The Connectivity Of The Leaf-Exchange Spanning Tree Graph Of A Graph

H.J. Broersma and Li Xueliang*
Department of Applied Mathematics
University of Twente
7500 AE Enschede
The Netherlands

ABSTRACT. Let G be a connected (multi)graph. We define the leaf-exchange spanning tree graph $T_l(G)$ of G as the graph with vertex set $V_l = \{T|T$ is a spanning tree of $G\}$ and edge set $E_l = \{(T,T')|E(T)\Delta E(T') = \{e,f\}, e \in E(T), f \in E(T')$ and e and f are incident with a vertex v of degree 1 in T and $T'\}$. $T_l(G)$ is a spanning subgraph of the socalled spanning tree graph of G, and of the adjacency spanning tree graph of G, which were studied by several authors. A variation on the leaf-exchange spanning tree graph appeared in recent work on basis graphs of branching greedoids. We characterize the graphs which have a connected leaf-exchange spanning tree graph and give a lower bound on the connectivity of $T_l(G)$ for a 3-connected graph G.

1 Introduction

We use [1] for terminology and notation not defined here and consider simple graphs only, although obvious analogues of the results also hold for multigraphs.

Let G be a connected graph. The *spanning tree graph* $T(G)$ of G has vertex set $V_T = \{T|T$ is a spanning tree of $G\}$ and edge set $E_T = \{T,T'|E(T)\Delta E(T') = \{e,f\}, e \neq f\}$. Spanning tree graphs were studied in several papers (e.g. [3,6]) and many of the results on spanning tree graphs have been generalized to results on basis graphs of matroids. In more recent papers [5,8], the spanning subgraph $T_a(G)$ of $T(G)$ was studied. The edge set of $T_a(G)$

*On leave from the Department of Mathematics, Xinjiang University, Urumchi, Xinjiang, P.R. China

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is the subset of E_T formed by pairs (T, T') such that $E(T)\Delta E(T') = \{e, f\}$ for a pair of adjacent edges e and f. For a (spanning) tree T (of G), a vertex of degree one in T will be called a *leaf* of T, and an edge of T incident with a leaf will be called a *leaf edge* of T. We define the *leaf-exchange spanning tree graph* $T_l(G)$ of G as the spanning subgraph of $T(G)$ for which the edge set is formed by pairs (T, T') with $E(T)\Delta E(T') = \{e, f\}$ such that e and f are leaf edges (of T and T', respectively). It is obvious that $T_l(G)$ is also a spanning subgraph of $T_a(G)$.

In Section 2 we characterize the graphs that have a connected leaf-exchange spanning tree graph, while in Section 3 we give a lower bound on the connectivity of $T_l(G)$ in terms of parameters of G. The latter result may be of some help in finding spanning trees with a given number of leaves, which is an NP-complete problem in general [7]. A slight variation on the leaf-exchange spanning tree graph appeared in recent work on basis graphs of branching greedoids [2,4,9]. In these graphs the spanning trees all have a common root vertex.

2 Graphs with a connected leaf-exchange spanning tree graph

A graph G is a *block* if either G is isomorphic to K_2 or G is 2-connected; G is a *multiblock* if it can be obtained from a number of blocks $\not\cong K_2$ by identifying precisely one vertex of each block; this vertex is called the *center* of the multiblock. A graph is called *tree-like* if it can be constructed from a tree T and a number of disjoint multiblocks (disjoint from T) by identifying the centers of the multiblocks with distinct vertices of T. Note that a 2-connected graph is a multiblock and that it is tree-like.

It is clear that the leaf-exchange spanning tree graph of a tree is K_1. In the sequel of this section, we assume G is not a tree.

Theorem 1. *$T_l(G)$ is connected if and only if G is tree-like.*

Proof: Suppose G is tree-like and $M_1, M_2, \ldots, M_m$ are the multiblocks used to construct G from a tree. Regard M_i as a rooted graph with the center as the root. The spanning trees of M_i are the bases of a branching greedoid (See [4]) and the basis graph $G(M_i)$ of this branching greedoid is connected. Since $T_l(G)$ has a spanning subgraph isomorphic to $G(M_1) \times G(M_2) \times \ldots \times G(M_m)$, this implies that $T_l(G)$ is connected.

Now suppose G is not tree-like. Then G contains a block $H \not\cong K_2$, such that H has at least two vertices u and v in common with two other blocks H_1 and H_2 (possibly K_2), respectively. Let $e_1, e_2, \ldots, e_p$ be the edges of H incident with v. Consider a spanning tree T_1 of G containing e_1 and none of $e_2, \ldots, e_p$, and a spanning tree T_p of G containing e_p and none of $e_1, \ldots, e_{p-1}$. Using the fact that there is a unique path in T_1 (and T_p) from u to v and that u and v are not leaves of T_1 (and T_p), it can be shown that

T_1 and T_p are not connected in $T_l(G)$. □

3 The connectivity of the leaf-exchange spanning tree graph

For a spanning tree T of a connected graph G, let $\Lambda(T)$ denote the set of leaves of T, and let $\lambda(G) = \min\{|\Lambda(T)| \; |T \text{ is a spanning tree of } G\}$. For a subset $S \subset V(G)$, let $\overline{S} = V(G) \backslash S$, and for a vertex $v_i \in S$, let $(v_i, \overline{S}) = \{v_i u | u \in \overline{S}\}$ and $s_i = s(v_i) = |(v_i, \overline{S})|$.

Note that $s_i \geq \delta - |S| + 1$. Let

$$\rho(t) = \min\left\{\sum_{i=1}^{t} s_i - t | S = \{v_1, v_2, \ldots, v_t\} \subset V(G)\right\}$$

Then $\rho(t) \geq t(\delta - t + 1) - t = t(\delta - t)$.

In this section we prove the following result.

Theorem 2. *Let G be a k-connected graph ($k \geq 3$). Then $T_l(G)$ is $\rho(t)$-connected, where $t = \min\{\lambda(G), \lfloor \frac{1}{2}(k-1) \rfloor\}$.*

To prove Theorem 2 in the sequel we are going to show there are at least $\rho(t)$ internally-disjoint paths between any two vertices X and Y of $T_l(G)$. We use several lemmas. First we give a lower bound on the number of spanning trees of G with t common leaves (Lemma 3). Using a variation on Menger's Theorem (Lemma 4) we prove that the set of spanning trees of G with t common leaves induces a $\rho(t)$-connected subgraph of $T_l(G)$ (Lemma 5). This implies the existence of $\rho(t)$ internally-disjoint paths between X and Y if the corresponding spanning trees in G have t common leaves. For the other case, we use the above results in combination with some counting lemmas (Lemmas 6 and 7) to prove that for any vertex T of $T_l(G)$ there is a set of disjoint paths (except for the origin T) to at least $\rho(t)$ distinct vertices of $T_l(G)$ corresponding to spanning trees of G with t common leaves (Lemma 8). Together with Lemma 5 this completes the proof of Theorem 2.

Lemma 3. *Let G be a k-connected graph ($k \geq 2$), and let $S = \{v_1, \ldots, v_t\} \subset V(G)$ for some $t \leq k-1$. Then G has at least $\Pi_{i=1}^{t} s_i$ spanning trees containing $v_1, \ldots, v_t$ as leaves.*

Proof: Since G is k-connected and $t \leq k-1$, $G - S$ is connected and contains a spanning tree. For every v_i we can use one of the s_i edges to extend this tree to a spanning tree of G containing $v_1, \ldots, v_t$ as leaves. This can be done in $\Pi_{i=1}^{t} s_i$ different ways. □

The following lemma is an easy consequence of Menger's Theorem.

Lemma 4. *Let G be a k-connected graph ($k \geq 1$), and let $A = \{u_1, \ldots, u_h\} \subset V(G)$, $B = \{v_1, \ldots, v_h\} \subset V(G)$ with $h \leq k$ and $C = A \cap B$. Then there* *are at least $|A \backslash C|$ internally-disjoint paths from $A \backslash C$ to $B \backslash C$ which do not contain vertices of C.*

Lemma 5. *Let G be a k-connected graph ($k \geq 3$), and let $S = \{v_1, \ldots, v_t\} \subset V(G)$ for some $t \leq k-2$. Let $B(S)$ be the set of spanning trees of G containing $v_1, \ldots, v_t$ as leaves. Then $B(S)$ induces a $\rho(t)$-connected subgraph of $T_l(G)$.*

Proof: Since G is k-connected and $t \leq k-2$, $G - S$ is 2-connected and hence $T_l(G-S)$ is connected by Theorem 1. For two elements X and Y of $B(S)$ let $T = X - S$ and $T' = Y - S$. Then T and T' are spanning trees of $G - S$. Hence there is a path $T = T_1 T_2 \ldots T_p = T'$ in $T_l(G-S)$. For all i with $1 \leq i \leq p$, let $\mathcal{T}_1$ denote the set of all spanning trees induced by $E(T_i) \cup \{e_{1i_1}, \ldots, e_{ti_t}\}$ with $e_{ji_j} \in (v_j, \overline{S})$. Clearly $\mathcal{T}_i$ induces a subgraph of $T_l(G)$ which is isomorphic to $K_{s_1} \times K_{s_2} \times \ldots \times K_{s_t}$, hence a $\left(\sum_{i=1}^{t}(s_i - 1)\right)$-connected graph and thus a $\rho(t)$-connected graph. Since at most one of the edges incident with v_j is incident with the leaf that makes T_i and T_{i+1} adjacent, there are at least $\Pi_{i=1}^{t}(s_i - 1)$ independent edges between $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ in $T_l(G)$. Since $t \leq k-2 \leq \delta - 2$, we have $s_i \geq \delta - t + 1 \geq 3$, so that $\Pi_{i=1}^{t}(s_i - 1) \geq \sum_{i=1}^{t}(s_i - 1) \geq \rho(t)$. Hence there are at least $\rho(t)$ independent edges between $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ $(1 \leq i \leq p-1)$. For a triple $\mathcal{T}_i, \mathcal{T}_{i+1}, \mathcal{T}_{i+2}$ $(1 \leq i \leq p-2)$, let A and B be subsets of the graph induced by $\mathcal{T}_{i+1}$ determined by $\rho(t)$ edges between $\mathcal{T}_i$ and $\mathcal{T}_{i+1}$ and $\mathcal{T}_{i+1}$ and $\mathcal{T}_{i+2}$, respectively, and let $C = A \cap B$. Then, by Lemma 4, there are at least $|A \backslash C|$ internally-disjoint paths from $A \backslash C$ to $B \backslash C$ which do not contain vertices of C. Using this at each stage we can find $\rho(t)$ internally-disjoint paths between X and Y. □

Lemma 6. *Let $x_i \in \mathbf{R}$ and $x_i \geq 2$ ($i = 1, 2, \ldots, r+t$ with $1 \leq r \leq t$ and $r, t \in \mathbf{N}$). Then*

$$\Pi_{i=1}^{r+t} x_i \geq \sum_{i=1}^{r+t} x_i + 2rt - (r+t).$$

Proof: Consider the function

$$f(x_1, \ldots, x_{r+t}) = \Pi_{i=1}^{r+t} x_i - \sum_{i=1}^{r+t} x_i - 2rt + (r+t).$$

Since $\frac{\partial f}{\partial x_i} = \Pi_{\substack{j=1 \\ j \neq i}}^{r+t} x_j - 1$ $(1 \leq i \leq r+t)$, it clearly suffices to show $f(2, \ldots, 2) = 2^{r+t} - 2rt - (r+t) = g(r,t) \geq 0$. Straightforward calculations show $g(1,1) = 0$, $g(2,2) = 4$ and $g(1,t) = 2^{t+1} - 3t - 1 = 3(2^{t-1} - t) + 2^{t-1} - 1 > 0$ if $t \geq 2$. For $t \geq r \geq 2$, we get

$$\frac{\partial g}{\partial r} = 2^{t+1}(2^{r-1} ln 2) - 2t - 1 > 2^{t+1} - 2t - 1 > 0,$$

and similarly $\frac{\partial q}{\partial t} > 0$. This completes the proof. □

Lemma 7. *Let G be a k-connected graph ($k \geq 3$), and let $S = S' \cup S''$, where $S' = \{u_1, \ldots, u_r\} \subset V(G)$, $S'' = \{v_1, \ldots, v_t\} \subset V(G)$ such that $S' \cap S'' = \emptyset$. Moreover, let*

$$s'_i = |(u_i, \overline{S'})| \text{and } s_i = |(u_i, \overline{S})| \quad (i = 1, \ldots, r), \text{and}$$
$$s''_j = |(v_j, \overline{S''})| \quad (j = 1, \ldots, t) \text{ and } s_p = |(v_{p-r}, \overline{S})| \quad (p = r+1, \ldots, r+t).$$

If $1 \leq r \leq t \leq \lfloor \frac{1}{2}(k-1) \rfloor$, then

$$\Pi_{i=1}^{r} s_i \Pi_{p=r+1}^{r+t} s_p \geq \sum_{i=1}^{r} s'_i + \sum_{j=1}^{t} s''_j - (r+t) \geq \rho(r) + \rho(t).$$

Proof: It is easy to see that

$$\sum_{i=1}^{r} s'_i + \sum_{j=1}^{t} s''_j \leq \sum_{i=1}^{r} s_i + \sum_{p=r+1}^{r+t} s_p + 2rt, \text{ hence}$$
$$\sum_{i=1}^{r} s'_i + \sum_{j=1}^{t} s''_j - (r+t) \leq \sum_{i=1}^{r} s_i + \sum_{p=r+1}^{r+t} s_p + 2rt - (r+t). \qquad (*)$$

Since $s_i \geq \delta - (r+t) + 1 \geq k - 2t + 1 \geq k - 2\lfloor \frac{1}{2}(k-1) \rfloor + 1 \geq 2$ and similarly $s_p \geq 2$, by Lemma 6,

$$\Pi_{i=1}^{r} s_i \Pi_{p=r+1}^{r+t} s_p \geq \sum_{i=1}^{r} s_i + \sum_{p=r+1}^{r+t} s_p + 2rt - (r+t). \qquad (**)$$

From (*), (**) and the definition of ρ we get the result. □

If $v, u_1, u_2, \ldots, u_n$ are distinct vertices in a graph G, then a *fan* between v and $\{u_1, u_2, \ldots, u_n\}$, denoted by $v - \{u_1, u_2, \ldots, u_n\}$, is a set of n paths having only v in common and connecting v to each vertex of $\{u_1, u_2, \ldots, u_n\}$.

Lemma 8. *Let G be a k-connected graph ($k \geq 3$), and let T be a spanning tree of G, and $\{v_1, \ldots, v_t\} \subset V(G)$ with $t \leq \min\{\lambda(G), \lfloor \frac{1}{2}(k-1) \rfloor\}$. Then there is a fan $T - \{T_1, T_2, \ldots, T_{\rho(t)}\}$ in $T_l(G)$ such that each T_i contains $v_1, \ldots, v_t$ as leaves ($i = 1, 2, \ldots, \rho(t)$).*

Proof: If $v_1, \ldots, v_t$ are leaves of T, then the result follows from Lemma 5. Since $t \leq \lambda(G)$, T has at least t leaves. Without loss of generality, assume $v_1, \ldots, v_r$ are not leaves of T with $1 \leq r \leq t$, and $u_1, \ldots, u_r, v_{r+1}, \ldots, v_t$ are t leaves of T. Since $r \leq t$ implies $r + t \leq 2t \leq 2\lfloor \frac{1}{2}(k-1) \rfloor \leq k-1$, by Lemma 3, $T_l(G)$ has at least $\Pi_{i=1}^{r+t} s_i$ spanning trees containing $u_1, \ldots, u_r$,

$v_1, \ldots, v_t$ as leaves, where $s_i = |(u_i, \{\overline{u_1, \ldots, u_r, v_1, \ldots, v_t}\})|$ $(i = 1, \ldots, r)$ and $s_p = |(v_{p-r}, \{\overline{u_1, \ldots, u_r, v_1, \ldots, v_t}\})|$ $(p = r+1, \ldots, r+t)$. By Lemma 7, $\Pi_{i=1}^{r+t} s_i \geq \rho(r) + \rho(t) \geq \rho(t)$, so $T_l(G)$ has at least $\rho(t)$ spanning trees $T_1, \ldots, T_{\rho(t)}$ containing $u_1, \ldots, u_r, v_1, \ldots, v_t$ as leaves. T and T_i have t common leaves $u_1, \ldots, u_r, v_{r+1}, \ldots, v_t$ and $T \neq T_i$ $(i = 1, \ldots, \rho(t))$, since at least one of $v_1, \ldots, v_r$ is a leaf of T_i but not of T. By Lemma 5, there is a fan $T - \{T_1, \ldots, T_{\rho(t)}\}$. □

Proof of Theorem 2: Consider arbitrary spanning trees X and Y of G. We want to show there are at least $\rho(t)$ internally-disjoint paths in $T_l(G)$ connecting X and Y.

If $|\Lambda(X) \cap \Lambda(Y)| \geq t$, this follows from Lemma 5. Assume $|\Lambda(X) \cap \Lambda(Y)| < t$. Since $t \leq \lambda(G)$, both X and Y have at least t leaves. Let $v_1, \ldots, v_t$ be leaves of Y and suppose $v_1, \ldots, v_q$ are leaves of X while $v_{q+1}, \ldots, v_t$ are not leaves of X for some $q < t$. By Lemma 8 there is a fan $X - \{Y_1, \ldots, Y_{\rho(t)}\}$ such that Y_i contains $v_1, \ldots, v_t$ as leaves $(i = 1, \ldots, \rho(t))$. By Lemma 5, for every Y_i there are $\rho(t)$ internally-disjoint paths from Y_i to Y. It is not difficult to see that this implies there are $\rho(t)$ internally-disjoint paths from X to Y. □

The following result is an immediate consequence of Theorem 2.

Corollary 7. *Let G be a k-connected graph $(k \geq 3)$ and suppose, for some $n_1, n_2 \in \mathbb{N}$ with $n_1 < n_2$, there are spanning trees T_1 and T_2 of G with n_1 and n_2 leaves, respectively. Then, for any $n \in \mathbb{N}$ with $n_1 < n < n_2$, there are at least $\rho(t)$ spanning trees of G with n leaves.*

If T_1 and T_2 in Corollary 7 have common leaves $v_1, \ldots, v_t$ for some $t \leq k-2$, then all $\rho(t)$ spanning trees can be chosen to have $v_1, \ldots, v_t$ as leaves. This follows from Lemma 5.

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