On A Generalisation Of Signed Dominating Functions Of Graphs

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ABSTRACT. For a positive integer k, a k-subdominating function of $G = (V, E)$ is a function $f: V \rightarrow \{-1, 1\}$ such that the sum of the function values, taken over closed neighborhoods of vertices, is at least one for at least k vertices of G. The sum of the function values taken over all vertices is called the aggregate of f and the minimum aggregate amongst all k-subdominating functions of G is the k-subdomination number $\gamma_{ks}(G)$. In the special cases where $k = |V|$ and $k = \lceil |V|/2 \rceil$, γ_{ks} is respectively the signed domination number [4] and the majority domination number [2]. In this paper we characterize minimal k-subdominating functions. By determining γ_{ks} for paths, we give a sharp lower bound for γ_{ks} for trees. We also determine an upper bound for γ_{ks} for trees which is sharp for $k \leq |V|/2$.

1 Introduction

For a graph $G = (V, E)$ and vertex $v \in V$, let $N(v) = \{u \in V: uv \in E\}$ and $N[v] = \{v\} \cup N(v)$ denote the open and closed neighborhoods, respectively, of v. For $k \in \mathbf{Z}^+$, a *k-subdominating function* (kSF) of G is a function $f: V \to \{-1, 1\}$ such that $f[v] = \sum_{u \in N[v]} f(u) \geq 1$ for at least k vertices v of G. The *aggregate* ag(f) of such a function is defined by $ag(f) = \sum_{v \in V} f(v)$ and the *k-subdomination number* $\gamma_{ks}(G)$ by $\gamma_{ks}(G) = \min\{ag(f): f \text{ is a kSF of } G\}$. In the special cases where $k = |V|$ and $k = \lceil |V|/2 \rceil$, γ_{ks} is respectively the signed domination number γ_s [4] and the majority domination number γ_{maj} [2].

In this paper we characterize minimal k-subdominating functions and give a sharp lower bound for the k-subdomination number of trees. Special cases of these results solve the open problems 2 and 3 posed in [2]. A sharp upper bound for the majority domination number was given by Alon [1]. This bound also gives an upper bound for γ_{ks} if $k \leq \lceil |V|/2 \rceil$. For trees we improve this bound and extend it to an upper bound for γ_{ks} for all $k \in \{1, \ldots, n\}$, where $n = |V|$.

2 Minimal k-subdominating functions

Let f be a kSF of G. We say $v \in V$ is *covered by* f (or simply *covered* if the function is clear from the context) if $f[v] \geq 1$ and denote the set of vertices covered by f, by C_f. Let $P_f = \{v \in V: f(v) = 1\}$ and $B_f = \{v \in V: f[v] \in \{1, 2\}$. Note that each $v \in B_f$ is covered. However, v is no longer covered if a function value of 1 in $N[v]$ is changed to -1. The kSF f is *minimal* if no $g < f$ is a kSF. For $A, B \subseteq V$, we say A *dominates* B, denoted by $A \succ B$, if for each $b \in B$, $N[b] \cap A \neq \emptyset$. If $A \succ V$, then A is a *dominating set* of G.

Theorem 1. *The kSF f is minimal if and only if for each k-subset K of C_f, $K \cap B_f \succ P_f$.*

Proof: Suppose f is a kSF satisfying the above condition but, contrary to the result, $g < f$ is a kSF with k-subset $K' \subseteq C_g \subseteq C_f$. Then there exists $v \in V$ with $g(v) < f(v)$, i.e., $g(v) = -1$ and $f(v) = 1$. By assumption $B_f \cap K' \succ \{v\}$, i.e., there exists $w \in B_f \cap K' \cap N[v]$. Now, $f[w] \in \{1, 2\}$ and $v \in N[w]$, hence $g[w] < 1$, a contradiction which shows that f is minimal.

Conversely, suppose that f is a minimal kSF and there exists a k-subset $K \subseteq C_f$ with $B_f \cap K \not\succ \{v\}$, where $v \in P_f$. Let $h: V \to \{-1, 1\}$ be defined by $h(v) = -1$ and $h(w) = f(w)$ for $w \in V - \{v\}$. If $w \in K \cap B_f$, then $w \notin N[v]$ so that $v \notin N[w]$ and $h[w] = f[w] \geq 1$. For $w \in K - B_f$, $f[w] \geq 3$. It is possible that $v \in N[w]$; however, $h[w] \geq f[w] - 2 \geq 1$. Thus h is a kSF, contrary to the minimality of f. □

We digress to remark that Theorem 1 may be embedded in a far more general setting by a simple transformation. Let $h: V \to \{0, 1\}$ satisfy

$$h[v] \geq \lceil (1 + |N[v]|)/2 \rceil \qquad \text{for each } v \in V.$$

Then f defined by

$$f(u) = \begin{cases} -1 & \text{if } h(u) = 0 \\ 1 & \text{if } h(u) = 1 \end{cases}$$

is a signed domination function (i.e., a kSF with $k = |V|$). The function h as defined here is a special case of an η-function (see [3]). Many results concerning signed domination and kSFs (including Theorem 1) may be generalized into the η-function framework. The details will be presented elsewhere.

3 Lower bound for the k-subdomination number of trees

Let γ be the minimum value of γ_{ks} taken over all n-vertex trees ($n \geq k$) and $\mathcal{G}$ be the set of such trees T with $\gamma_{ks}(T) = \gamma$. Further, let $\alpha(T)$ be the degree sum of all vertices of T with degree at least three and define $\mathcal{T} = \{T \in \mathcal{G} : \alpha(T) \text{ is minimum }\}$. An endvertex of T is also called a *leaf* of T. Let P_n denote the path on n vertices.

Proposition 2. *For any n, $\mathcal{T} = \{P_n\}$.*

Proof: Suppose, to the contrary, that $T \in \mathcal{T}$ has vertex v with $\deg(v) \geq 3$ and set of neighbors $N(T, v)$. Let f be a kSF of T with $ag(f) = \gamma$. Consider T to be rooted at v and for $u \in N(T, v)$, let $T(u)$ denote the subtree of T induced by u and its descendants.

For any $\{x, y\} \subset N(T, v)$ we show the existence of a tree $T' = T'\{x, y\}$ such that (i) $V(T') = V(T)$, (ii) $N(T', v) = N(T, v)$, (iii) f is a kSF of T', (iv) $T'(z) = T(z)$ for each $z \in N(T, v) - \{x, y\}$ and (v) $f(\ell) = 1$ for some leaf ℓ of $T'(x)$.

The tree T itself satisfies conditions (i) - (iv). If it does not satisfy (v) for some $\{x, y\} \subset N(T, v)$, then f takes value -1 for every leaf of $T(x)$ and $T(y)$ (since the roles of x and y can be interchanged). (Note that if u is a leaf with $f(u) = -1$, then $u \notin C_f$.) In this case, set $T_0 = T$ and form a sequence of trees $T_0, T_1, \ldots, T_j = T'$ recursively as follows. Choose leaves x_0 of $T_0(y)$ and x_1 of $T_0(x)$. Form T_1 from T_0 by deleting the edge $x_1 w_1$ and adding a new edge $x_0 x_1$. Observe that $\alpha(T_1) \leq \alpha(T_0)$ and (since the same vertices are covered) f is a kSF of T_1. The minimality of $\alpha(T_0)$ implies that $\alpha(T_1) = \alpha(T_0)$. Hence $x_1 \neq x$, $T_1 \in \mathcal{T}$ and satisfies (i) - (iv).

This process is now continued if necessary until a tree $T_j = T'$ is formed with $f = 1$ on some leaf of $T_j(x)$. Specifically, at the ith stage, select a leaf x_i of $T_{i-1}(x)$ and form T_i from T_{i-1} by removing the edge $x_i w_i$ and adding a new edge $x_{i-1} x_i$. At each stage $\alpha(T_i) \leq \alpha(T_{i-1})$, f is a kSF for T_i and the minimality of $\alpha(T_0)$ implies $\alpha(T_i) = \alpha(T_0)$. Hence $x_i \neq x$, each $T_i \in \mathcal{T}$ and satisfies (i) - (iv). Finiteness ensures that the process terminates and hence (v) is also satisfied, say, for $T_j = T'$.

Suppose there exists $u \in N(T,v)$ with $f(u) = -1$. Choose $\{x,y\} \subseteq N(T,v) - \{u\}$ and form $T' = T'\{x,y\}$ as above. Now construct T^* from T' by deleting uv and adding the new edge $u\ell$. It is easily verified that $T^* \in \mathcal{G}$ with $\alpha(T^*) < \alpha(T') = \alpha(T)$, contradicting the minimality of $\alpha(T)$. If $f(u) = 1$ for all $u \in N(T,v)$, then $f[v] \geq 2$. Select any $u \in N(T,v)$ and form T^* as above. Again $T^* \in \mathcal{G}$and the same contradiction is obtained. □

Let $\mathcal{F} = \{f : f$ is a kSF of P_n with $ag(f) = \gamma\}$.

Proposition 3. *Let the vertex sequence of P_n be $1, 2, \ldots, n$. There exists $f^* \in \mathcal{F}$ such that $\{1, 2, \ldots, k\} \subseteq C_{f^*}$.*

Proof: Let $s = \min\{i : i \in P_f\}$, $t = \max\{i : i \in P_f\}$ and choose $f \in \mathcal{F}$ such that $w(f) = t - s$ is minimum. Suppose that the vertices in C_f are not consecutive on P_n. Then there exists i satisfying $i - 1 \in C_f$, $i \notin C_f$ and $\ell \in C_f$ for some $\ell > i$. Now, for any $u \in V(P_n)$, $u \in C_f$ it at least two vertices in $\{u-1, u, u+1\}$ are in P_f. This implies that $f(i+1) = -1$. Let $j = \min\{\ell : \ell > i+1$ and $f(\ell) = 1\}$. Define $f' : : V \to \{-1, 1\}$ by $(f'(1), f'(2), \ldots, f'(n)) = (f(1), \ldots, f(i), f(j), f(j+1), \ldots, f(t), -1, \ldots, -1)$. Then f' is a kSF of P_n with $ag(f') = ag(f) = \gamma$ and $w(f') < w(f)$, contrary to the minimality of $w(f)$. Hence the vertices in C_f are consecutive on P_n. If $1 \in C_f$, then $f^* = f$ satisfies the requirements. Now suppose that $c > 1$ is the first vertex of P_n covered by f.
If $f(c-1) = -1$, define $f^* : V \to \{-1, 1\}$ by

$$(f^*(1), f^*(2), \ldots, f^*(n)) = (f(c), f(c+1), \ldots, f(t), -1, \ldots, -1).$$

If $f(c-1) = 1$ (this implies $f(c) = -1$ and $f(c+1) = 1$), then define $f^* : V \to \{-1, 1\}$ by

$$(f^*(1), f^*(2), \ldots, f^*(n)) = (1, f(c+1), f(c+2), \ldots, f(t), -1, \ldots, -1).$$

In each case f^* is a kSF with $ag(f^*) = ag(f) = \gamma$ and $\{1, 2, \ldots, n\} \subseteq C_{f^*}$ as required. □

Theorem 4. *For $n \geq 2$ and $1 \leq k \leq n$,*

$$\gamma_{ks}(P_n) = 2\lfloor (2k+4)/3 \rfloor - n.$$

Proof: For $k = n$ the result is proved in [4], hence we may assume $k < n$. By Proposition 3, we require a kSF f with minimum size P_f covering $\{1, \ldots, k\}$. Observe that f has value 1 on at least two of any consecutive triple of vertices in $\{1, \ldots, k+1\}$ and that $f(1) = f(2) = 1$. Let σ be the infinite sequence $1, 1, -1, 1, 1, -1, \ldots$. Let $(f(1), f(2), \ldots, f(n))$ be the first $k+1$ terms of σ followed by $(n-k-1) - 1$'s . Then f is a kSF of P_n with minimum aggregate and $|P_f| = \lfloor (2k+4)/3 \rfloor$. The result now follows. □

Note that Theorem 4 generalizes the corresponding results of [2] and [4].

Corollary 5. *For any n-vertex tree T and $k \le n$, where $n \ge 2$,*

$$\gamma_{ks}(T) \ge 2\lfloor (2k+4)/3 \rfloor - n$$

with equality for $T = P_n$. □

4 Upper bounds for the k-subdomination number of trees

Alon [1] showed that if G is a connected graph, then $\gamma_{maj}(G) \le 2$. We record this short and elegant proof here.

Theorem 6 [1]. *For any connected n-vertex graph G,*

$$\gamma_{maj}(G) \le \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Proof: Suppose firstly that n is odd; say $n = 2k+1$. Amongst all partitions $\{A', B'\}$ of $V(G)$ with $|A'| = k+1$ and $|B'| = k$, let $\{A, B\}$ be one such that the number of edges joining vertices in A to vertices in B is minimum. Then each vertex $v \in A$ is adjacent to at least as many vertices in A as to vertices in B, for otherwise $\{B \cup \{v\}, A - \{v\}\}$ contradicts the choice of $\{A, B\}$. Define $f: V(G) \to \{-1, 1\}$ by

$$f(u) = \begin{cases} 1 & \text{if } u \in A \\ -1 & \text{if } u \in B. \end{cases}$$

Clearly, f is a majority dominating function and it follows that $\gamma_{maj}(G) \le k+1-k = 1$.

Now suppose n is even, let $v \in V(G)$ be arbitrary and define the partition $\{A, B\}$ of $G - v$ as above. Clearly, the function $g: V(G) \to \{-1, 1\}$ defined by

$$g(u) = \begin{cases} 1 & \text{if } u \in A \cup \{v\} \\ -1 & \text{if } u \in B \end{cases}$$

is a majority dominating function and $\gamma_{maj}(G) \le k+2-k = 2$. □

Obviously, if f is a majority dominating function, then f is a k-subdominating function for each $k \le \lceil |V|/2 \rceil$. Hence we have

Corollary 7. *For any connected n-vertex graph G and integer $k \le \lceil \frac{1}{2}n \rceil$,*

$$\gamma_{ks}(G) \le \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

□

That this bound is sharp can be seen by noting that $\gamma_{ks}(K_{2p+1}) = 1$ and $\gamma_{ks}(K_{2p}) = 2$ for each $k \leq p+1$. For trees we now improve this bound and extend it to an upper bound for γ_{ks} for all $k \in \{1, \ldots, n\}$. We need the following definitions.

A vertex x of a tree T is said to be *remote* if x is adjacent to a leaf of T. A remote vertex x is *very remote* if x is adjacent to at most one vertex of T that is not a leaf. Note that each tree T has at least one very remote vertex: Let $r(T)$ denote the radius and $C(T)$ the center of T. Let $z \in C(T)$ and consider any leaf y at distance $r(T)$ from z. Say $N(y) = \{x\}$. Then x is a very remote vertex of T. A tree T' is a *full subtree* of T if $T' = T$ or T' is a component of $T - e$ for some edge e of T. In the latter case, if $e = uv$ where $u \in V(T')$, we say that T' is *attached at* u. If $T' \approx K_{1m}$ is a full subtree of T, then T' is called a *full substar* of T. Note that if T' is a full subtree attached at a very remote vertex x of T, then T' is a full substar of T with center x.

Let L denote the set of leaves of T. For each $v \in V$, define $L(v) = N(v) \cap L$ and $\ell(v) = |L(v)|$. If confusion is possible, we also write $L(T)$, $L_T(v)$ and $\ell_T(v)$ to emphasize that T is the tree under consideration, and if $T = S_i$ (say), we write $\ell_i(v)$ for $\ell_T(v)$.

Theorem 8. *For any n-vertex tree T and integer $k \in \{1, \ldots, n\}$, $\gamma_{ks} \leq 2(k+1) - n$.*

Proof: The result clearly holds if $T = K_2$ or if $k = 1$; thus we assume that $k \geq 2$ and $n \geq 3$. Set $S_0 = T$ and $s_0 = k$. We construct a sequence $T_1, \ldots, T_n$ of disjoint subtrees of T as follows: If S_0 contains a full substar G_1 with center v_1 such that $s_0 \leq \ell_0(v_1)$, let T_1 be the subtree of S_0 induced by v_1 and any s_0 leaves of G_1, and set $s_1 = -1$. Otherwise, let T_1 be a (nontrivial) full subtree of S_0 of order $k_1 \leq s_0$ attached at v_1 (if $T_1 \neq S_0$) and define $s_1 = s_0 - k_1$. Continuing in this way, if $s_i > 0$, define $S_i = S_{i-1} - T_i$. If S_i contains a full substar G_{i+1} with center v_{i+1}, where $s_i \leq \ell_i(v_{i+1})$, let T_{i+1} be the subtree of S_i induced by v_{i+1} and any s_i leaves of G_{i+1}, and set $s_{i+1} = -1$. Otherwise, let T_{i+1} be a full subtree of S_i of order $k_{i+1} \leq s_i$ attached at v_{i+1} (if $T_{i+1} \neq S_i$) and set $s_{i+1} = s_i - k_{i+1}$. We thus obtain a finite sequence of disjoint subtrees $T_1, \ldots, T_r$ of T and a sequence of integers $s_0 > s_1 > \cdots > s_r$, where $s_r \in \{0, -1\}$.

Let F be the (possibly disconnected) subgraph of T induced by $\cup_{i=1}^{r} V(T_i)$. Note that $|V(F)| = k + 1$ if $s_r = -1$ and $|V(F)| = k$ otherwise. Define $f\colon V(T) \to \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in V(F) \\ -1 & \text{otherwise.} \end{cases}$$

Note that for each $i = 1, \ldots, r$, v_i is the only vertex of T_i that is possibly adjacent to a vertex of $S_r = T - F$, and since T_i is full (except possibly T_r

if $s_r = -1$), v_i (except possibly v_r) is adjacent to at most one vertex of S_r. Moreover, if v_i ($i = 1, \ldots, r$) is adjacent to a vertex of S_r, then v_i is not a leaf of T (since T_i is nontrivial). Hence f covers each vertex of F except possibly v_r if $s_r = -1$. In either case, f covers at least k vertices of T and $ag(f) \leq 2(k+1) - n$. □

That this bound is exact for n-vertex trees when $k \leq \frac{1}{2}n$ follows easily since $\gamma_{ks}(K_{1,n-1}) = 2(k+1) - n$ if $k \leq \frac{1}{2}n$. However, we have not been able to find a tree (or any other connected graph) of order n for which $\gamma_{ks} = 2(k+1) - n$ if $k > \frac{1}{2}n$. Hence we formulate the following conjectures.

Conjecture 1: For any n-vertex tree and any k with $\frac{1}{2}n < k \leq n$, $\gamma_{ks} \leq 2k - n$.

Conjecture 2: For any connected graph of order n and any k with $\frac{1}{2}n < k \leq n$, $\gamma_{ks} \leq 2k - n$.

If either of these conjectures is false, there still remains the problem of determining the smallest integer $p = p(n)$ such that $\gamma_{ks} \leq 2k - n$ for all graphs (trees) of order n and all $k \geq p$.

In the rest of this section we determine conditions on k such that $\gamma_{ks} \leq 2k - n$ for certain classes of n-vertex trees. For a rooted tree T and any vertex u of T, let $T(u)$ denote the subtree of T induced by u and its descendants. (Then $T(u)$ is a full subtree of T attached at u. However, not all full subtrees of T are of the form $T(u)$ for some vertex u, with respect to a fixed root of T.)

Theorem 9. *Let T be an n-vertex tree rooted at v, where $deg(v) = s$ and $\ell(v) = t$; say $N(v) = \{w_1, \ldots, w_t, u_1, \ldots, u_{s-t}\}$ where $L(v) = \{w_1, \ldots, w_t\}$ and $|V(T(u_1))| \leq \cdots \leq |V(T(u_{s-t}))|$. If $r = \lceil \frac{1}{2}(s+2) \rceil \leq s - t$ and $n \geq k \geq |V(T(u_1))| + \cdots + |V(T(u_r))|$, then $\gamma_{ks} \leq 2k - n$.*

Proof: Let $i \geq r$ be the largest integer such that $k \geq |V(T(u_1))| + \cdots + |V(T(u_i))| = m$ and let F' be the subforest of T of order m with $F' = T(u_1) \cup \cdots \cup T(u_i)$. Let $k' = k - m - 1$. If $i = s$–t and $k' > 0$, let F be the substar of T induced by $\{v, w_1, \ldots, w_{k'}\}$. If $i < s - t$ and $k' > 0$, let F be the subforest of $T(u_{i+1})$ of order $k' + 1$ or k' constructed as described in the proof of Theorem 8. Define F^* by

$$F^* = \begin{cases} F' & \text{if } k' \leq 0 \\ F' \cup F & \text{if } k' > 0 \end{cases}$$

and $f\colon V(T) \to \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in V(F^*) \\ -1 & \text{otherwise.} \end{cases}$$

It follows from the proof of Theorem 8 and the fact that per definition of u_j, $T(u_j) \not\cong K_1$ ($j = 1, \ldots, s-t$), that $V(F^*) - \{z\} \subseteq C_f$, where z is the vertex of F corresponding to the vertex v_r in the proof of Theorem 8, in the case where $i < s-t$ and $|V(F)| = k'+1$. Also, $\{u_1, \ldots, u_r\} \subseteq P_f$ and thus $f[v] \geq r-(s-r)-1 = 2\lceil s/2 \rceil + 2 - s - 1 \geq 1$. Therefore $v \in C_f$ so that $|C_f| \geq 1+m+k' = k$. Hence f is a kSF with $ag(f) \leq 2(m+k'+1) - n = 2k-n$ and the result follows. □

For any vertex v of T, define $\eta(v)$ by

$$\eta(v) = \begin{cases} \frac{d+2}{2d_1-\ell(v)} & \text{if } deg(v) = 2d+1 \text{ for some integer } d \geq 1 \\ \frac{d+1}{2d-\ell(v)} & \text{if } deg(v) = 2d \text{ for some integer } d \geq 1. \end{cases}$$

Corollary 10. *If T is an n-vertex tree such that $\eta(v) \leq 1$ and $\eta(v)(n-\ell(v)-1) \leq k \leq n$ for some vertex v of T, then $\gamma_{ks} \leq 2k-n$.*

Proof: We use the notation of Theorem 9. If $\eta(v) \leq 1$, then $r \leq s-t$ and if $k \geq \eta(v)\ (n-t-1)$, then $k \geq |V(T(u_1))| + \cdots + |V(T(u_r))|$ so that the result follows from Theorem 9. □

Obviously, the bound on k given by Theorem 9 can be smaller than $\frac{1}{2}n$. However, if the subtrees $T(u_i)$, $i = 1, \ldots, s-t$, are of comparable size, then this bound may exceed $\lceil \frac{1}{2}n \rceil$, even if $\ell(v) = 0$. Consider, for example, the $(6d+1)$-vertex tree T which has a central vertex v of degree $2d$ such that each neighbor of v is adjacent to two leaves and to no other vertices. By Theorem 9 (or, equivalently in this case, Corollary 10), $\gamma_{ks}(T) \leq 2k-n$ if $k \geq 3d+3$. However, $\lceil \frac{1}{2}n \rceil = 3d+1 < 3d+3$. Note that $3d+1$ is also not the smallest value of p such that $\gamma_{ks}(T) \leq 2k-n$ for all $k \geq p$, for it can easily be shown that $\gamma_{ks}(T) \leq 2k-n$ for all $k \geq 2d+1$.

In general, if more is known about the structure of T, the techniques used in the proof of Theorem 9 can be refined to find smaller lower bounds p for k as described above. We illustrate this by considering full m-ary trees. A *full m-ary tree of height h* is a rooted tree such that each vertex which is not a leaf has exactly m children, and all leaves are at distance h from the root. For each $i = 0, \ldots, h$, the *ith level* of a rooted tree consists of all vertices at distance i from the root.

We need the following result from number theory:

Lemma 11. *For any integer $\ell \geq 1$, each integer $k \geq 2\ell+4$ can be written as*

$$k = \alpha_1(\ell+2) + \alpha_2(\ell+3) + \cdots + \alpha_{\ell+1}(2\ell+2), \tag{A}$$

where each α_i ($i = 1, \ldots, \ell+1$) is a non-negative integer.

Proof (by induction on ℓ): If $\ell = 1$, we must show that each integer $k \geq 6$ can be written as $k = 3\alpha_1 + 4\alpha_2$ for some non-negative integers α_1 and α_2. This is an easy exercise, using induction on k.

Suppose the lemma holds for some fixed integer $\ell \geq 1$. Using induction on k we now prove that each integer $k \geq 2\ell + 6$ can be written as

$$k = \alpha_1(\ell+3) + \alpha_2(\ell+4) + \cdots + \alpha_{\ell+2}(2\ell+4), \tag{B}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{\ell+2}$ are non-negative integers.

(B) obviously holds for $k = 2\ell + 6$, so assume it holds for some fixed $k \geq 2\ell + 6$. If $\alpha_i \neq 0$ for each $i \in \{1, 2, \ldots, \ell + 1\}$, then

$$\begin{aligned} k+1 &= \alpha_1(\ell+3) + \alpha_2(\ell+4) + \cdots + \alpha_{\ell+2}(2\ell+4) + 1 \\ &= \alpha_1(\ell+3) + \alpha_2(\ell+4) + \cdots + (\alpha_i - 1)(\ell+i+2) \\ &\quad + (\alpha_{i+1}+1)(\ell+i+3) + \cdots + \alpha_{\ell+2}(2\ell+4). \end{aligned}$$

If $\alpha_i = 0$ for each $i \in \{1, 2, \ldots, \ell + 1\}$, then $\alpha_{\ell+2} \geq 2$ since $k \geq 2\ell + 6$. In this case,

$$\begin{aligned} k+1 &= (\alpha_{\ell+2} - 2)(2\ell+4) + 4\ell + 8 + 1 \\ &= (\alpha_{\ell+2} - 2)(2\ell+4) + 2(\ell+3) + (2\ell+3). \end{aligned}$$

Hence (B) holds for $k + 1$ and hence, by the induction principle, for all $k \geq 2\ell + 6$. The lemma now follows, also by the induction principle. □

Theorem 12. *For any full m-ary tree with n vertices,* $\gamma_{ks} \leq 2k - n$ *whenever* $2\lceil\frac{1}{2}(m+3)\rceil \leq k \leq n$.

Proof: Let $T = (V, E)$ be a full m-ary tree of, say, height h (≥ 1) and consider any $k \geq 2\lceil\frac{1}{2}(m+3)\rceil$. Note that T has m^i vertices at level i; in particular, T has m^h leaves and m^{h-1} remote vertices. Let V_i denote the set of vertices of T at level i, $i = 0, \ldots, h$. For any function $f\colon V \to \{-1, 1\}$ and any integer $i \in \{1, \ldots, h\}$, if $V_i \cup V_{i-1} \subseteq P_f$, then $V_{i-1} \subseteq C_f$. Hence if $k \geq m^h + m^{h-1}$, let $j \geq 1$ be the largest integer such that $k \geq m^h + \cdots + m^{h-j}$ and let W be any subset of $V_{h-(j+1)}$ of cardinality $k - (m^h + \cdots + m^{h-j})$. Define $f\colon V \to \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in V_h \cup \cdots \cup V_{h-j} \cup W \\ -1 & \text{otherwise.} \end{cases}$$

Since $j \geq 1$, $V_h \subseteq C_f$ and by the above, $V_i \subseteq C_f$ for each $i \in \{h_j, \ldots, h-1\}$. Moreover, for each $w \in W$, $|N[w] \cap P_f| = m+1$ and $|N[w] - P_f| \leq 1$; hence $f[w] \geq m \geq 1$. Thus $W \subseteq C_f$ so that $|C_f| \geq k$. It follows that f is a kSF with $ag(f) \leq 2k - n$.

Now suppose $2\lceil\frac{1}{2}(m+3)\rceil \leq k < m^h + m^{h-1}$. If $m = 1$, then T is a path and the result follows from Theorem 4. If $m \geq 2$, then $m = 2\ell$ or $m = 2\ell + 1$ for some integer $\ell \geq 1$ so that by Lemma 11, k can be written as described in (A).

Case 1. $m = 2\ell + 1$.

Then $2\ell + 2 = m + 1$. Of all possible partitions of k of the form (A), choose one in which $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell+1}$ is minimum. We show that $\alpha \leq m^{h-1}$: Suppose $\alpha > m^{h-1}$. We may assume that amongst all partitions of k of the above type we have chosen one for which $\alpha_{\ell+1}$ is maximum. Since $k < m^{h-1}(m+1)$ and $2\ell + 2 = m + 1$, it follows that $\alpha_{\ell+1} \leq m^{h-1} - 1$ and thus $\beta = \alpha - \alpha_{\ell+1} \geq 2$. If $\beta = 2$, then $\alpha_{\ell+1} = m^{h-1} - 1$ and $k \geq 2(\ell+2) + (m^{h-1}-1)(2\ell+2) > m + 1 + (m^{h-1} - 1)(m+1) = m^{h-1}(m+1)$, a contradiction. Hence, $\beta \geq 3$ and we may write $\Sigma_{i=1}^{\ell}\alpha_i(\ell+i+1) = \Sigma_{i=1}^{\ell}\beta_i(\ell+i+1) + r$, where $\Sigma_{i=1}^{\ell}\beta_i = \beta - 3$ and $3(\ell+2) \leq r \leq 3(2\ell+1)$. If $r \leq 4\ell + 4$, then $r = (2\ell+2) + (\ell+t)$ for some $t \in \{4, \ldots, \ell+2\}$, so that $k = \Sigma_{i=1}^{\ell+1}\alpha_i'(\ell+i+1)$ with $\Sigma_{i=1}^{\ell+1}\alpha_i' = \alpha - 1$, contradicting the minimality of α. If $r \geq 4\ell+6$, then $k = \Sigma_{i=1}^{\ell+1}\alpha_i''(\ell+i+1)$ with $\Sigma_{i=1}^{\ell+1}\alpha_i'' = \alpha$ and $\alpha_{\ell+1}'' > \alpha_{\ell+1}$, contradicting the maximality of $\alpha_{\ell+1}$. Thus $r = 4\ell + 5$. If $\beta = 3$, then $m^{h-1} - 2 \leq \alpha_{\ell+1} \leq m^{h-1} - 1$ and $k \geq 4\ell+5+(m^{h-1}-2)(2\ell+2) > 2(m+1)+(m^{h-1}-2)(m+1) = m^{h-1}(m+1)$, a contradiction. Thus $\beta \geq 4$ and $\Sigma_{i=1}^{\ell}\alpha_i(\ell+i+1) = \Sigma_{i=1}^{\ell}\beta_i'(\ell+i+1)+r+r'$, where $\Sigma_{i=1}^{\ell}\beta_i' = \beta-4$ and $\ell+2 \leq r' \leq 2\ell+1$. Then $5\ell+7 \leq r+r' \leq 6\ell+6$ and as before there exists a partition of k which contradicts the maximality of $\alpha_{\ell+1}$. Therefore $\alpha \leq m^{h-1}$.

For $i \in \{1, 2, \ldots, \ell+1\}$, choose pairwise disjoint sets W_i as follows. Let $W_i \subseteq V_h \cup V_{h-1}$ consist of α_i vertices of V_{h-1} together with $\ell + i$ leaves adjacent to each of these vertices and let $W = \cup_{i=1}^{\ell+1} W_i$. Note that $|W_i| = \alpha_i(\ell+i+1)$ so that $|W| = k$. Define $f\colon V \to \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in W \\ -1 & \text{otherwise.} \end{cases}$$

If $x \in W \cap V_h$, then $f[x] = 2$. If $x \in W \cap V_{h-1}$, then $|N[x] \cap P_f| \geq \ell + 2$ and $|N[x] - P_f| \leq m + 2 - \ell - 2 = \ell + 1$. Hence $f[x] \geq \ell + 2 - (\ell+1) = 1$. Thus $W \subseteq C_f$ so that f is a kSF with $ag(f) = 2k - n$.

Case 2. $m = 2\ell$.

Then $2\ell + 2 = m + 2$. Consider any partition of k of the form (A) in which $\alpha_{\ell+1}$ is minimum. Then $\alpha_{\ell+1} \leq 2$, for otherwise k can also be written as $k = k - 3(2\ell+2) + 2(\ell+2) + 2(2\ell+1)$, a contradiction. Moreover, if $\ell \geq 2$, then $\alpha_{\ell+1} \leq 1$, for otherwise $k = k - 2(2\ell+2) + 2(\ell+2) + 2\ell$. Finally, if $\ell \geq 3$ and $\alpha_{\ell+1} = 1$, then $\alpha_i \neq 0$ for some $i \in \{1, 2, \ldots, \ell\}$ since $k \geq 2\ell + 4$. If $i < \ell$, then we may write k as $k = k - (2\ell+2) - (\ell+i+1) + (2\ell+1) + (\ell+i+2)$, contradicting the choice of $\alpha_{\ell+1}$. If $i = \ell$, then $k = k-(2\ell+2)-(2\ell+1)+2(\ell+2)+(2\ell-1)$, again a contradiction. Hence $\alpha_{\ell+1} = 0$ if $\ell \geq 3$.

Amongst all partitions of k of the form (A) in which $\alpha_{\ell+1}$ is minimum, choose one such that $\alpha_1+\alpha_2+\cdots+\alpha_{\ell+1}$ is minimum. By further assuming that this partition has been chosen such that α_ℓ is maximum, it follows, as in Case 1, that $\alpha_1 + \alpha_2 + \cdots + \alpha_{\ell+1} \le m^{h-1}$. If $\alpha_{\ell+1} = 0$, define W and f as in Case 1. Then f is a kSF with $ag(f) = 2k - n$.

Suppose $\alpha_{\ell+1} \ne 0$. Note that $\alpha_1 + \cdots + \alpha_\ell \le m^{h-1} - (\alpha_{\ell+1} + 1)$. If $\alpha_{\ell+1} = 2$, then $\ell = 1$ and $k \ge 6$. Since also $k < 2^h + 2^{h-1}$, it follows that $h \ge 3$. Let $x \in V_{h-2}$ with y_1 and y_2 the children of x. Choose $y_3 \in V_{h-1} - \{y_1, y_2\}$ and let the children of y_i be z_{i1} and z_{i2}, $i \in \{1,2,3\}$. Let $W_2 = \{x, y_1, y_2, y_3, z_{11}, z_{21}, z_{31}, z_{32}\}$. Then $|W_2| = 8 = 2(2\ell + 2)$. Choose W_1 disjoint from W_2 as in Case 1 and let $W = W_1 \cup W_2$. If f is defined as in Case 1, then it is easy to check that $W \subseteq C_f$.

If $\alpha_{\ell+1} = 1$, then $\alpha_i \ne 0$ for some $i < \ell + 1$. For $\ell = 1$, let $W_2 = \{x, y_1, y_2, z_{11}, z_{12}, z_{21}, z_{22}\}$ and let W_1, disjoint from W_2, consist of $\alpha_1 - 1$ vertices of V_{h-1} together with their adjacent leaves. For $\ell = 2$, note that $\alpha_1 = 0$, for otherwise $k = k - 4 - 6 + 2(5)$, contradicting the choice of α_3. Also, $h \ge 2$. Let $u \in V_{h-2}$ have children $v_1, \ldots, v_4$ and let v_i ($i \in \{1, \ldots, 4\}$) have children $w_{i1}, \ldots, w_{i4}$. Let $W_3 = \{u, v_1, v_2, v_3, w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{31}, w_{33}\}$ and let W_2, disjoint from W_3, consist of $\alpha_2 - 1$ vertices of V_{h-1} together with their children. In either case, if W and f are defined as before, it is straightforward to verify that f is a kSF with $ag(f) = 2k - n$. □

Note that the lower bound for k given in Theorem 12 depends only on m and not on the order of the tree.

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