Complete m-partite decompositions of complete multigraphs

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Graham and Pollak [1] proved that $n-1$ is the minimum number of complete bipartite subgraphs into which the edges of K_n can be decomposed. Subsequently, simple proofs have been given by Tverberg [9], Lovász (see [7]) and Peck [7].

Let $[i,j]$ denote the integer interval including i and j. Let $K(n|t)$ denote the complete multigraph with vertex set $[1,n]$, containing exactly t edges between every pair of distinct vertices (but containing no loops). For m disjoint nonempty subsets $A_1, A_2, \ldots, A_m$ of $[1,n]$, the graph $K(A_1, A_2, \ldots, A_m)$ is called a *complete m-partite graph*, or a *CmPG* for short, where the A_i's are called its *parts*. Let $f(n,t,m)$ be the minimum number of CmPG's into which the edges of $K(n|t)$ can be decomposed if a decomposition exists, otherwise letting $f(n,t,m) = \infty$. If $K(n|t)$ has a decomposition of CmPG's, we call it a *complete m-partite decomposition* of $K(n|t)$, or a *CmPD*.

Pritikin [8] proved that $f(n,t,2) \geq \max\{n-1, t\}$, and that if $n = 2, 3, 5$, $f(n,2,2) = n$; otherwise $f(n,2,2) = n-1$. In [3] and [4], we proved that $f(n,1,m) \geq \lceil (n-1)/(m-1) \rceil$, and that when n is large enough, $f(n,1,m) = \lceil (n-1)/(m-1) \rceil$. In [5], for $f(n,t,2)$ we gave some results. This note is an addendum to [5]. Here, using the methods of [5], for general t and $m \geq 3$ we obtain several results of $f(n,t,m)$.

First, combining the proof of Theorem 1 of [3] with that of Theorem 1 of [8], we obtain the following

Theorem 1. $f(n,t,m) \geq \max\{\lceil (n-1)/(m-1) \rceil, t\}$. □

Let D be an affine 2-(v,k,λ) design. Then D admits an inner and outer σ-resolution with c block classes (see p. 154 of [6]), with $\sigma = 1$ and inner constant $\rho = 0$. Let μ be its outer constant, and let m be the block number in each block class. Thus, $v = \mu m^2$, $k = \mu m$, $\lambda = (\mu m - 1)/(m-1)$ and $c = (\mu m^2 - 1)/(m-1)$ (see Theorem 5.8 of p. 164 of [6]). Therefore, D is a 2-$(\mu m^2, \mu m, (\mu m - 1)/(m-1))$ design.

Using the method of Lemma 3.14 of [5], we may easily prove the following.

Lemma 2. *If an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design exists, then* $f(\mu m^2, \mu m, m) = (\mu m^2-1)/(m-1)$ $(= \lceil (\mu m^2-1)/(m-1) \rceil)$. □

We note that in the CmPD of $K(\mu m^2|\mu m)$ in the foregoing method, each part has μm vertices, implying the following.

Lemma 3. *If an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design exists, then* $f(\mu m^2 - i, \mu m, m) = (\mu m^2-1)/(m-1)$ $(= \lceil (\mu m^2 - i - 1)/(m-1) \rceil)$ *for* $i = 0, 1, \ldots, m-2$; *and when* $i = m-1, m, \ldots, \mu m - 1$, $f(\mu m^2 - i, \mu m, m) \leq (\mu m^2-1)/(m-1)$. □

Applying the method of Lemma 2.2 of [5], we have

Lemma 4. *If* $K(n_i|t)$ *can be decomposed into* p_i *CmPG's for* $i = 1, 2$, *then* $K(n_1 + n_2 - 1|t)$ *can be decomposed into* $p_1 + p_2$ *such graphs; in particular,* $K(2n_i - 1|t)$ *has a CmPD of* $2p_i$ *CmPG's..* □

Applying Lemma 4 to Lemma 3, we may obtain the following two results:

Theorem 5. *Suppose that there is an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design. Then for* $k = 1, 2, \ldots,$ *and* $i = 0, 1, \ldots, m-2$, $f(k(\mu m^2 - 1) + 1 - i, \mu m, m) = k(\mu m^2-1)/(m-1)$ $(= \lceil (k(\mu m^2-1) + 1 - i - 1)/(m-1) \rceil)$. □

Theorem 6. *Suppose that there is an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design, and let* $s = \lceil (\mu m^2-1)/(\mu m-1) \rceil - 1$. *If one of the following conditions holds:*

(i) $k = 1, 2, \ldots, s$, *and* $i = m-1, m, \ldots, k(\mu m - 1)$,

(ii) $k = s+1, s+2, \ldots,$ *and* $i = m-1, m, \ldots, \mu m^2 - 2$;

then $f(k(\mu m^2 - 1) + 1 - i, \mu m, m) \leq k(\mu m^2-1)/(m-1)$. □

Note that when m is a prime power, an affine plane 2-$(m^2, m, 1)$ (outer constant $\mu = 1$) and designs $\mathcal{A}_n(m)$ $(= \mathcal{A}_{n,n-1}(m))$ are all affine designs. Since $\mathcal{A}_n(m)$ is a 2-$(m^n, m^{n-1}, (m^{n-1}-1)/(m-1))$ design $(\mu = m^{n-2})$, by Theorem 5 and Theorem 6, we have the following two results:

Theorem 7. *If m is a prime power, then* $f(k(m^n-1) + 1 - i, m^{n-1}, m) = k(m^n-1)/(m-1)$ $(= \lceil (k(m^n-1)+1-i-1)/(m-1) \rceil)$, *where* $n = 2, 3, \ldots,$ $k = 1, 2, \ldots,$ *and* $i = 0, 1, \ldots, m-2$. □

Theorem 8. *Let m be a prime power,* $n = 2, 3, \ldots,$ *and* $s = \lceil (m^n - 1)/(m^{n-1}-1) \rceil - 1$. *If one of the following conditions holds:*

(i) $k = 1, 2, \ldots, s$, *and* $i = m-1, m, \ldots, k(m^{n-1} - 1)$,

(ii) $k = s+1, s+2, \ldots,$ *and* $i = m-1, m, \ldots., m^n - 2$;

then $f(k(m^n - 1) + 1 - i, m^{n-1}, m) \leq k(m^n-1)/(m-1)$. □

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