Path-Width and Tree-Width of the Join of Graphs*

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ABSTRACT. Let $PW(G)$ and $TW(G)$ denote the path-width and tree-width of a graph G, respectively. Let $G+H$ denote the join of two graphs G and H. We show in this paper that

$$PW(G+H) = \min\{|V(G)| + PW(H), |V(H)| + PW(G)\}$$

and

$$TW(G+H) = \min\{|V(G)| + TW(H), |V(H)| + TW(G)\}$$

1 Introduction

Graphs considered in this paper are finite, and may have loops or multiple edges. For a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively.

To study the theory of graph minors, the concepts of path-width and tree-width were introduced in [1,2,3].

A path-decomposition of a graph G is a sequence $(X_1, X_2, \ldots, X_m)$ of subsets of $V(G)$ such that

(i) $X_1 \cup X_2 \cup \cdots \cup X_m = V(G)$;

(ii) For every $e \in H(G)$, there exists i with $1 \leq i \leq m$ such that X_i contains both ends of e;

(iii) For $1 \leq i \leq j \leq k \leq m$, $X_i \cap X_k \subseteq X_j$.

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The path-width $PW(G)$ of G is the minimum of $\max(|X_i|-1\colon 1 \le i \le m)$, taken over all path-decompositions $(X_1, \ldots, X_m)$ of G. (The null graph has path-width 0).

A tree-decomposition of a graph G is a pair $(T, \mathcal{X})$, where T is a tree and $\mathcal{X} = (X_t\colon t \in V(T))$ is a family of subsets of $V(G)$, with the following properties:

(W1) $\cup(X_t\colon t \in V(T)) = V(G)$;

(W2) For every $e \in H(G)$ there exists $t \in V(T)$ such that e has both ends in X_t;

(W3) For t, t', $t'' \in V(T)$, if t' is on the path of T between t and t'', then $X_t \cap X_{t''} \subseteq X_{t'}$.

The width of the tree-decomposition $(T, \mathcal{X})$ of G is defined as

$$TW(G,T,\mathcal{X}) = \max\{|X_t| - 1 | t \in V(T)\}$$

And the tree-width of G is defined as

$$TW(G) = \min\{TW(G,T,\mathcal{X}) | (T,\mathcal{X}) \text{ is a tree-decomposition of } G\}.$$

We can easily see that the definition of path-decomposition and path-width can be equivalently restated as:

(I) A path-decomposition of a graph G is a tree-decomposition $(T, \mathcal{X})$ of G such that T is a path;

(II) $PW(G) = \min\{TW(G,T,\mathcal{X}) | (T,\mathcal{X}) \text{ is a path decomposition of } G\}$.

The join $G+H$ of two graphs G and H (suppose that G and H are disjoint) is the graph obtained from G and H by adding all edges between $V(G)$ and $V(H)$.

The main results of this paper are as follows:

$$PW(G+H) = \min\{|V(G)| + PW(H), |V(H)| + PW(G)\}$$

and

$$TW(G+H) = \min\{|V(G)| + TW(H), |V(H)| + TW(G)\}$$

Terminology and notation not defined in this paper can be found in [1-3].

2 Foundamental Lemmas

A graph H is a minor of a graph G if either $H = G$ or H can be obtained from G by using (may be repeatedly) the following graph operations:

(i) Deleting a vertex;

(ii) Deleting an edge;

(iii) Contracting an edge.

When a tree H is a minor of a graph G, we say that H is a minor subtree of G.

Lemma 2.1. *Suppose that $(T, \mathcal{X})$ is a tree-decomposition of the join $G+H$ of two graphs G and H, where $\mathcal{X} = (X_t\colon t \in V(T))$. Then there is a minor subtree T' of T together with a family $\mathcal{X}' = (X'_t\colon t \in V(T'))$ satisfying the following three conditions.*

(1) $(T', \mathcal{X}')$ is a tree-decomposition of $G + H$;

(2) Either $V(G) \subseteq \cap(X'_t\colon t \in V(T'))$ or $V(H) \subseteq \cap(X'_t\colon t \in V(T'))$;

(3) $TW(G + H, T', \mathcal{X}') \leq TW(G + H, T, \mathcal{X})$.

Proof: Note that $(T, \mathcal{X})$ is a tree-decomposition satisfying (1) and (3). Choose a minor subtree T' of T together with a family $\mathcal{X}' = (X'_t\colon t \in V(T'))$ which satisfies the above conditions (1) and (3) such that $|V(T')|$ is as small as possible.

If $|V(T')| = 1$, then (2) holds trivially.

Suppose $|V(T')| \geq 2$, and set

$$V_0 = \{t \in V(T')|t \text{ is a vertex of } T' \text{ of degree } 1\}.$$

Statement 1 For each $t \in V_0$, either $V(G) \subseteq X'_t$ or $V(H) \subseteq X'_t$.

Otherwise, there exist $t \in V_0$, $y \in V(G)$, $z \in V(H)$, such that $y, z \in X'_t$. Suppose that t' is the vertex adjacent to t in T'. For each $u \in V(G) \cap X'_t$, $uz \in H(G + H)$. By (W2), there is a certain $t'' \in V(T')$, such that $u, z \in X'_{t''}$. Now, t' lies on the path of T between t and t''. According to (W3), $u \in X'_t \cap X'_{t'}G \subseteq X'_{t'}$. Hence, $V(G) \cap X'_t \subseteq X'_{t'}$. Similarly, we can prove that $V(H) \cap X'_t \subseteq X'_{t'}$ and then $X'_t = X'_t \cap (V(G) \cup V(H)) \subseteq X'_{t'}$. Set $T'' = T' - t$, $\mathcal{X}'' = \mathcal{X}' - X'_t$. Then we can easily see that T'' is also a minor subtree of T, and T'' together with $\mathcal{X}''$ satisfies the conditions (1) and (3). But $|V(T'')| < |V(T')|$, contrary to the choice of T'.

Statement 2 Either $V(G) \subseteq \cap(X'_t: t \in V_0)$ or $V(H) \subseteq \cap(X'_t: t \in V_0)$.

Otherwise by Statement 1, there exist $t', t'' \in V_0$ such that

$$\begin{array}{ll} V(G) \subseteq X'_{t'}, & V(G) \not\subseteq X'_{t''}, \\ V(H) \subseteq X'_{t''}, & V(H) \not\subseteq X'_{t'}. \end{array}$$

Suppose that $(t_1, t_2, \ldots, t_r)$ is the path of T' between t' and t'', where $t_1 = t'$, $t_r = t''$. By (W2), we can deduce

$$V(H) \cap X'_{t_i} \subseteq V(H) \cap X'_{t_j}$$

and

$$V(G) \cap X'_{t_j} \subseteq V(G) \cap X'_{t_i},$$

for $1 \leq i \leq j \leq r$. Now we consider the following three cases.

Case 1 $V(G) \subseteq X'_{t_2}$.

In this case, $X'_{t_1} \subseteq X'_{t_2}$. Set $T'' = T' - t_1$, $\mathcal{X}'' = \mathcal{X}' - X'_{t_1}$. As in Statement 1, we can get a contradiction.

Case 2 $V(H) \cap X'_{t_1} = V(H) \cap X'_{t_2}$ and $V(G) \not\subseteq X'_{t_2}$.

In this case, $X'_{t_2} \subseteq X'_{t_1}$. Set

$$X''_{t_1} = X'_{t_2}, X''_{t_2} = X'_{t_1} \text{ and } X''_t = X'_t$$

for $t \in V(T') \setminus \{t_1, t_2\}$. Then T' together with $\mathcal{X}'' = (X''_t: t \in V(T'))$ is still a tree-decomposition of $G + H$ and $TW(G + H, T', \mathcal{X}') = TW(G + H, T', \mathcal{X}'')$. But $V(G) \not\subseteq X''_{t_1}$ and $V(H) \not\subseteq X''_{t_1}$, this contradicts our Statement 1.

Case 3 $V(H) \cap X'_{t_1} \neq V(H) \cap X'_{t_2}$ and $V(G) \not\subseteq X'_{t_2}$

In this case, there exists $uv \in E(G + H)$ such that $u \in V(G) \setminus X'_{t_2} \subseteq X'_{t_1} \setminus X'_{t_2}$ and $v \in (V(H) \cap X'_{t_2}) \setminus X'_{t_1} \subseteq X'_{t_2} \setminus X'_{t_1}$. Especially, $u \not\subseteq X'_{t_1}$ and $\{u, v\} \not\subseteq X'_{t_1}$. By (W2), there is a certain $t \in V(T')$ such that $u, v \in X'_t$. Because t_2 lies on the path of T' between t_1 and t, by using (W3), we have $u \in X'_{t_1} \cap X'_t \subseteq X'_{t_2}$, a contradiction. This completes the proof of Statement 2.

Without loss of generality, we can suppose that $V(H) \subseteq \cap(X'_t: t \in V_0)$ For each vertex $t \in V(T') \setminus V_0$, there exist two vertices $t', t'' \in V_0$ such that t lies on the path of T' between t' and t''. By using (W3) again, we deduce $V(H) \subseteq X'_{t'} \cap X'_{t''} \subseteq X'_t$. Hence

$$V(H) \subseteq \cap(X'_t: t \in V(T'))$$

This completes the proof of our Lemma.

Because a path-decomposition is in fact a special tree-decomposition. We can see that the following Lemma is also true.

Lemma 2.2. *Let $(T, \mathcal{X})$ be a path-decomposition of $G + H$. Then the result of Lemma 2.1 is still true, and in this case $(T', \mathcal{X}')$ (in Lemrna 2.1) is also a path-decomposition of $G + H$.*

3 Proof of the Main Results

Theorem 3.1. $TW(G+H) = \min\{|V(G)| + TW(H), |V(H)| + TW(G)\}$

Proof: Given a tree-deccmpesition $(T, \mathcal{X})$ of H such that $TW(H, T, \mathcal{X}) = TW(H)$. Set $T' = T$, $X_t' = X_t \cup V(G)$ and $\mathcal{X}' = (X_t' \colon t \in V(T'))$. We can easily see that $(T', \mathcal{X}')$ is a tree-decomposition of $G + H$ with width $|V(G)| + TW(H)$, so $TW(G + H) \leq |V(G)| + TW(H)$. Symmetricaly, we also have $TW(G + H) \leq |V(H)| + TW(G)$. Hence

$$TW(G+H) \leq \min\{|V(G)| + TW(H), |V(H)| + TW(G)\}$$

On the other hand, suppose that $(T, \mathcal{X})$ is a tree-decomposition of $G+H$ such that $TW(G + H, T, \mathcal{X}) = TW(G + H)$. According to Lemma 2.1, we can further suppose that either $V(G) \subseteq \cap(X_t \colon t \in V(T))$ or $V(H) \subseteq \cap(X_t \colon t \in V(T))$. If $V(G) \subseteq \cap(X_t \colon t \in V(T))$, then set $T' = T$, $X_t' = X_t \setminus V(G)$, for $t \in V(T)$, and $\mathcal{X}' = (X_t' \colon t \in V(T'))$. We can easily see that $(T', \mathcal{X}')$ is a tree-decomposition of H with width $TW(H, T', \mathcal{X}') = TW(G+H) - |V(G)|$. In this case, we have $TW(G+H) \geq |V(G)| + TW(H)$. If $V(H) \subseteq \cap(X_t \colon t \in V(T))$, we also can prove that $TW(G+H) \geq |V(H)| + TW(G)$. Hence we always have

$$TW(G+H) \geq \min\{|V(G)| + TW(H), |V(H)| + TW(G)\}$$

This completes our proof.

By using Lemma 2.2 and the similar technique as in Theorem 3.1, we can prove the following Theorem.

Theorem 3.2. $PW(G+H) = \min\{|V(G)| + PW(H), |V(H)| + PW(G)\}$.

Using the above results, we can deduce the tree-width and path-width of the join of k graphs.

Theorem 3.3. *Let $G = G_1 + G_2 + \cdots + G_k$ be the join of k graphs $G_1, G_2, \ldots, G_k$. Then*

$$TW(G) = |V(G)| - \max\{|V(G_i)| - TW(G_i) | 1 \leq i \leq k\},$$

and

$$PW(G) = |V(G)| - \max\{|V(G_i)| - PW(G_i) | 1 \leq i \leq k\}$$

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