without loss of generality, we can assume $C_1(\mathcal{A}_1) = \{\{(a_1, y_t), (a_2, y_{t+1}), \ldots, (a_k, y_{t+k-1})\} : \{a_1, a_2, \ldots, a_k\} \in \mathcal{A}_1, 1 \le t \le u\}$, where $t+i$, the index of y, is taken modulo u. Let $\mathcal{D}_m(A) = \{\{(a_1, y_t), (a_2, y_{t+1}), \ldots, (a_k, y_{t+k-1})\} : t = m, k+m, 2k+m, \ldots, (u-k)+m, A = \{a_1, a_2, \ldots, a_k\} \in \mathcal{A}_1\}$, $1 \le m \le k$. Then the elements of $\mathcal{D}_m(A)$ are pairwise disjoint. Finally let $\mathcal{D}'_m = \cup_{B \in \mathcal{D}_m(A)} B$ and $\mathcal{E}_m(A) = \cup_{(x_i, y_j) \in \mathcal{D}'_m(A)} (\mathcal{A}_i \times \{y_j\})$. Then $|\mathcal{D}'_m(A) \cap (\mathcal{X} \times \{y_s\})| = 1$, $1 \le s \le u$. Therefore, $\mathcal{D}_m(A) \cup \mathcal{E}_m(A)$ is a parallel class on $\mathcal{Z}$.

For $2 \le i \le u$, we may take $C_1(\mathcal{A}_i) = \{A \times \{y\} : y \in \mathcal{Y}, A \in \mathcal{A}_i\}$. Since it holds that

$$\bigcup_{1 \le m \le k, A \in \mathcal{A}_1} \mathcal{D}_m(A) = C_1(\mathcal{A}_1),$$

$$\bigcup_{B \in C_1(\mathcal{A}_1)} B = (\mathcal{X} - \{x_1\}) \times \mathcal{Y},$$

$$\bigcup_{1 \le m \le k, A \in \mathcal{A}_1} \mathcal{E}_m(A) = \bigcup_{1 \le m \le k, A \in \mathcal{A}_1} \bigcup_{(x_i, y_j) \in \mathcal{D}'_m(A)} (\mathcal{A}_i \times \{y_j\})$$

$$= \bigcup_{(x_i, y_j) \in (\mathcal{X} - \{x_1\}) \times \mathcal{Y}} (\mathcal{A}_i \times \{y_j\}) = \bigcup_{2 \le i \le v} (\mathcal{A}_i \times \mathcal{Y}),$$

we have

$$\bigcup_{1 \le i \le v} C_1(\mathcal{A}_i) = C_1(\mathcal{A}_1) \cup \Big(\bigcup_{2 \le i \le v} (C_1(\mathcal{A}_i)) \Big)$$

$$= \Big(\bigcup_{1 \le m \le k, A \in \mathcal{A}_1} \mathcal{D}_m(A) \Big) \cup \Big(\bigcup_{2 \le i \le v} (\mathcal{A}_i \times \mathcal{Y}) \Big)$$

$$= \bigcup_{1 \le m \le k, A \in \mathcal{A}_1} (\mathcal{D}_m(A) \cup \mathcal{E}_m(A)).$$

Hence $\bigcup_{1 \le i \le v} C_1(\mathcal{A}_i)$, where $C_1(\mathcal{A}_i)$, $1 \le i \le v$, is the partial parallel class with respect to $\{x_i\} \times \mathcal{Y}$ as mentioned before, can be partitioned into $k \cdot (v-1)/k = v-1$ parallel classes on $\mathcal{Z}$. Therefore the required semiframe has been obtained. □

Construction 3.6 requires a resolvable TD(k, u). It may very well be the case that the resolvable TD(k, u) is not known to exist. The following construction covers this case when k is a prime power. Note that a resolvable balanced incomplete bock design RB$(k, \lambda; v)$ is a (k, λ)-RGDD of type 1^v.

Construction 3.7. Let k be any prime power. Then the existence of an $\mathrm{RB}(k,1;u)$ and an $\mathrm{AR}(k,v)$ implies the existence of a $(k,k-1)$-$\mathrm{SF}(v-1,u-1;u^v)$ in which the $v-1$ parallel classes form a 1-balanced set.

Proof: Let $(\mathcal{X},\mathcal{A})$ be the $\mathrm{AR}(k,v)$ with point set $\mathcal{X}=\{x_1,x_2,\ldots,x_v\}$ and partial parallel classes $\mathcal{A}_m$ which partition $\mathcal{X}-\{x_m\}$, $1\le m\le v$. Let $(\mathcal{Y},\mathcal{B})$ be the $\mathrm{RB}(k,1;u)$ with point set $\mathcal{Y}=\{y_1,y_2,\ldots,y_u\}$ and parallel classes $\mathcal{B}_n$, $1\le n\le (u-1)/(k-1)$. As the point set of the semiframe we take $\mathcal{Z}=\mathcal{X}\times\mathcal{Y}$.

For each $A\in\mathcal{A}$ and $B\in\mathcal{B}$ we construct a resolvable $\mathrm{TD}(k,k)$ on $A\times B$ with groups $\{\{a\}\times B\colon a\in A\}$ and parallel classes $C_i(A\times B)$, $1\le i\le k$. Let

$$C_i(\mathcal{A}_m\times\mathcal{B}_n)=\bigcup_{\substack{A\in\mathcal{A}_m\\ B\in\mathcal{B}_n}} C_i(A\times B)$$

$$1\le i\le k,\quad 1\le m\le v, 1\le n\le\frac{u-1}{k-1}.$$

Since $A\cap A'=B\cap B'=\phi$ for any distinct A, $A'\in\mathcal{A}_1$ and any distinct B, $B'\in\mathcal{B}_1$, without loss of generality we may assume

$$C_1(A^t\times B)=\{(a_t,b_1),(a_{t+1},b_2),\ldots,(a_{t+k-1},b_k)\}, 1\le t\le k,$$

where $\{a_1,a_2,\ldots,a_k\}=A\in\mathcal{A}_1$ and $\{b_1,b_2,\ldots,b_k\}=B\in\mathcal{B}_1$, and the indices $t+i$ are taken modulo k.

Let

$$C_1(A\times B)=\bigcup_{1\le t\le k} C_1(A^t\times B),$$

$$C_1(A^t\times\mathcal{B}_1)=\bigcup_{B\in\mathcal{B}_1} C_1(A^t\times B),$$

$$C_1(\mathcal{A}_1\times\mathcal{B}_1)=\bigcup_{A\in\mathcal{A}_1,B\in\mathcal{B}_1} C_1(A\times B).$$

For each $(A,B)\notin\mathcal{A}_1\times\mathcal{B}_1$, assume $C_1(A\times B)=\{A\times\{b\}\colon b\in B\}$, and let $C_1(\mathcal{A}_m\times\mathcal{B}_n)=\cup_{A\in\mathcal{A}_m,B\in\mathcal{B}_n}C_1(A\times B)$ for $(m,n)\ne(1,1)$. Let

$$\mathcal{C}(\mathcal{A}\times\mathcal{B})=\bigcup_{\substack{1\le i\le k\\ 1\le m\le v\\ 1\le n\le (u-1)/(k-1)}} C_i(\mathcal{A}_m\times\mathcal{B}_n)\setminus\bigcup_{\substack{1\le m\le v\\ 1\le n\le (u-1)/(k-1)\\ (m,n)\ne(1,1)}} C_1(\mathcal{A}_m\times\mathcal{B}_n)$$

and

$$\mathcal{C}=\mathcal{C}(\mathcal{A}\times\mathcal{B})\bigcup\Big(\bigcup_{2\le m\le v}(\mathcal{A}_m\times\mathcal{Y})\Big),$$

where

$$\mathcal{A}_m\times\mathcal{Y}=\bigcup_{y\in\mathcal{Y}}(\mathcal{A}_m\times\{y\}).$$

Then $(Z,\mathcal{C})$ is the required semiframe, where

$$\mathcal{C} = \{C(\mathcal{A}\times\mathcal{B})\setminus C_1(\mathcal{A}_1\times\mathcal{B}_1)\}\bigcup C_1(\mathcal{A}_1\times\mathcal{B}_1)\bigcup\{\bigcup_{2\le m\le v}(\mathcal{A}_m\times\mathcal{Y})\}$$

$$=(\bigcup_{\substack{2\le i\le k\\1\le m\le v\\1\le n\le(u-1)/(k-1)}} C_i(\mathcal{A}_m\times\mathcal{B}_n))\bigcup(\bigcup_{\substack{A\in\mathcal{A}_1\\1\le t\le k}} C_1(A^t\times\mathcal{B}_1))$$

$$\bigcup(\bigcup_{\substack{\{x_m,y_n\}\in C_1(A^t,\mathcal{B}_1)\\A\in\mathcal{A}_1\\1\le t\le k}}(\mathcal{A}_m\times\{y_n\})).$$

$$=(\bigcup_{\substack{2\le i\le k\\1\le m\le v\\1\le n\le(u-1)/(k-1)}} C_i(\mathcal{A}_m\times\mathcal{B}_n))\bigcup\bigcup_{\substack{A\in\mathcal{A}_1\\1\le t\le k}}[C_1(A^t\times\mathcal{B}_1)$$

$$\bigcup(\bigcup_{\{x_m,y_n\}\in C_1(A^t\times\mathcal{B}_1)}(\mathcal{A}_m\times\{y_n\}))].$$

Hence, $\mathcal{C}$ can be partitioned into $(k-1)\cdot(u-1)/(k-1) = u-1$ partial parallel classes $C_i(\mathcal{A}_m\times\mathcal{B}_n)$ with respect to group $\{x_m\}\times\mathcal{Y}$ for $1\le m\le v$, and $k\cdot(v-1)/k = v-1$ parallel classes $C_1(A^t\times\mathcal{B}_1)\cup(\cup_{\{x_m,y_n\}\in C_1(A^t\times\mathcal{B}_1)}\mathcal{A}_m\times\{y_n\})$ which form a 1-balanced set, where $C_1(\mathcal{A}_1\times\mathcal{B}_1)$ is a partial parallel class with respect to $\{x_1\}\times\mathcal{Y}$, and $\mathcal{A}_m\times\mathcal{Y}$ is a partial parallel class with respect to $\{x_m\}\times\mathcal{Y}$, $2\le m\le v$. □

We can also use free difference families in rings to construct semiframes. By a ring R we mean a ring with an identity being not zero. Recall that, $U(R)$, the units of R, forms a group under ring multiplication.

Let G be an additive abelian group and $B=\{b_1,b_2,\ldots,b_k\}$ be a subset of G. Define the *development* of B as $devB = \{B+g\colon g\in G\}$, where $B+g=\{b_1+g,b_2+g,\ldots,b_k+g\}$ for $g\in G$. Let $\mathcal{F}=\{B_1,B_2,\ldots,B_t\}$ be a family of subsets of G and define the *development* of $\mathcal{F}$ as $dev\mathcal{F} = \cup_{1\le i\le t}\, devB_i$. If $dev\mathcal{F}$ is a (k,λ)-GDD of type 1^v, it is said that $\mathcal{F}$ is a $(k,\lambda;v)$ *difference family*, denoted by DF$(k,\lambda;v)$, where the B_i's, $1\le i\le t$, are the *base blocks*. If the base blocks are mutually disjoint, the difference family is said to be *free*, and denoted by FDF.

It is easy to see that in an FDF$(k,\lambda;v)$, $\lambda\le k$, where $\lambda=k$ if and only if $v=k$.

Construction 3.8. Suppose there exist an RB$(k,\lambda;u)$ and an FDF$(k,\lambda;v)$ over a ring R, where $v\ne k$. If there exists a set of u distinct units u_i, $0\le i\le u-1$, whose differences are still units of R, then there exists a (k,λ)-SF$((v-1)(\lambda u/(k-1)-1),1;u^v)$.

Proof: By the assumption that $v\ne k$, we have $\lambda\le k-1$, which implies $\sum_{0\le i\le s-1}|A_i| = sk = \lambda(v-1)/(k-1) < v$, where $\{A_0,\ldots,A_{s-1}\}$ is the

FDF$(k,\lambda;v)$. Thus, without loss of generality, we may assume that $0 \notin A_i$ for any i, $0 \le i \le s-1$.

We construct an RB$(k,\lambda;u)$ on $I_u = \{0,1,\ldots,u-1\}$. Let $\mathcal{V} = R \times I_u$ and $\mathcal{G} = \{\{x\} \times I_u : x \in R\}$. As some base blocks for a (k,λ)-SF$((v-1)(\lambda u/(k-1)-1),1;u^v)$, we choose

$$B_j^i = A_j \times \{i\} = \{(a_1^j,i),(a_2^j,i),\ldots,(a_k^j,i)\}, i \in I_u, j = 0,1,\ldots,s-1,$$

where $A_j = \{a_1^j, a_2^j, \ldots, a_k^j\}$ is a base block of the given free difference family. Now replace the base blocks B_j^i by $u_i B_j^i$.

In order to get further base blocks, for any blocks $B = \{b_1, b_2, \ldots, b_k\}$ of the RB$(k,\lambda;u)$ $(I_u, \mathcal{B})$, we put

$$C_B(x) = \{(u_{b_1}, b_1), (u_{b_2}, b_2), \ldots, (u_{b_k}, b_k)\} \cdot x, \quad x \in R - \{0\},$$

where $(u,i) \cdot x$ means (ux, i).

For any parallel class P of the RB$(k,\lambda;u)$, put

$$C_P(x) = \bigcup_{B \in P} C_B(x), \quad x \in R - \{0\}.$$

We now have an appropriate set $\mathcal{S}$ of base blocks

$$\mathcal{S} = \{u_i B_j^i : i \in I_u, j = 0,1,\ldots,s-1\}$$
$$\bigcup\{C_B(x) : x \in R - \{0\}, B \in \mathcal{B}\}.$$

The pure differences all arise from the blocks $u_i B_j^i$, and the mixed differences all arise from $C_B(x), x \in R - \{0\}, B \in \mathcal{B}$. By hypothesis,

$$\sum_j \Delta_{ii}(u_i B_j^i) = \sum_j u_i \Delta_{ii} B_j^i = u_i \sum_j \Delta_{ii} A_j = u_i \cdot \lambda(R - \{0\})$$
$$= \lambda(R - \{0\}).$$

Furthermore, for $i < j$,

$$\Delta_{ij}\{C_B(x) : x \in R - \{0\}, B \in \mathcal{B}\} = \lambda(u_i - u_j)(R - \{0\}) = \lambda(R - \{0\}).$$

Hence $\Delta_{ii}\mathcal{S} = \lambda(R - \{0\})$, $\Delta_{ij}\mathcal{S} = \lambda(R - \{0\})$ for $i \ne j$. Thus $(\mathcal{V}, \mathcal{G}, dev\mathcal{S})$ is a (k,λ)-GDD of type u^v. It remains to show that this GDD is a semiframe.

We have to partition the blocks into v partial parallel classes and $(v-1)(\lambda u/(k-1)-1)$ parallel classes.

As the partial parallel class Q_0 with respect to $\{0\} \times I_u$, take all blocks $u_i B_j^i, i \in I_u$, $j = 0,1,\ldots,s-1$, and the blocks $C_{P_0}(x)$, where P_0 is one parallel class of the RB$(k,\lambda;u)$, and x is nonzero and distinct from all

$a_i^j, i \in I_k$, $j = 0, 1, \ldots, s-1$. Other partial parallel classes are given by $Q_g = \tau_g Q_0$, where $\tau_g : (x, i) \mapsto (x+g, i), g \in R$. That is, $Q_g = \{\tau_g(B) : B \in Q_0\}$. We now construct the parallel classes.

Let $R_P(x) = \{\tau_g C_P(x) : g \in R\}$ with $x \in \cup_{0 \le j \le s-1} A_j$, and P is any parallel class of the RB$(k, \lambda; u)$, and $S_P(x) = \{\tau_g C_P(x) : g \in R\}$ with $x \in R - \{0\} - \cup_{0 \le j \le s-1} A_j$, and P is any parallel class of the RB$(k, \lambda; u)$ being not P_0. Both $R_P(x)$ and $S_P(x)$ are parallel classes. The total number of the parallel class is $\lambda(u-1)ks/(k-1) + [\lambda(u-1)/(k-1) - 1](v-1-ks) = (v-1)(\lambda u/(k-1) - 1)$, as required. □

If we restrict ourselves to $\lambda = k - 1$, then, in the proof of Construction 3.8, $ks = v - 1$, and hence $\cup_{0 \le i \le s-1} A_i = R - \{0\}$. This implies that the partial parallel class Q_0 does not contain $C_{P_0}(x)$ for any x, and hence so is Q_g for any $g \in R$. In this instance, we permit $P = P_0$ in the expression of $S_P(x)$. Now we re-partition the blocks $C_B(x), B \in \mathcal{B}, x \in R - \{0\}$. For any parallel class P of the RB$(k, k-1; u)$, $C_P = \cup_{x \in R - \{0\}} C_P(x)$ is a partial parallel class with respect to $\{0\} \times I_u$. Other partial parallel classes are given by $\tau_g C_P$, $g \in R$. Hence the parallel classes $R_P(x)$, $S_P(x)$ defined above form 1-balanced sets for any P.

Construction 3.9. Suppose there exist an RB$(k, k-1; u)$ and an FDF$(k, k-1; v)$ over a ring R. If there exists a set of u distinct units such that their differences are still units of R, then there exists a $(k, k-1)$-SF$((v-1)(u-1), 1; u^v)$ in which the parallel classes can be partitioned into $u - 1$ subsets of $v - 1$ parallel classes each of which forms a 1-balanced set.

4 Series of semiframes

Here we give some series of designs with $k = 3$. Other cases on values of k (≥ 4) can be similarly discussed. First a brief description of some preliminary results is given. The necessary condition for the existence of an AR(k, v) is $v \equiv 1 \pmod{k}$. It is known (see, for example, [4]) that this condition is also sufficient for $k = 3, 4, 5$ and almost sufficient for $6 \le k \le 10$. It is also well known that the existence of a TD(k, v) is equivalent to that of $k - 2$ mutually orthogonal Latin squares of order v. For more information on the lower bound of the number of mutually orthogonal Latin squares of order v, the reader is referred to, for example, [2] for details.

The only systematic work on (4,1)-RGDDs is due to Shen [15, 16], and Rees and Stinson [13] who proved that a (4,1)-RGDD of type 3^u exists if and only if $u \equiv 0 \pmod{4}$, $u \ge 8$, except possibly for $u \in \{28, 44, 88, 124, 152, 184, 220, 268, 284\}$. The existence problem of RB$(3, \lambda; v)$ was completely settled by Hanani, and Ray-Chaudhuri and Wilson (see [1]).

For detailed information on units in rings and free difference family the reader is referred to [6, 7].

Theorem 4.1. *Let $u \equiv 1 \pmod 3$, $0 < p/(u-1) \leq g/3$, $g \neq 6, 18$. Then the necessary condition for the existence of a (3,2)-SF$(p, d; g^u)$ is also sufficient.*

Proof: By Lemma 2.2, if a (3,2)-SF$(p, d; g^u)$ exists, then $g \equiv 0 \pmod 3$, $d = g - p/(u-1)$ and $u \geq 4$. Let $g = 3n$ for $n \neq 2, 6$ and $p = m(u-1)$. Then $0 < m \leq n$. So put $m = n - i$, $0 \leq i < n$. Since there exist an AR$(3, u)$ ([4]) and a resolvable TD(3,3) ([2]), we have a (3,2)-SF$(u-1, 2; 3^u)$ in which $u-1$ parallel classes form a 1-balanced set by Construction 3.6. Applying Construction 3.3 with a resolvable TD$(3, n)$ ([2]), we obtain a (3,2)-SF$(n(u-1) - i(u-1), 2n+i; (3n)^u)$ for each $i = 0, 1, \ldots, n-1$, which is a (3,2)-SF$(m(u-1), 3n - m; (3n)^u)$, i.e. a (3,2)-SF$(p, d; g^u)$. □

Theorem 4.2. *Let $0 < p/3 \leq m$, $m \neq 2, 6$ and $n \in N - \{7, 11, 22, 31, 38, 46, 55, 67, 71\}$. Then there exists a (3,2)-SF$((4n-1)p, 3d; (9m)^{4n})$.*

Proof: By Theorem 4.1, there exists a (3,2)-SF$(p, d; (3m)^4)$. Apply Construction 3.4 with a (4,1)-RGDD of type 3^{4n}. □

Theorem 4.3. *Let $u \equiv 3 \pmod 6$, and $q \equiv 1 \pmod 6$ be a prime power. Then there exists a (3,1)-SF$((q-1)(u/2-1), 1; u^q)$.*

Proof: An RB$(3, 1; u)$ exists for any $u \equiv 3 \pmod 6$ (see [18]), and an FDF$(3, 1; q)$ exists for any $q \equiv 1 \pmod 6$ if q is a prime power (see [8]). Apply Construction 3.8. □

Theorem 4.4. *Let $u \equiv 0 \pmod 3$, $u \neq 6$, and $v = \prod_i q_i^{n_i}$ be the prime power factorization of v such that $q_i \equiv 1 \pmod 3$ for all i and $\min_i\{q_i^{n_i} - 1\} \geq u$. Then there exists a (3,2)-SF$((v-1)(u-1-j), 1+j; u^v)$ for $j = 0, 1, \ldots, u-1$.*

Proof: An RB$(3, 2; u)$ exists for any $u \equiv 0 \pmod 3$, $u \neq 6$ (see [4]), and an FDF$(3, 2; v)$ exists for any of such v (see [7, 8]). Apply Construction 3.9. Then apply Construction 3.3 with $n = \mu = 1$. In this instance, there is no need for the existence of a resolvable TD$(k, 1)$. We can obtain a (3,2)-SF$((v-1)(u-2), 2; u^v)$ from a (3,2)-SF$((v-1)(u-1), 1; u^v)$. Repeat this procedure. □

Theorem 4.4 can be used to establish the following.

Theorem 4.5. *Let $v = \prod_i q_i^{n_i}$ be the prime power factorization of v such that $q_i \equiv 1 \pmod 3$ for all i and $\min_i\{q_i^{n_i} - 1\} \geq u$ where $u \neq 6$. Then the necessary conditions for the existence of a (3,2)-SF$(p, d; u^v)$ are also sufficient.*

Proof: The necessity follows from Lemma 2.3, where $u \equiv 0 \pmod 3$, $d = u - p/(v-1)$, and $v \geq 4$. The sufficiency follows from Theorem 4.4. □

5 Cyclic semiframes

We have described some constructions for semiframes in Section 3 and illustrated them with some examples in Section 4. Here we shall deal with a special type of semiframe which we call a cyclic semiframe. The ideas of the constructions here are based on those in [5], and hence sketches for their proofs will be only given.

Let $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ be a (k, λ)-GDD of type g^u, and σ be a permutation on $\mathcal{X}$. For any block $B = \{b_1, \ldots, b_k\} \in \mathcal{B}$ and any group $G = \{x_1, \ldots, x_g\} \in \mathcal{G}$, define $B^\sigma = \{b_1^\sigma, \ldots, b_k^\sigma\}$ and $G^\sigma = \{x_1^\sigma, \ldots, x_g^\sigma\}$. If $\mathcal{B}^\sigma = \{B^\sigma : B \in \mathcal{B}\} = \mathcal{B}$ and $\mathcal{G}^\sigma = \{G^\sigma : G \in \mathcal{G}\} = \mathcal{G}$, then σ is called an *automorphism* of $(\mathcal{X}, \mathcal{G}, \mathcal{B})$. If there is an automorphism σ of order $gu = |\mathcal{X}|$, then the (k, λ)-GDD is said to be *cyclic*, and denoted by (k, λ)-CGDD. For a (k, λ)-CGDD of type g^u, the point set $\mathcal{X}$ can be identified with Z_{gu}. In this case, the design has an automorphism $\sigma: i \mapsto i + 1 \pmod{gu}$, and each group must be the subgroup uZ_g of Z_{gu} or its cosets (see [9]).

For a partial parallel class P_i, let $P_i^\sigma = \{B^\sigma : B \in P_i\}$, and for a parallel class Q_j, let $Q_j^\sigma = \{B^\sigma : B \in Q_j\}$. If the underlying GDD of a (k, λ)-SF$(p, d; g^u)$ is cyclic with respect to an automorphism σ of order $gu = |\mathcal{X}|$, such that $\mathcal{P}^\sigma = \{P_1^\sigma, P_2^\sigma, \ldots\} = \mathcal{P}$ and $\mathcal{Q}^\sigma = \{Q_1^\sigma, Q_2^\sigma, \ldots\} = \mathcal{Q}$, then the semiframe is said to be *cyclic* with respect to σ, and denoted by (k, λ)-CSF$(p, d; g^u)$.

At first, we give a construction for such semiframes by using a kind of difference method similar to that of [19].

Let k be odd, say $k = 2m+1$. We shall say that a prime power p satisfies the condition R_k if and only if $p \equiv 1 \pmod{k(k-1)}$ and there exists a primitive kth root of unity, ε, in GF(p) such that $\{\varepsilon - 1, \ldots, \varepsilon^m - 1\}$ is a system of representatives for the m cosets modulo $H^m = \{1, \alpha^m, \ldots, \alpha^{(f-1)m}\}$ for a primitive element α of GF(p), where $f = (p-1)/m$. Note that the condition R_k is well defined. In fact, it does not depend on the choice of the primitive kth root of unity in GF(p).

Construction 5.1. If a prime p satisfies the condition R_k, then there exists a $(k, 1)$-CSF$((p-1)/(k-1), 1; k^p)$.

Proof: Arbitrarily choose a primitive element α of GF$(p) = Z_p$ such that $\alpha \not\equiv 0 \pmod{k}$, and let $p = nk(k-1)+1$. Without loss of generality, let $\varepsilon = \alpha^{2mn}$ since α^{2mn} is a primitive kth root of unity. Let $A = \{1, \varepsilon, \ldots, \varepsilon^{2m}\} \pmod{p}$ and $A_j = A\alpha^{jm}$ for $j = 0, 1, \ldots, n-1$. It is known that A_j's, $j = 0, 1, \ldots, n-1$, form the base blocks of a cyclic balanced incomplete block design CB$(k, 1; p)$ and $\cup_{j=0}^{n-1} \Delta A_j = kZ_p - \{0\} \pmod{kp}$.

Next, let $B_i = \{\alpha^i, \varepsilon\alpha^i + p, \ldots, \varepsilon^{2m}\alpha^i + 2mp\} \pmod{kp}$ for $i = 0, 1, \ldots, 2mn-1$. Then $\cup_{i=0}^{2mn-1} \Delta B_i = Z_{kp} - kZ_p - pZ_k$ which implies that the base blocks B_i's together with the base blocks A_j's generate a $(k, 1)$-CGDD of type k^p.

The blocks of this design can be partitioned into partial parallel classes and parallel classes. The partial parallel class with respect to $\{h, p+h, \ldots, (k-1)p+h\} = pZ_k + h$ are $P_h = P_0 + h \pmod{kp}$ for $h \in Z_p$, where $P_0 = \{A_j + \ell p, B_i + \ell p \pmod{kp}: j = 0, 1, \ldots, n-1; i = 0, 1, \ldots, 2mn-1; i \neq 0, m, \ldots, (n-1)m; \ell = 0, 1, \ldots, k-1\}$. Furthermore, the parallel classes are $Q_{j\ell} = Q_{j0} + \ell \pmod{kp}$ for $j = 0, 1, \ldots, n-1; \ell = 0, 1, \ldots, k-1$, where $Q_{j0} = \{B_{jm} + hk \pmod{kp}: h \in Z_p\}$ for $j = 0, 1, \ldots, n-1$. □

This result can be generalized to the following.

Construction 5.2. If two primes p and q satisfy the condition R_k respectively, then there exists a $(k,1)$-CSF$((pq-1)/(k-1), 1; k^{pq})$.

Proof: Let $p = n'k(k-1)+1$ and $q = n''k(k-1)+1$. By virtue of Construction 5.1, there exists a $(k,1)$-CSF$((p-1)/(k-1), 1; k^p)$ with partial parallel classes $P'_0, P'_1, \ldots, P'_{p-1}$ and parallel classes $Q'_{00}, Q'_{01}, \ldots, Q'_{n'-1,k-1}$ where $P'_0 = \{A_i + \ell p \pmod{kp}: i = 0, 1, \ldots, n'(k-1)-1; \ell = 0, 1, \ldots, k-1\}$, $Q'_{j0} = \{A'_i + hk \pmod{kp}: j = 0, 1, \ldots, n'-1; h \in Z_p\}$, and a $(k,1)$-CSF$((q-1)/(k-1), 1; k^q)$ with partial parallel classes $P''_0, P''_1, \ldots, P''_{q-1}$ and parallel classes $Q''_{00}, Q''_{01}, \ldots, Q''_{n''-1,k-1}$ where $P''_0 = \{B_i + \ell q \pmod{kq}: i = 0, 1, \ldots, n''(k-1)-1; \ell = 0, 1, \ldots, k-1\}$, $Q''_{j0} = \{B_j + hk \pmod{kq}: j = 0, 1, \ldots, n''-1; h = 0, 1, \ldots, k-1\}$.

Now for each block $A_i = \{a_{i1}, a_{i2}, \ldots, a_{ik}\}$, we define the following q base blocks

$$A_i^{(t)} = \{a_{is} + \overline{st}kp \pmod{kpq}: s = 1, 2, \ldots, k\}$$

for $t = 0, 1, \ldots, q-1$, where $\overline{st} \equiv st \pmod q$ and $0 \leq \overline{st} \leq q-1$. Similarly, $A_j'^{(t)}$ for each block A'_j is defined. Then it can be checked that any point of pZ_{kq} does not occur with the point 0, while every point of $Z_{kpq} - pZ_{kq}$ occurs once together with the point 0.

Next, for each block $B_i = \{b_{i1}, b_{i2}, \ldots, b_{ik}\}$, let $pB_i = \{pb_{i1}, pb_{i2}, \ldots, pb_{ik}\}$ (mod kpq) be a base block of the desired semiframe, and similarly let pB'_j's be also base blocks of the semiframe. Then any point of pqZ_k does not occur with the point 0, while every point of $pZ_{kq} - pqZ_k$ occurs once with the point 0.

By developing the base blocks $A_i^{(t)}$, $A_j'^{(t)}$, qB_i and qB'_j, a $(k,1)$-CGDD of type k^{pq} is obtained.

Finally, we partition the blocks into partial parallel classes and parallel classes .

Let $P_0 = \{A_i^{(t)} + \ell pq \pmod{kpq}: t = 0, 1, \ldots, q-1; i = 0, 1, \ldots, (k-1)n'-1; \ell = 0, 1, \ldots, k-1\} \cup pP''_0$. Since P''_0 is a partial parallel class with respect to qZ_k of the $(k,1)$-CSF$((q-1)/(k-1), 1; kq)$, every point of $p(Z_{kq} - qZ_k) = pZ_{kq} - pqZ_k$ is contained in pP''_0 exactly once. Furthermore, $\{A_i^{(t)} + \ell pq \pmod{kpq}: t = 0, 1, \ldots, q-1; i = 0, 1, \ldots, (k-1)n'-1; \ell = 0, 1, \ldots, k-1\} = Z_{kpq} - pZ_{kq}$. Then P_0 is a partial parallel class with respect to pqZ_k. More pq partial parallel classes can be obtained by letting $P_h = P_0 + h \pmod{kpq}$ for $h = 0, 1, \ldots, pq-1$.

Let $Q_{(jq+t)k} = \{A_i'^{(t)} + hk \pmod{kpq} : h = 0, 1, \ldots, pq-1\}$ for $t = 0, 1, \ldots, q-1$ and $j = 0, 1, \ldots, n'-1$, and let $\tilde{Q}_{j,k} = \{vB_j' + hk \pmod{kpq} : h = 0, 1, \ldots, pq-1\}$ for $j = 0, 1, \ldots, n''-1$. Then $Q_{(jq+t)k}$ and $\tilde{Q}_{j,k}$ are parallel classes respectively. Then we can obtain $n'qk + n''k$ parallel classes by defining $Q_{(jq+t)k+\ell} = Q_{(jq+t)k} + \ell \pmod{kpq}$ and $\tilde{Q}_{jk+\ell} = \tilde{Q}_{jk} + \ell$ respectively. □

Note that the partial parallel classes and parallel classes in Construction 5.2 have same properties as those in Construction 5.1, and hence we can extend Construction 5.2 to the cases when the number of primes exceeds two.

Now we present a recursive construction for cyclic semiframes by using a difference matrix. It looks similar to that of Construction 3.1.

Let $D = (d_{ij}), i = 1, 2, \ldots, k; j = 1, 2, \ldots, g\lambda$, be a matrix with entries from Z_g such that the list $(d_{i\ell} - d_{j\ell})_{\ell=1,\ldots,g\lambda}$ contains each element of Z_g precisely λ times, whenever $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$. Then D is called a *difference matrix* and denoted by DM$(k, \lambda; g)$. A DM$(k, \lambda; g)$ is said to be *homogeneous*, denoted by HDM$(k, \lambda; g)$, if the list $(d_{i\ell})_{\ell=1,\ldots,g\lambda}$ contains each element of Z_g precisely λ times for $i = 1, 2, \ldots, k$. It is well known that an HDM$(k, \lambda; g)$ is equivalent to a DM$(k+1, \lambda; g)$.

Construction 5.3. Let p be a prime satisfying the condition R_k. If there exists an HDM$(k, 1; g)$, then there exists a $(k, 1)$-CSF$(g(p-1)/(k-1), g; (gk)^p)$.

Proof: Classes $P_0', \ldots, P_{p-1}', Q_{00}', \ldots, Q_{n-1,k-1}'$ are defined just as in the proof of Construction 5.2, while for each A_i, we define $A_i^{(t)} = \{a_{is} + d_{st}kp \pmod{kpg} : s = 1, 2, \ldots, k\}$ for $t = 0, 1, \ldots, g-1$, and similarly we define $A_j'^{(t)}$ for each A_j'. Then the blocks developed from the base blocks $A_i^{(t)}$ and $A_j'^{(t)}$ form the collection of blocks of a $(k, 1)$-CGDD of type $(kg)^p$, where $P_h = \{A_i^{(t)} + \ell pq + h \pmod{kpq} : t = 0, 1, \ldots, g-1; i = 0, 1, \ldots, (k-1)n' + 1; \ell = 0, 1, \ldots, k-1\}$, $h \in Z_{pg}$, forms a partial parallel class with respect to $pZ_{kg} + h$ and $Q_{(jg+t)k+\ell} = \{A_j'^{(t)} + hk + \ell \pmod{kpq} : h = 0, 1, \ldots, pg-1\}$ for $t = 0, 1, \ldots, g-1; j = 0, 1, \ldots, n-1$, and $\ell = 0, 1, \ldots, k-1$, forms a parallel class. □

An immediate consequence of these constructions is the following.

Theorem 5.4. *Let g be a positive integer not divisible by 2 and 3, and p_i's are primes such that $p_i \equiv 1 \pmod 6$ for $i = 1, 2, \ldots, n$. Then there exists a $(3, 1)$-CSF$((p_1 p_2 \cdots p_n - 1)g/2, g; (3g)^{p_1 p_2 \cdots p_n})$.*

Proof: By [3], an HDM$(3, 1; g)$ always exists for such g. Furthermore, every prime $p_i \equiv 1 \pmod 6$ satisfies the condition R_k, that is, $\varepsilon - 1 \in H^1$. □

6 Another application of semiframes

Rees [10, 11] listed some applications of semiframes for the construction of other types of designs. In this final section, we describe one more application for the construction of RGDDs. This shows that there is some possibility of constructing RGDDs and frames, which are the extreme cases of semiframes, from proper semiframes.

Construction 6.1. Let $d \leq \lambda g(n-1)/(k-1)$. Then the existence of a (k,λ)-SF$(p,d;(ng)^u)$ and a (k,λ)-RGDD of type g^n implies the existence of a (k,λ)-RGDD of type g^{nu}.

Proof: Let $(\mathcal{X},\mathcal{G},\mathcal{B})$ be the semiframe in which $\mathcal{B}$ is written as a disjoint union $\mathcal{B} = \mathcal{P} \cup \mathcal{Q}$ where $\mathcal{P}$ is partitioned into partial parallel classes P_G^i, $1 \leq i \leq d$, $G \in \mathcal{G}$, with respect to $G \in \mathcal{G}$, and $\mathcal{Q}$ is partitioned into parallel classes Q_j, $1 \leq j \leq p$. Let $(\mathcal{G},\mathcal{H}_G,\mathcal{B}_G)$ be the RGDD of type g^n in which $\mathcal{B}_G$ is partitioned into parallel classes B_G^ℓ, $1 \leq \ell \leq \lambda g(n-1)/(k-1)$, of $G \in \mathcal{G}$. Then $(\mathcal{X}, \cup_{G\in\mathcal{G}}\mathcal{H}_G, \mathcal{B}\cup\cup_{G\in\mathcal{G}}\mathcal{B}_G)$ is the required design, where Q_j, $P_G^i \cup B_G^i$, $\cup_{G\in\mathcal{G}} B_G^\ell$, $G \in \mathcal{G}$, $1 \leq i \leq d$, $1 \leq j \leq p$, $d+1 \leq \ell \leq \lambda g(n-1)/(k-1)$, are the $\lambda g(nu-1)/(k-1)$ parallel classes. □

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